Lesson 5: Reasoning about square roots

Goals

- Comprehend that $-\sqrt{a}$ represents the opposite of $\sqrt{a}$.
- Determine a solution to an equation of the form $x^2 = a$ and represent the solution as a point on the number line.
- Identify the two whole number values that a square root is between and explain (orally) the reasoning.

Learning Targets

- When I have a square root, I can reason about which two whole numbers it is between.

Lesson Narrative

The purpose of this lesson is to encourage students to reason about square roots and reinforce the idea that they are numbers on a number line. This lesson continues students' move from geometric to algebraic characterisations of square roots.

Addressing

- Use square root and cube root symbols to represent solutions to equations of the form $x^2 = p$ and $x^3 = p$, where p is a positive rational number. Evaluate square roots of small perfect squares and cube roots of small perfect cubes. Know that $\sqrt{2}$ is irrational.
- Use rational approximations of irrational numbers to compare the size of irrational numbers, locate them approximately on a number line diagram, and estimate the value of expressions (e.g., π^2). For example, by truncating the decimal expansion of $\sqrt{2}$, show that $\sqrt{2}$ is between 1 and 2, then between 1.4 and 1.5, and explain how to continue on to get better approximations.

Instructional Routines

- Stronger and Clearer Each Time
- Clarify, Critique, Correct
- Think Pair Share
- True or False

Student Learning Goals

Let's approximate square roots.

5.1 True or False: Squared

Warm Up: 5 minutes

The purpose of this warm-up is for students to analyse symbolic statements about square roots and decide if they are true or not based on the meaning of the square root symbol.

Instructional Routines

- True or False

Launch

Display one problem at a time. Tell students to give a signal when they have an answer and a strategy. After each problem, give students 1 minute of quiet think time and follow with a whole-class discussion.

Student Task Statement

Decide if each statement is true or false.

$\left(\sqrt{5}\right)^2 = 5$

$\left(\sqrt{9}\right)^2 = 3$

$7 = \left(\sqrt{7}\right)^2$

$\left(\sqrt{10}\right)^2 = 100$

$\left(\sqrt{16}\right) = 2^2$

Student Response

true, false, true, false, true

Activity Synthesis

Poll students on their responses for each problem. Record and display their responses for all to see. If all students agree, ask 1 or 2 students to share their reasoning. If there is disagreement, ask students to share their reasoning until an agreement is reached.

5.2 Square Root Values

10 minutes

The purpose of this activity is for students to think about square roots in relation to the two whole number values they are closest to. Students are encouraged to use numerical approaches, especially the fact that $\sqrt{a}$ is a solution to the equation $x^2 = a$, rather than less efficient geometric methods (which may not even work). Students can draw a number line if that helps them reason about the magnitude of the given square roots, but this is not required. However the reason, students must construct a viable argument.

Instructional Routines

- Clarify, Critique, Correct
- Think Pair Share

Launch

Do not give students access to calculators. Students in groups of 2. 2 minutes of quiet work time followed by a partner then a whole-class discussion.

Representation: Internalise Comprehension. Activate or supply background knowledge. Provide students with access to a number line that includes rational numbers to support information processing.
Supports accessibility for: Visual-spatial processing; Organisation

Student Task Statement

What two whole numbers does each square root lie between? Be prepared to explain your reasoning.

1. $\sqrt{7}$
2. $\sqrt{23}$
3. $\sqrt{50}$
4. $\sqrt{98}$

Student Response

1. 2 and 3. 2^2 is 4 and 3^2 is 9, so $\sqrt{7}$ is between 2 and 3.
2. 4 and 5. 4^2 is 16 and 5^2 is 24, so $\sqrt{23}$ is between 4 and 5.
3. 7 and 8. 7^2 is 49 and 8^2 is 64, so $\sqrt{50}$ is between 7 and 8.
4. 9 and 10. 9^2 is 81 and 10^2 is 100, so $\sqrt{98}$ is between 9 and 10.

Are You Ready for More?

Can we do any better than “between 3 and 4” for $\sqrt{12}$? Explain a way to figure out if the value is closer to 3.1 or closer to 3.9.

Student Response

Answers vary. Sample response: Since $3.5^2 = 12.25$, we know that it is somewhere between 3 and 3.5. That tells us that it is closer to 3.1 than 3.9.

Activity Synthesis

Discuss:

- "What strategy did you use to figure out the two whole numbers?" (I made a list of perfect squares and then found which two the number was between.)
- "Did anyone use inequality symbols when writing their answers?" (Yes, for the first problem, I wrote $2 < \sqrt{5} < 3$.)

Once the class is satisfied with which two whole numbers the square roots lie between, ask students to think more deeply about their relationship. Give 1–2 minutes for students to pick one of the last two square roots and figure out which whole number the square root is closest to and to be ready to explain how they know. One possible misconception that could be covered here is that if a number is exactly halfway between two perfect squares, then the square root of that number is also halfway between the square root of the perfect squares. For example, students may think that $\sqrt{26}$ is halfway between 4 and 6 since 26 is halfway between 16 and 36. It's close, since $\sqrt{26} \approx 5.099$, but it's slightly larger than "halfway."

This is a good opportunity to remind students of the graph they made earlier showing the relationship between the side length and area of a square. The graph showed a non-proportional relationship, so making proportional assumptions about relative sizes will not be accurate.

Reading, Writing, Speaking: Clarify, Critique, Correct. Before students share their strategies for figuring out the two whole numbers that each square root lies between, present an incorrect solution based on a misconception about the definition of exponents. For example, "$\sqrt{7}$ is in between 2 and 4, because 2^2 is 4, and 4^2 is 8"; or "$\sqrt{23}$ is in between 11 and 12, because 11^2 is 22, and 12^2 is 24." Ask students to identify the error, critique the reasoning, and revise the original statement. As students discuss in pairs, listen for students who clarify the meaning of a number raised to the power of 2. This routine will engage students in meta-awareness as they critique and correct the language used to relate square roots to the two whole number values they are closest to.
Design Principles(s): Cultivate conversation; Maximise meta-awareness

5.3 Solutions on a Number Line

10 minutes

The purpose of this activity is for students to use rational approximations of irrational numbers to place both rational and irrational numbers on a number line and to reinforce the definition of a square root as a solution to the equation of the form $x^2 = a$. This is also the first time that students have thought about negative square roots.

Instructional Routines

- Stronger and Clearer Each Time
- Think Pair Share

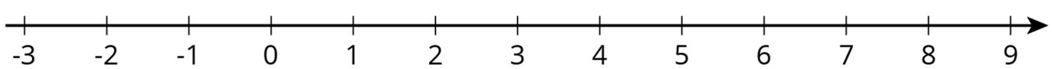
Launch

Do not provide students with access to calculators. Students in groups of 2. 2 minutes of quiet work time followed by a partner, then a whole-class discussion.

Representation: Develop Language and Symbols. Use virtual or concrete manipulatives to connect symbols to concrete objects or values. For example, use a kinaesthetic representation of the number line on a clothesline. Students can place and adjust numbers on folder paper or card on the clothesline in a hands-on manner.
Supports accessibility for: Conceptual processing Writing, Speaking, Listening: Stronger and Clearer Each Time. After students have had time to plot x, y, and z on the number line, ask them to write a brief explanation of their reasoning for each number on their paper. Ask each student to meet with 2–3 other partners in a row for feedback. Provide students with prompts for feedback that will help them strengthen their ideas and clarify their language (e.g., "How do you know that $z = \sqrt{30}$?" and "How do you know that $\sqrt{30}$ is between 5 and 6?", etc.). Students can borrow ideas and language from each partner to refine and clarify their original explanation. This will help students revise and refine both their ideas and their verbal and written output.
Design Principles(s): Optimise output (for explanation); Maximise meta-awareness

Student Task Statement

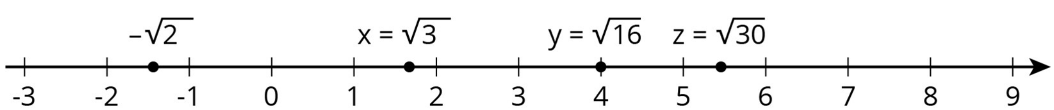
The numbers x, y, and z are positive, and $x^2 = 3$, $y^2 = 16$, and $z^2 = 30$.



1. Plot x, y, and z on the number line. Be prepared to share your reasoning with the class.

2. Plot -$\sqrt{2}$ on the number line.

Student Response



Activity Synthesis

Display the number line from the activity for all to see. Select groups to share how they chose to place values onto the number line. Place the values on the displayed number line as groups share, and after each placement poll the class to ask if students used the same reasoning or different reasoning. If any students used different reasoning, invite them to share with the class.

Conclude the discussion by asking students to share how they placed -$\sqrt{2}$ and why.

Lesson Synthesis

To approximate a square root, start by finding the whole numbers it lies between, and then try to get more accurate approximations.

- "How can we find the whole numbers that a square root lies between?" (Look at the squares of whole numbers whose squares are greater than and less than the number inside the square root symbol, like 121 and 144 for $\sqrt{130}$.)
- "How can we get a better approximation?" (Test values between those two whole numbers.)

5.4 Betweens

Cool Down: 5 minutes

Student Task Statement

Which of the following numbers are greater than 6 and less than 8? Explain how you know.

- $\sqrt{7}$
- $\sqrt{60}$
- $\sqrt{80}$

Student Response

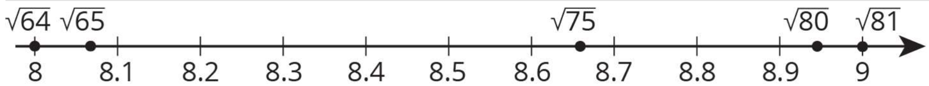
Only $\sqrt{60}$.

Since $6^2 = 36$ and $8^2 = 64$, the number inside the square root must be between 36 and 64.

Student Lesson Summary

In general, we can approximate the values of square roots by observing the whole numbers around it, and remembering the relationship between square roots and squares. Here are some examples:

- $\sqrt{65}$ is a little more than 8, because $\sqrt{65}$ is a little more than $\sqrt{64}$ and $\sqrt{64} = 8$.
- $\sqrt{80}$ is a little less than 9, because $\sqrt{80}$ is a little less than $\sqrt{81}$ and $\sqrt{81} = 9$.
- $\sqrt{75}$ is between 8 and 9 (it's 8 point something), because 75 is between 64 and 81.
- $\sqrt{75}$ is approximately 8.67, because $8.67^2 = 75.1689$.



If we want to find a square root between two whole numbers, we can work in the other direction. For example, since $22^2 = 484$ and $23^2 = 529$, then we know that $\sqrt{500}$ (to pick one possibility) is between 22 and 23.

Many calculators have a square root command, which makes it simple to find an approximate value of a square root.

Lesson 5 Practice Problems

1. **Problem 1 Statement**

   a. Explain how you know that $\sqrt{37}$ is a little more than 6.

   b. Explain how you know that $\sqrt{95}$ is a little less than 10.

   c. Explain how you know that $\sqrt{30}$ is between 5 and 6.

   Solution

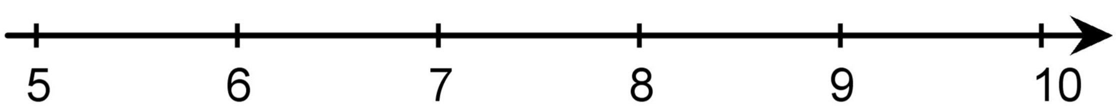
   a. $\sqrt{36}$ is exactly 6, and $\sqrt{37}$ is a little more than that.

   b. $\sqrt{100}$ is exactly 10, and $\sqrt{95}$ is a little less than that.

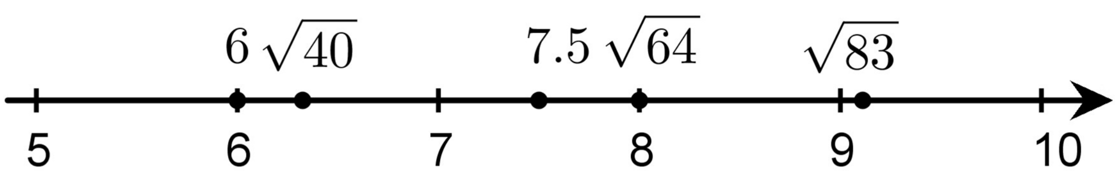
   c. $\sqrt{25} = 5$, $\sqrt{36} = 6$, and $\sqrt{30}$ is in between.

2. **Problem 2 Statement**

   Plot each number on the number line: 6, $\sqrt{83}$, $\sqrt{40}$, $\sqrt{64}$, 7.5



   Solution



3. **Problem 3 Statement**

The equation $x^2 = 25$ has *two* solutions. This is because both $5 \times 5 = 25$, and also -5 × -5 = 25. So, 5 is a solution, and also -5 is a solution.

Select **all** the equations that have a solution of -4:

a. $10 + x = 6$

b. $10 - x = 6$

c. $-3x = -12$

d. $-3x = 12$

e. $8 = x^2$

f. $x^2 = 16$

Solution ["A", "D", "F"]

4. **Problem 4 Statement**

Find all the solutions to each equation.

a. $x^2 = 81$

b. $x^2 = 100$

c. $\sqrt{x} = 12$

Solution

a. 9 and -9

b. 10 and -10

c. 144

5. **Problem 5 Statement**

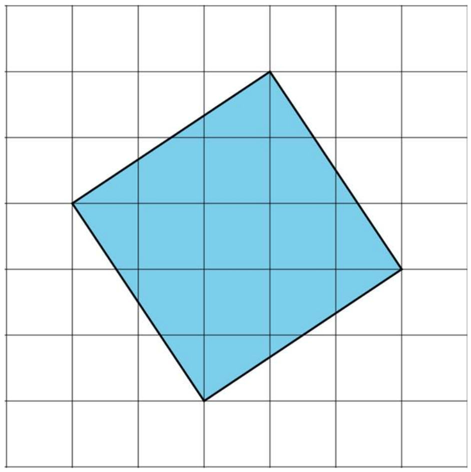
Select all the irrational numbers in the list. $\frac{2}{3}, \frac{-123}{45}, \sqrt{14}, \sqrt{64}, \sqrt{\frac{9}{1}}, -\sqrt{99}, -\sqrt{100}$

Solution

$\sqrt{14}, -\sqrt{99}$

6. **Problem 6 Statement**

Each grid square represents 1 square unit. What is the exact side length of the shaded square?



Solution

$\sqrt{13}$ units

7. **Problem 7 Statement**

For each pair of numbers, which of the two numbers is larger? Estimate how many times larger.

a. 0.37×10^6 and 700×10^4

b. 4.87×10^4 and 15×10^5

c. $500\,000$ and 2.3×10^8

Solution

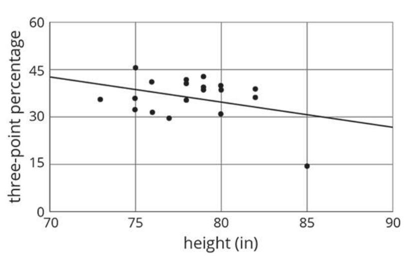
a. 700×10^4, about 20 times larger

b. 15×10^5, about 30 times larger

c. 2.3×10^8, about 500 times larger

8. **Problem 8 Statement**

The scatter plot shows the heights (in inches) and three-point percentages for different basketball players last season.



a. Circle any data points that appear to be outliers.

b. Compare any outliers to the values predicted by the model.

Solution

a. The point at (85, 14) is an outlier.

b. This point represents a player who had a significantly worse (by about 15% of the attempts) three-point percentage than the model predicts for his height.