Lesson 7: A proof of Pythagoras' theorem

Goals

- Calculate an unknown side length of a right-angled triangle using Pythagoras' theorem, and explain (orally) the reasoning.
- Explain (orally) an area-based algebraic proof of Pythagoras' theorem.

Learning Targets

- I can explain why Pythagoras' theorem is true.

Lesson Narrative

In the warm-up of this lesson, students study a diagram they will use to prove Pythagoras' theorem. In the first activity they prove Pythagoras' theorem using the diagram. Then they apply Pythagoras' theorem in the next activity. The final activity before the cool down is an optional look at a transformational proof of Pythagoras' theorem.

Addressing

- Understand and apply Pythagoras' theorem.
- Explain a proof of Pythagoras' theorem and its converse.
- Apply Pythagoras' theorem to determine unknown side lengths in right-angled triangles in real-world and mathematical problems in two and three dimensions.

Building Towards

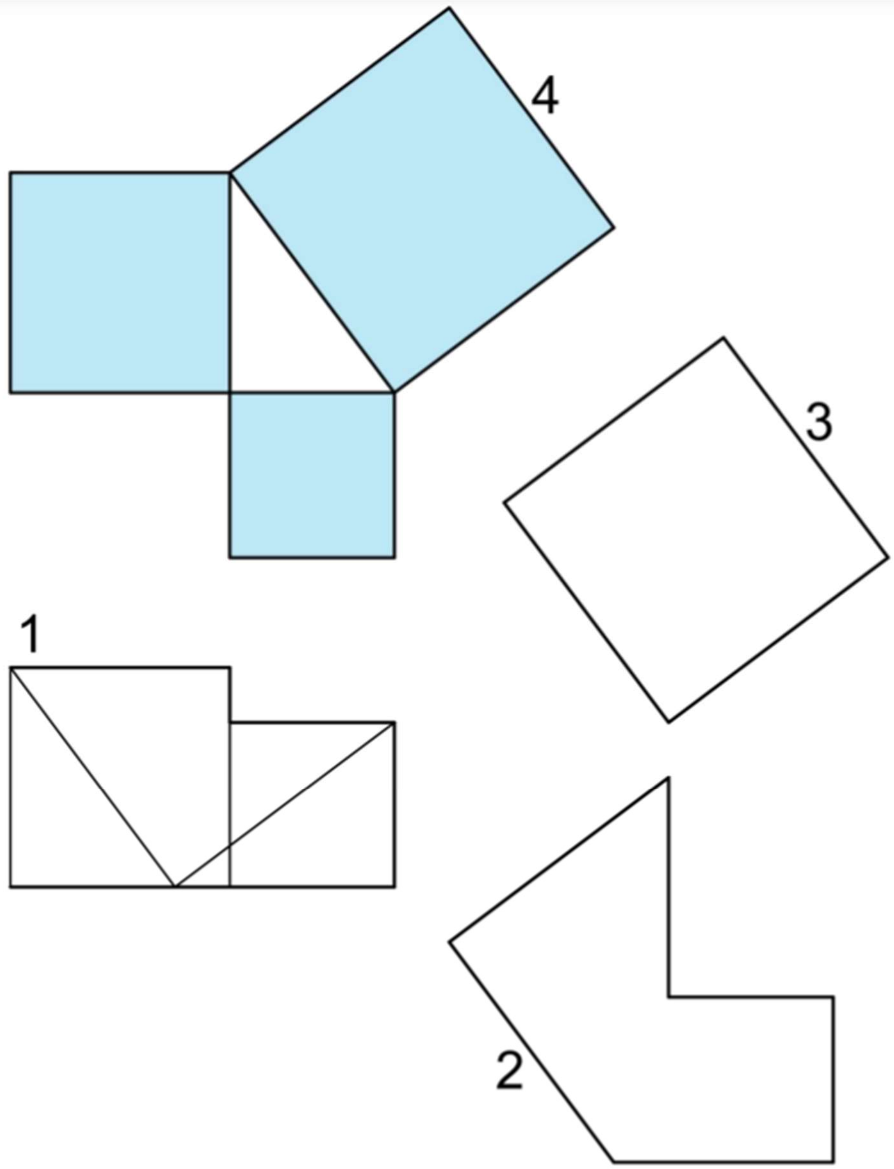
- Explain a proof of Pythagoras' theorem and its converse.

Instructional Routines

- Collect and Display
- Clarify, Critique, Correct
- Co-Craft Questions
- Notice and Wonder
- Think Pair Share

Required Materials

Copies of blackline master
A Transformational Proof



Required Preparation

If you choose to do the optional activity, you will need the 5 cut-out shapes from the Making Squares blackline master used in the first lesson of this unit—1 set of 5 for every 2 students. You will also need copies of the A Transformational Proof blackline master—1 copy for every 2 students.

Student Learning Goals

Let's prove Pythagoras' theorem.

7.1 Notice and Wonder: A Square and Four Triangles

Warm Up: 5 minutes

The purpose of this warm-up is to give students a chance to study a diagram that they will need to understand for an upcoming proof of Pythagoras' theorem. The construction depends on the triangles being right-angled triangles, so students get to contrast it with a similarly constructed figure with non-right-angled triangles. In that case, the composite figure is not a square.

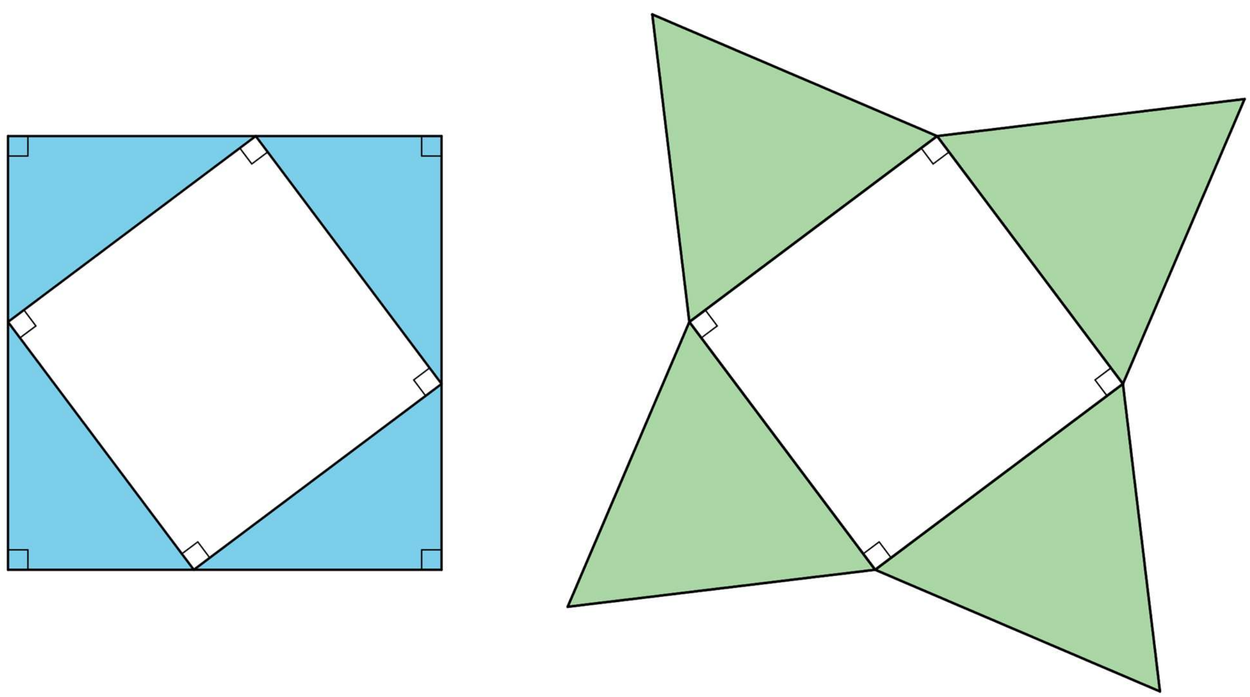
Instructional Routines

- Notice and Wonder

Launch

Arrange students in groups of 2. Display the diagram for all to see. Give students 1 minute of quiet work time to identify at least one thing they notice and at least one thing they wonder about the diagram. Ask students to give a signal when they have noticed or wondered about something. When the minute is up, give students 1 minute to discuss their observations and questions with their partner. Follow with a whole-class discussion.

Student Task Statement



What do you notice? What do you wonder?

Student Response

Answers vary. Sample response:

I notice that:

- There are two figures both made up of a square and four triangles.
- The triangles in the left-hand figure are right-angled triangles and the other triangles are not.
- The figure on the left is a square.
- The figure on the right is not a square.
- The small squares in the middle look the same size.

I wonder:

- Are the squares in the middle the same size?
- Are the blue triangles all the same size?
- Are the green triangles all the same size?
- Does it matter if the triangles are right-angled triangles?
- Does it matter if they are equilateral triangles?
- What are these figures going to be used for?

Activity Synthesis

Ask students to share the things they noticed and wondered. Record and display their responses for all to see. If possible, record the relevant reasoning on or near the image. After each response, ask the class whether they agree or disagree and to explain alternative ways of thinking, referring back to the images each time.

Tell students that when you take a square and put a congruent right-angled triangle on each side as shown on the left, they form a larger square (they will be able to prove this later in KS4). But it doesn't work if the triangles are not right-angled triangles. We will use this construction in the next activity.

7.2 Adding Up Areas

15 minutes

The purpose of this activity is for students to work through an area-based algebraic proof of Pythagoras' theorem. One of the figures used in this particular proof, G, was first encountered by students at the start of the year during a unit on transformations and again in a recent lesson where they reasoned about finding the area of the triangles.

While there are many proofs of Pythagoras' theorem similar to the one in this activity, they often rely on $(a+b)^2 = a^2 + 2ab + b^2$, which is material beyond the scope of KS3.

For this proof, students reason about the areas of the two squares with the same dimensions. Each square is divided into smaller regions in different ways and it is by using the equality of the total area of each square that they uncover Pythagoras' theorem. The extension uses this same division to solve a challenging area composition and decomposition problem.

Instructional Routines

- Co-Craft Questions
- Think Pair Share

Launch

Begin by explaining to students how the two figures are constructed. Each figure starts with a square with side length $a + b$.

- Figure F partitions the square into two squares and two rectangles.
- Figure G takes a right-angled triangle with shorter sides a and b and puts one identical copy of it in each corner of the square. The copies touch each other because the short leg of one and the long leg of the one next to it add up to $a + b$, so they fit exactly into a side. So they form a quadrilateral in the middle. We know the quadrilateral is a square because
    - The corners must be 90 degree angles:
        - The two acute angles in each triangle must sum to 90 degrees because the sum of the angles in a triangle is 180 degrees, and the third angle is 90 degrees.
        - The two smaller angles along with one of the corners of the quadrilateral form a straight angle with a measure of 180 degrees, that means that the angle at the corner must also be 90 degrees.
    - All four sides are the same length: they all correspond to a hypotenuse of one of the congruent right-angled triangles.

Arrange students in groups of 2. Give 3 minutes of quiet work time for the first two problems. Ask partners to share their work and come to an agreement on the area of each figure and region before moving on to the last problem. Follow with a whole-class discussion.

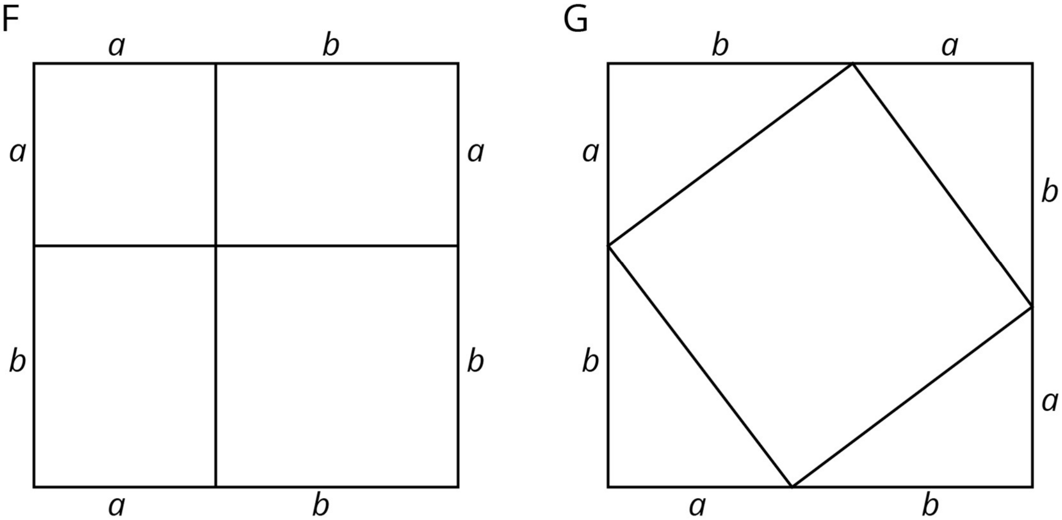
Representation: Internalise Comprehension. Differentiate the degree of difficulty or complexity by beginning with an example with more accessible values. For example, consider demonstrating how to calculate the area of a figure made of various shapes using numbers instead of variables for the side lengths. Highlight connections between this simpler figure and the one used in the activity by highlighting corresponding side lengths. *Supports accessibility for: Conceptual processing Conversing, Writing: Co-Craft Questions.*

Before revealing the questions in this activity, display the image of the squares with a side length of $a + b$ and invite students to write possible mathematical questions about the diagram. Ask students to compare the questions they generated with a partner before sharing questions with the whole class. Listen for and amplify questions about the total area for each square or the area of each of the nine smaller regions of the squares. If no student asks about the area of each smaller region, ask students to adapt a question to align with the learning goals of this lesson. Then reveal and ask students to work on the actual questions of the task. This routine will help develop students' meta-awareness of language as they generate questions about area in preparation for the proof of Pythagoras' theorem.
Design Principle(s): Maximise meta-awareness

Student Task Statement

Both figures shown here are squares with a side length of $a + b$. Notice that the first figure is divided into two squares and two rectangles. The second figure is divided into a square and four right-angled triangles with shorter sides of lengths a and b. Let's call the hypotenuse of these triangles c.

F   G



1. What is the total area of each figure?

2. Find the area of each of the 9 smaller regions shown in the figures and label them.

3. Add up the area of the four regions in Figure F and set this expression equal to the sum of the areas of the five regions in Figure G. If you rewrite this equation using as few terms as possible, what do you have?

Student Response

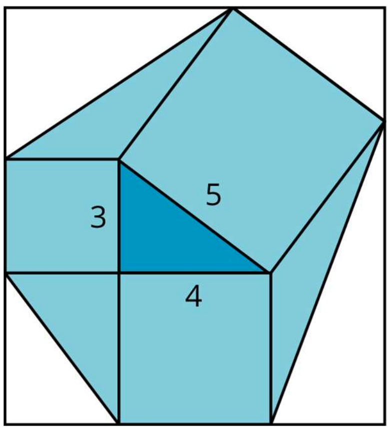
1. $(a + b)^2$

2. Figure F: a^2, ab, ab, b^2. Figure G: $\frac{1}{2}ab$, $\frac{1}{2}ab$, $\frac{1}{2}ab$, $\frac{1}{2}ab$, c^2.

3. $a^2 + b^2 = c^2$. The sum of the area of regions in F is $a^2 + 2ab + b^2$, and the sum of the area of regions in G is $4\left(\frac{1}{2}ab\right) + c^2 = 2ab + c^2$. Then $a^2 + 2ab + b^2 = 2ab + c^2$ implies $a^2 + b^2 = c^2$.

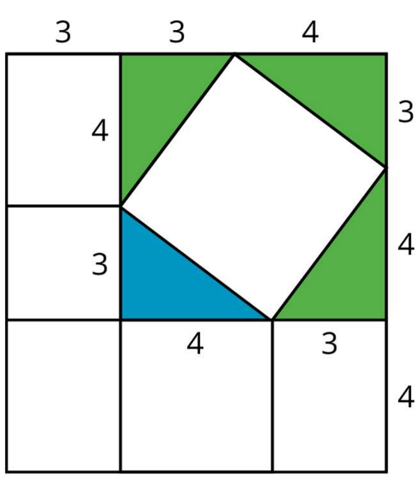
Are You Ready for More?

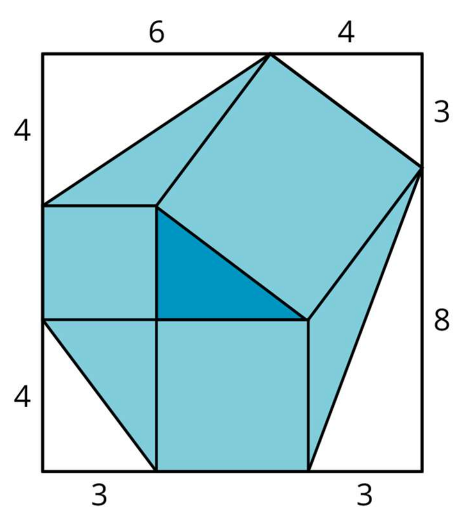
Take a 3-4-5 right-angled triangle, add on the squares of the side lengths, and form a hexagon by connecting vertices of the squares as in the image. What is the area of this hexagon?



Student Response

We can find the area of the shaded region if we can find the area of the surrounding rectangle and subtract the area of the four unshaded triangles. To find the dimensions of the rectangle, add in three more copies of the right-angled triangle as in our proof of Pythagoras' theorem. By writing down all the side lengths (the image on the left), we can see that the width of the rectangle is 10 and its height is 11, and so has an area of 110.



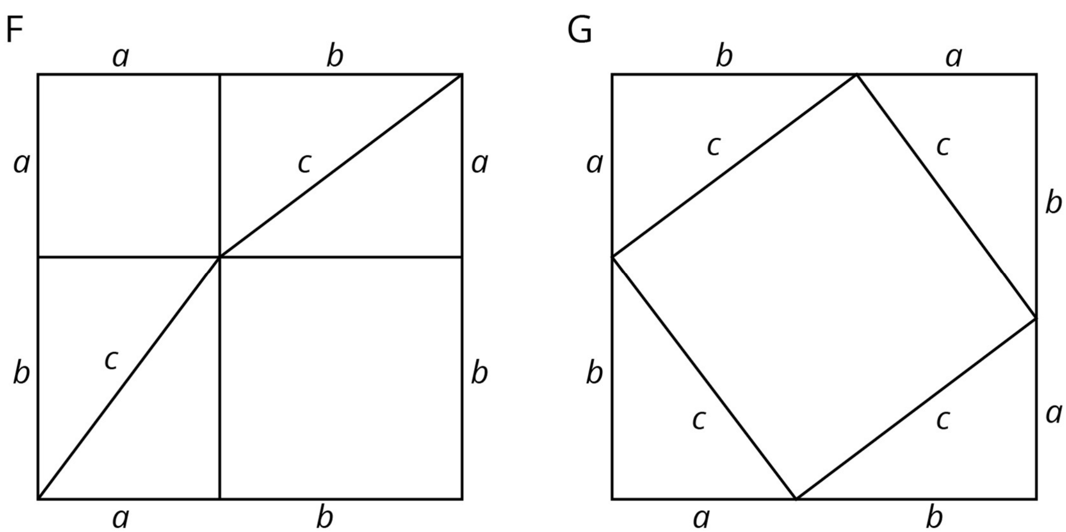


Returning to the original image (the image on the right), we see that the four triangles we have to subtract have areas 12, 6, 6, and 12. The area of the shaded region is $110 - 12 - 6 - 6 - 12 = 74$.

Activity Synthesis

Begin the discussion by selecting 2–3 groups to share their work and conclusion for the third question. Make sure the last group presenting concludes with $a^2 + b^2 = c^2$ or something close enough that the class can get there with a little prompting. For example, if groups are stuck with the equation looking something like $a^2 + ab + b^2 + ab = \frac{1}{2}ab + \frac{1}{2}ab + \frac{1}{2}ab + \frac{1}{2}ab + c^2$, encourage them to try and combine like terms and remove any quantities both sides have in common on each side.

After groups have shared, ask students how they see the regions in each figure matching the regions in the other figure. For example, since the two small squares in Figure F match the one large square in Figure G, how do the rectangles and triangles match? After some quiet think time, select 1–2 students to explain how they see it. (The area of the two rectangles is the same as the area of four of the triangles since, if I put two of the triangles together, I get a rectangle that is a wide and b long.) Show students an image with the diagonals added in, such as the one shown here, to help make the connection between the two figures clearer.



Note how these figures can be made for any right-angled triangle with shorter sides a and b and hypotenuse c.

7.3 Let's Take it for a Spin

10 minutes

Before this lesson, students could only find the length of a segment between the intersection of grid lines in a square grid by computing the area of a related square.

Pythagoras' theorem makes it possible to find the length of any segment that is a side of a right-angled triangle.

Instructional Routines

- Clarify, Critique, Correct
- Think Pair Share

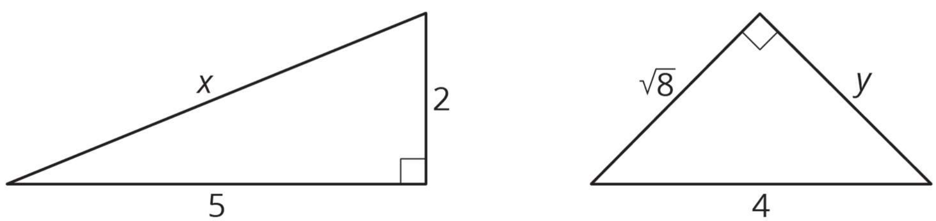
Launch

Arrange students in groups of 2. Give students 3 minutes of quiet work time followed by partner and then whole-class discussions.

Engagement: Develop Effort and Persistence. Encourage and support opportunities for peer interactions. Prior to the whole-class discussion, invite students to share their work with a partner, detailing the steps they took to solve for the missing side length. Display sentence frames to support student conversation such as: "First, I _____ because . . .", "Then I . . .", and "Finally, in order to solve, I _____ because . . . ."
Supports accessibility for: Language; Social-emotional skills

Student Task Statement

Find the unknown side lengths in these right-angled triangles.



Student Response

$x = \sqrt{29}$

$y = \sqrt{8}$

Activity Synthesis

Invite a few students to share their reasoning with the class for each unknown side length. As students share, record their steps for all to see, showing clearly the initial setup with $a^2 + b^2 = c^2$.

Reading, Writing, Speaking: Clarify, Critique, Correct. Before students share their methods for finding the **side length** of a right-angled triangle, present an incorrect solution based on a common error you observe in the class. For example, "I know that $a = \sqrt{8}$, $b = 4$, and $c = y$, so when I use Pythagoras' theorem, I get the equation $(\sqrt{8})^2 + 4^2 = y^2$. This equation

simplifies to $8 + 16 = y^2$. When I solve for y, I get $y = \sqrt{24}$." Ask students to identify the error, critique the reasoning, and revise the original statement. As students discuss in partners, listen for students who clarify the meaning of each term in the equation $a^2 + b^2 = c^2$. In Pythagoras' theorem, a^2 and b^2 represent the square of the shorter sides of the right-angled triangle, whereas c^2 represents the square of the hypotenuse. This routine will engage students in meta-awareness as they critique and correct a common error when applying Pythagoras' theorem.
Design Principles(s): Cultivate conversation; Maximise meta-awareness

7.4 A Transformational Proof

Optional: 15 minutes (there is a digital version of this activity)

In this activity, students explore a transformations-based proof of Pythagoras' theorem, calling back to their work with transformations earlier in the year. Since this proof is not one students are expected to derive on their own, the focus of this activity is on understanding the steps and why they are possible from a transformations perspective.

Listen for students using precise language when describing their transformations.

The digital activity demonstrates the same proof in a slightly different way. Students have the opportunity to explore three different right-angled triangles in the applets.

Instructional Routines

- Collect and Display

Launch

Tell students that today we are going to think about how to use transformations to show the relationship between the sides of right-angled triangles. Briefly review the language of rigid transformations (translation, rotation, reflection) with students using the 5 pre-cut pieces from the Making Squares blackline master from the first lesson in this unit.

Arrange students in groups of 2. Before students begin, remind them that if a problem asks them to explain, then they are expected to use precise language when describing the transformation of the shapes. Distribute pre-cut shapes from the Making Squares blackline master and the A Transformational Proof blackline master to each group. Leave 3–4 minutes for a whole-class discussion.

For students using the digital activity, there are no paper copies needed. Have students work in groups of two with the digital applet to explore the relationship between the squares and Pythagoras' theorem.

Action and Expression: Internalise Executive Functions. Chunk this task into more manageable parts to support students who benefit from support with organisational skills in problem solving. For example, present one question at a time and monitor students to ensure they are making progress throughout the activity.
Supports accessibility for: Organisation; Attention Conversing, Reading: Collect and Display.

As students work in pairs on the task, circulate and listen as they discuss their observations about the relationship between the squares and Pythagoras' theorem. Write down the words and phrases students use on a visual display. As students review the language collected in the visual display, encourage students to clarify the meaning of a word or phrase. For example, a phrase such as: "The small squares add up to the big square." can be restated as "The sum of the areas of the small squares, $a^2 + b^2$, is equal to the area of the large square, c^2.". Encourage students to refer back to the visual display during whole-class discussions throughout the lesson and unit. This routine will provide feedback to students in a way that supports sense-making while simultaneously increasing meta-awareness of language.
Design Principle(s): Support sense-making; Maximise meta-awareness

Student Task Statement

Your teacher will give your group a sheet with 4 figures and a set of 5 cut out shapes labelled D, E, F, G, and H.

1. Arrange the 5 cut out shapes to fit inside Figure 1. Check to see that the pieces also fit in the two smaller squares in Figure 4.
2. Explain how you can transform the pieces arranged in Figure 1 to make an exact copy of Figure 2.
3. Explain how you can transform the pieces arranged in Figure 2 to make an exact copy of Figure 3.
4. Check to see that Figure 3 is congruent to the large square in Figure 4.
5. If the right-angled triangle in Figure 4 has shorter sides a and b and hypotenuse c, what have you just demonstrated to be true?

Student Response

1. No response necessary.
2. Rotate Triangle G 90 degrees counter clockwise about it's uppermost point.
3. Rotate the triangle created by shapes G and H 90 degrees clockwise about the uppermost point of H.
4. No response necessary.
5. Since these shapes can be arranged to be either the area $a^2 + b^2$ of the two smaller squares or the area c^2 of the larger square, we have demonstrated that for a right-angled triangle with shorter sides a and b and hypotenuse c, $a^2 + b^2 = c^2$ is true.

Activity Synthesis

Select previously identified groups to share their explanations for each transformation and their conclusion to the last problem. If possible, have them show each transformation for all students to see.

An important takeaway for this activity is that this proof can be generalised to any right-angled triangle. To help students see why, ask them to consider how the diagonal lines in Figure 1 were created. Give 1–2 minutes for partners to discuss and then select 2–3 groups to share their ideas. Hopefully, at least one group noticed that the diagonals create two congruent right-angled triangles with sides of length a, b, and c—the same as the original right-angled triangle. This means that this process could be duplicated to show that for any right-angled triangle with shorter sides a and b and hypotenuse c, $a^2 + b^2 = c^2$ is true.

Lesson Synthesis

Review the proof of Pythagoras' theorem.

7.5 When is it True?

Cool Down: 5 minutes

Student Task Statement

Pythagoras' theorem is

1. True for all triangles
2. True for all right-angled triangles
3. True for some right-angled triangles
4. Never true

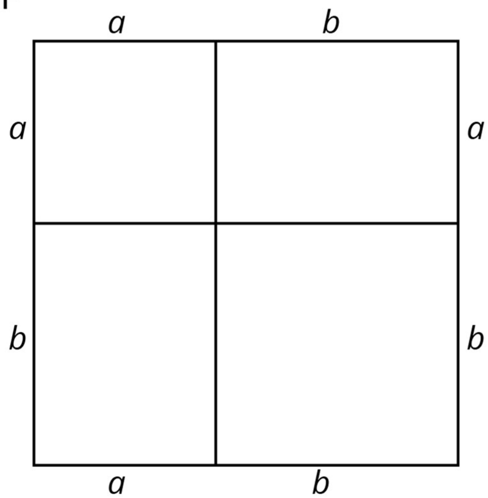
Student Response

True for all right-angled triangles

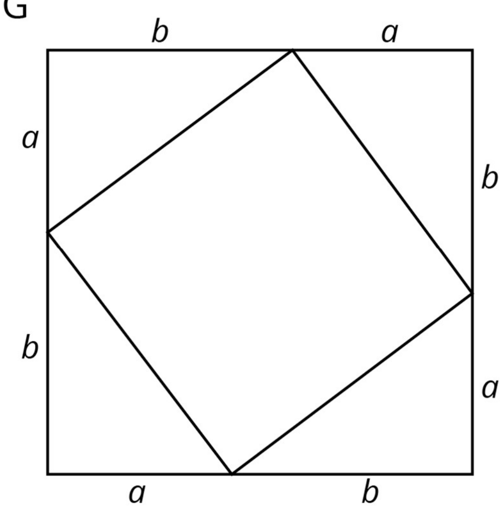
Student Lesson Summary

The figures shown here can be used to see why Pythagoras' theorem is true. Both large squares have the same area, but they are broken up in different ways. (Can you see where the triangles in Square G are located in Square F? What does that mean about the smaller squares in F and H?) When the sum of the four areas in Square F are set equal to the sum of the 5 areas in Square G, the result is $a^2 + b^2 = c^2$, where c is the hypotenuse of the triangles in Square G and also the side length of the square in the middle. Give it a try!

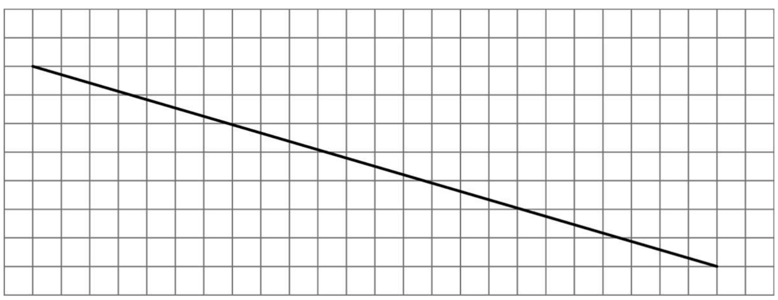
F



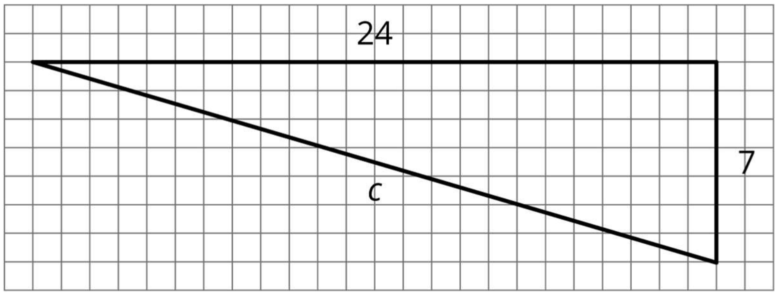
G



This is true for any right-angled triangle. If the shorter sides are a and b and the hypotenuse is c, then $a^2 + b^2 = c^2$. This property can be used any time we can make a right-angled triangle. For example, to find the length of this line segment:



The grid can be used to create a right-angled triangle, where the line segment is the hypotenuse and the shorter sides measure 24 units and 7 units:

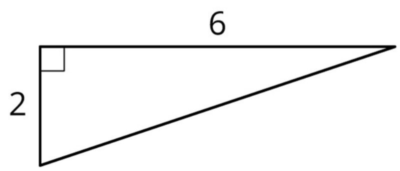


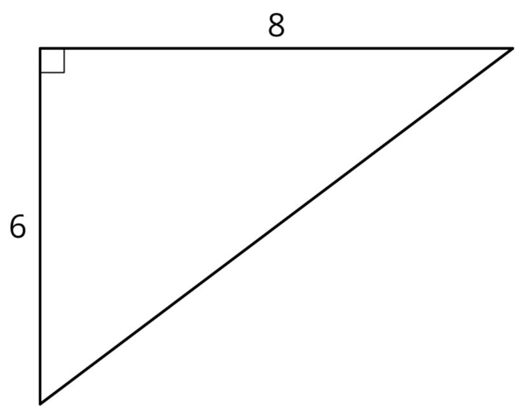
Since this is a right-angled triangle, $24^2 + 7^2 = c^2$. The solution to this equation (and the length of the line segment) is $c = 25$.

Lesson 7 Practice Problems

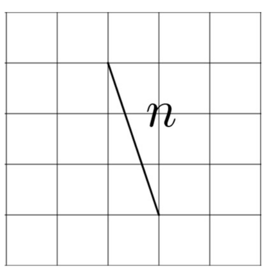
1. **Problem 1 Statement**

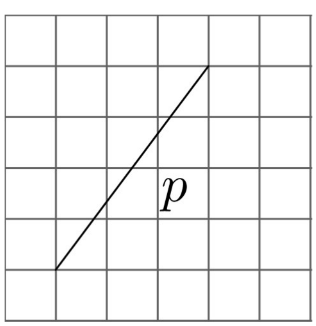
   a. Find the lengths of the unlabelled sides.





   b. One segment is n units long and the other is p units long. Find the value of n and p. (Each small grid square is 1 square unit.)





Solution

   a.

      i. $\sqrt{40}$, approximately 6.3

ii. $\sqrt{100}$, exactly 10

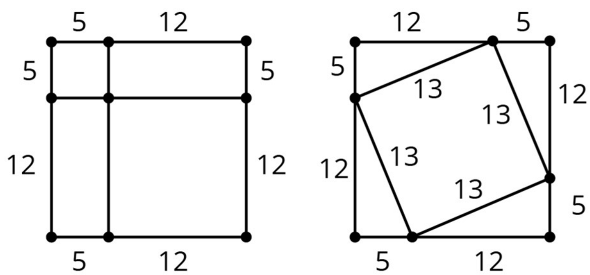
b.

i. $\sqrt{10}$ because $1^2 + 3^2 = 10$

ii. $\sqrt{25}$ (or 5) because $3^2 + 4^2 = 25$

2. **Problem 2 Statement**

Use the areas of the two identical squares to explain why $5^2 + 12^2 = 13^2$ without doing any calculations.



Solution

Answers vary. Sample explanation: The areas of the two large squares are the same since they are both 17 by 17 units. The area of the two rectangles on the left square are the same as the area of the 4 triangles in the right square (each pair of triangles makes a rectangle). So the area of the two smaller squares on the left must be the same as the area of the smaller square on the right. This means $5^2 + 12^2 = 13^2$.

3. **Problem 3 Statement**

Each number is between which two consecutive integers?

a. $\sqrt{10}$

b. $\sqrt{54}$

c. $\sqrt{18}$

d. $\sqrt{99}$

e. $\sqrt{41}$

Solution

a. 3 and 4

b. 7 and 8

c. 4 and 5

d. 9 and 10

e. 6 and 7

4. **Problem 4 Statement**

a. Give an example of a rational number, and explain how you know it is rational.

b. Give three examples of irrational numbers.

Solution

a. Answers vary. Sample response: $\frac{2}{3}$ is a rational number because rational numbers are fractions and their opposites and $\frac{2}{3}$ is a fraction.

b. Answers vary. Sample response: $\sqrt{2}$, -$\sqrt{12}$, $\sqrt{1.5}$

5. **Problem 5 Statement**

Write each expression as a single power of 10.

a. $10^5 \times 10^0$

b. $\frac{10^9}{10^0}$

Solution

a. 10^5

b. 10^9

6. **Problem 6 Statement**

Andre is ordering ribbon to make decorations for a school event. He needs a total of exactly 50.25 metres of blue and green ribbon. Andre needs 50% more blue ribbon than green ribbon for the basic design, plus an extra 6.5 metres of blue ribbon for accents. How much of each colour of ribbon does Andre need to order?

Solution

Andre needs to order 17.5 metres of green ribbon and 32.5 metres of blue ribbon. Strategies vary. Sample strategy: Let b represent the length of blue ribbon and g represent the length of green ribbon. Then $b = 1.5g + 6.25$ and $b + g = 50.25$. Substituting $1.5g + 6.25$ in for b in the second equation gives $(1.5g + 6.25) + g =$

50.25. Solving for g gives $g = 17.5$. Since the two kinds of ribbon must combine to make 50.25 metres, then $b = 50.25 - 17.5$, so $b = 32.5$ metres.