Lesson 10: Meet gradient

Goals

- Comprehend the term “gradient” to mean the quotient of the vertical distance and the horizontal distance between any two points on a line.
- Draw a line on a coordinate grid given its gradient and describe (orally) observations about lines with the same gradient.
- Justify (orally) that all “gradient triangles” on one line are similar by using transformations or Angle-Angle Similarity.

Learning Targets

- I can draw a line on a grid with a given gradient.
- I can find the gradient of a line on a grid.

Lesson Narrative

A gradient triangle for a line is a triangle whose longest side lies on the line and whose other two sides are vertical and horizontal. This lesson establishes the remarkable fact that the quotient of the vertical side length and the horizontal side length does not depend on the triangle: this number is called the **gradient** of the line. The argument builds on many key ideas developed in this unit:

- The enlargement of a gradient triangle, with centre of enlargement on the line, is a gradient triangle for the same line.
- Triangles sharing two common angles are similar.
- Quotients of corresponding sides in similar polygons are equal.

In future lessons, they will use gradient to write equations for lines.

Building On

- Add, subtract, multiply, and divide decimals to hundredths, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used.
- Interpret a fraction as division of the numerator by the denominator $\left(\frac{a}{b} = a \div b\right)$. Solve word problems involving division of whole numbers leading to answers in the form of fractions or mixed numbers, e.g., by using visual fraction models or equations to represent the problem. For example, interpret $\frac{3}{4}$ as the result of dividing 3 by 4, noting that $\frac{3}{4}$ multiplied by 4 equals 3, and that when 3 wholes are shared equally

among 4 people each person has a share of size $\frac{3}{4}$. If 9 people want to share a 50-pound sack of rice equally by weight, how many pounds of rice should each person get? Between what two whole numbers does your answer lie?

- Understand that a two-dimensional shape is similar to another if the second can be obtained from the first by a sequence of rotations, reflections, translations, and enlargements; given two similar two-dimensional shapes, describe a sequence that exhibits the similarity between them.

Addressing

- Use similar triangles to explain why the gradient m is the same between any two distinct points on a non-vertical line in the coordinate plane; derive the equation $y = mx$ for a line through the origin and the equation $y = mx + b$ for a line intercepting the vertical axis at b.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Collect and Display
- Co-Craft Questions
- Discussion Supports

Required Materials

Straightedges
A rigid edge that can be used for drawing line segments. Sometimes a ruler is okay to use as a straightedge, but sometimes it is preferable to use an unruled straightedge, like a blank index card.

Required Preparation

If using the print version of the materials, students need a straightedge in order to draw lines. If using the digital version, an applet is made available for this purpose.

Student Learning Goals

Let's learn about the gradient of a line.

10.1 Equal Quotients

Warm Up: 5 minutes

In this lesson, students will need to recognise equivalent fractions and decimals. This warm-up is designed to activate prior understanding.

Launch

Ask students if they can think of some different ways to write numbers that are equal to $1 \div 2$. Some examples might be $\frac{4}{8}$ or 0.5. Tell them that in this warm-up, they will think of numbers that equal $15 \div 12$. Give students 30–60 seconds to write as many as they can, or ask them to come up with at least three or four different numbers equal to $15 \div 12$.

Anticipated Misconceptions

Students may believe that this question must have only one acceptable response. To convince them that equivalent values are acceptable, you may need to appeal to the fact that one location on a number line can be expressed many different ways, depending on how you decide to subdivide a portion of the line.

Student Task Statement

Write some numbers that are equal to $15 \div 12$.

Student Response

Answers vary. Possible responses: $\frac{15}{12}, \frac{5}{4}, \frac{10}{8}, \frac{50}{40}$, 1.25, 1.2500.

Activity Synthesis

Ask students to share their strategies for finding numbers equal to $15 \div 12$. Record and display their answers and strategies for all to see. To involve more students in the conversation, consider asking:

- "Who can restate ___'s reasoning in a different way?"
- "Did anyone solve the problem the same way but would explain it differently?"
- "Did anyone solve the problem in a different way?"
- "Does anyone want to add on to _____'s strategy?"
- "Do you agree or disagree? Why?"

10.2 Similar Triangles on the Same Line

20 minutes

In this activity, students explain why certain triangles with one side along the same line are similar. This fact about the triangles will be used to define the gradient of the line. Students may show that the triangles are similar by describing a sequence of transformations and enlargements or by AA (or AAA). Alternatively, they may use the fact that grid lines are parallel and use what they know about the angles where a transversal meets a pair of parallel lines.

Monitor for students who use different methods for showing that the triangles are similar. Look for these methods in particular:

- Describing a sequence of transformations
- AA (or AAA) arguments using what students know about angles made by parallel lines with a transversal

Students need to use the structure of the grid for either argument. For the similarity argument, they need to use the grid to describe transformations and enlargements. For the AA argument, they need to use the fact that vertical or horizontal grid lines are parallel. In both cases, constructing a viable argument will require care and focus.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Co-Craft Questions

Launch

Before starting the activity, if possible, display this link https://ggbm.at/UxvbFxVQ for all students to see. Tell students that the goal is to make one triangle match up with the other triangle. Demonstrate how the controls work. Then, invite students to describe how you should manipulate the controls to make the triangles match up. Remind students to use words associated with transformations like *translate* and *scale factor*.

Arrange students in groups of 2. Assign one partner triangles ABC and CDE and the other partner ABC and FGH. Give students 5 minutes of quiet time to construct an argument for why their two triangles are similar. Remind students that if they claim something is true, they should explain how they can be sure it is true.

After a few minutes of quiet work time, ask students to share their reasoning with their partner and listen to their partner's explanation for why their triangles are similar. Then tell them to work with their partner to finish the activity.

Action and Expression: Develop Expression and Communication. To help get students started, display sentence frames to support students when they explain their strategy. "First, I _____ because….", "I noticed _____ so I….", "Why did you...?", "I agree/disagree because…." Encourage students to use what they know about transformations, corresponding angles and sides, and scale factor in their explanations.
Supports accessibility for: Language; Organisation Reading, Writing: Co-Craft Questions. To help students comprehend the diagram in this task, show students just the graph with the three gradient triangles. Ask students to write down possible mathematical questions that might be asked about the situation. Students may create questions about finding the length of missing lengths of line segments, or more broadly, about which triangles are similar. Ask students to compare the questions they generated with a partner before sharing questions with the whole class. Reveal the actual question that students are to work on, which is to explain why two of these triangles are similar. Through this routine, students are able to

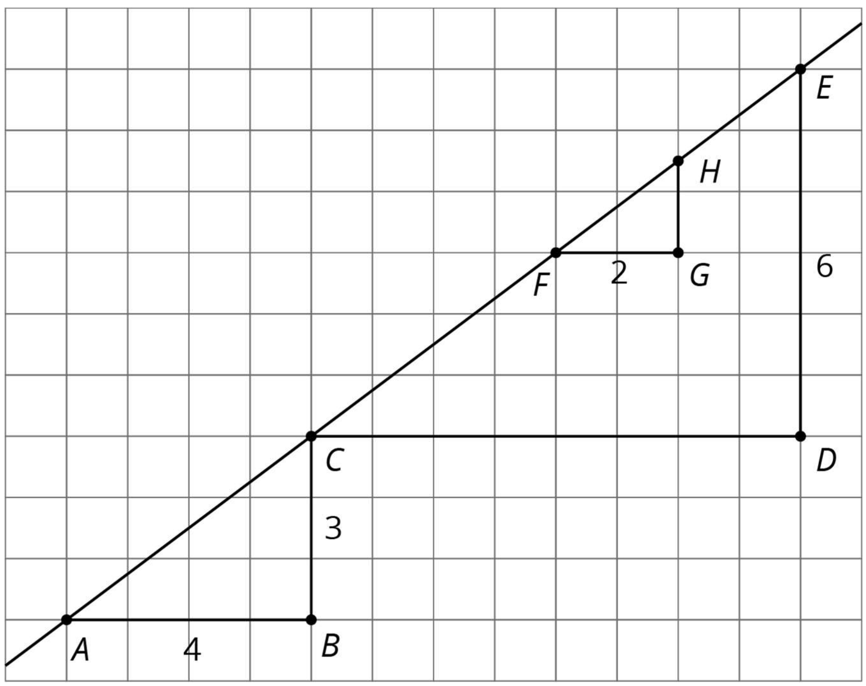
use conversation skills to generate, choose (argue for the best one), and improve questions, as well as develop meta-awareness of the language used in mathematical questions.
Design Principle(s): Cultivate conversation; Support sense-making, Meta-awareness of the language

Anticipated Misconceptions

If students struggle getting started, ask them what it means for two triangles to be similar (there is a sequence of translations, rotations, reflections, and enlargements taking one to the other).

Student Task Statement

1. The diagram shows three right-angled triangles, each with its longest side on the same line. Your teacher will assign you two triangles. Explain why the two triangles are similar.



2. Complete the table.

triangle	length of vertical side	length of horizontal side	(vertical side) ÷ (horizontal side)
ABC			
CDE			
FGH			

Student Response

1. Answers vary. Possible strategies:

   - Triangle ABC is similar to triangle CDE because you can translate triangle ABC so that A goes to C. Then perform an enlargement using scale factor 2 and C as the centre of enlargement.

   - Triangle FGH is similar to triangle CDE by AA. We know this first because angle FGH and angle CDE are both right angles so they are congruent to each other. We also know all of the vertical sides of the triangles are along the grid so they are all parallel. Since angle GFH and angle DCE are corresponding angles for transversal $\overleftrightarrow{EA}$ (and parallel grid lines GF and DC) they are congruent. A similar argument can be made about angles CED and FHG.

triangle	length of vertical side	length of horizontal side	(vertical side) ÷ (horizontal side)
ABC	3	4	$\frac{3}{4}$ or 0.75
CDE	6	8	$\frac{6}{8}$ or 0.75
FGH	1.5	2	$\frac{3}{4}$ or 0.75

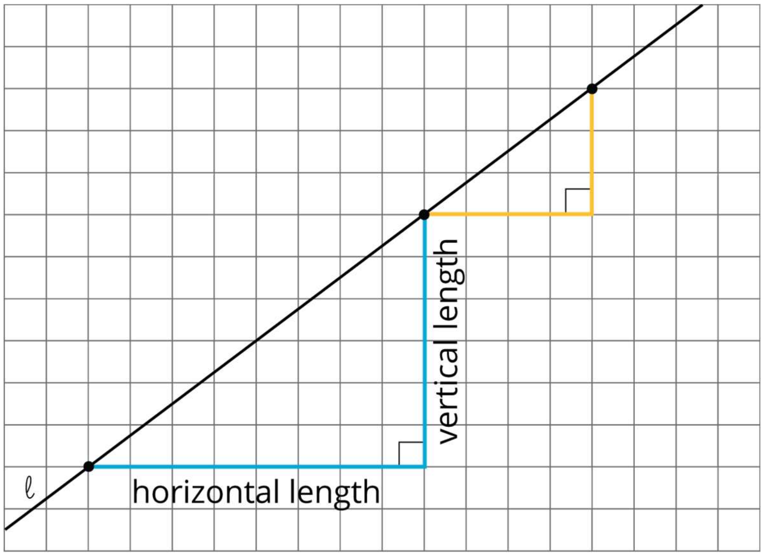
Activity Synthesis

Invite selected students to share their methods for showing that the triangles are similar. Sequence them so that a similarity transformation argument goes first, followed by an AA (or AAA) argument. For these particular triangles, describing a sequence of transformations is quick and efficient: a translation and enlargement (or an enlargement followed by a translation) naturally present themselves for these triangles. The AA argument requires a little extra work because no angles are labelled: finding angles uses the structure of the grid and previous results about corresponding angles created by parallel lines and a transversal. Make sure students understand that since triangles ABC and FGH are each similar to triangle CDE, they are similar to each other.

Relate the work on finding the quotients, in the second problem, back to the work in the last lesson looking at internal ratios of corresponding sides of similar triangles. Explain that whenever we have a (non-vertical, non-horizontal) line, we can construct triangles like these where one side is horizontal and one side is vertical, and the quotient of the length of the vertical side and the horizontal side will always be the same. This number is called the **gradient** of the line. The gradient of the line in this activity can be written as 0.75 or $\frac{3}{4}$ (or any value equal to these).

Make clear to students that the mathematical convention is to define gradient using vertical length divided by horizontal length and not the other way around. Display the diagram below, or create a similar diagram with the same information. Post this diagram for reference for several days, well into the next unit of study, along with accompanying text:

The gradient is vertical length ÷ horizontal length. The gradient of line ℓ can be written as $\frac{6}{8}$, $\frac{3}{4}$, 0.75, or any equal value.



10.3 Multiple Lines with the Same Gradient

10 minutes (there is a digital version of this activity)

The previous activity introduced the *gradient* of a line. In this activity, students practise graphing lines with a given gradient. They observe two important properties of gradient:

- Lines with the same gradient are parallel.
- As the gradient of a line increases so does its steepness (from left to right).

Look for students who draw gradient triangles to help construct their lines. Other students may count off horizontal and vertical distances. Select students using each method to share during the discussion. Also watch for students who notice the facts listed above about parallel lines and the impact of the gradient on the steepness of the line.

Instructional Routines

- Discussion Supports

Launch

Tell students that they are going to apply the new idea of gradient introduced in the previous activity as they draw and study properties of some lines with different gradients.

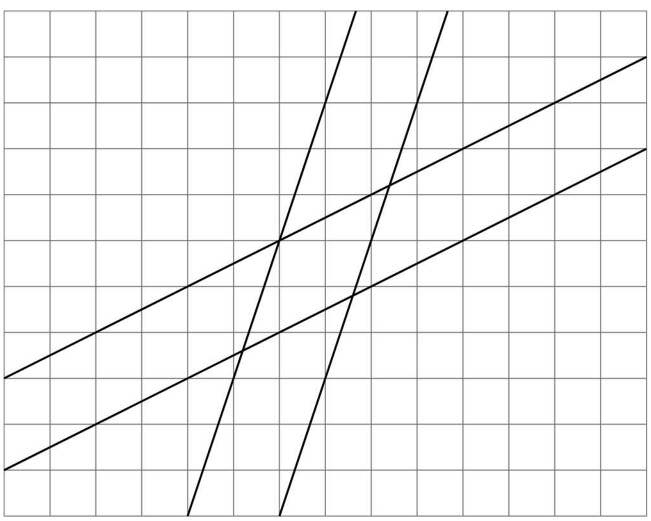
If using the print version of the materials, distribute straightedges. If using the digital version, students have access to a digital tool to draw lines.

Student Task Statement

1. Draw two lines with gradient 3. What do you notice about the two lines?

2. Draw two lines with gradient $\frac{1}{2}$. What do you notice about the two lines?

Student Response



The pairs of lines with the same gradient are parallel.

Are You Ready for More?

As we learn more about lines, we will occasionally have to consider perfectly vertical lines as a special case and treat them differently. Think about applying what you have learned in the last couple of activities to the case of vertical lines. What is the same? What is different?

Student Response

Geometrically, vertical lines are no different than any other line. You can rotate a vertical line to a non-vertical line. Just as lines with the same gradient are parallel, all vertical lines are also parallel. But some things are quite different. For example, the notion of a gradient triangle doesn't make much sense since there is no "horizontal distance" to use as the base of the triangle, and trying to define a gradient using gradient $= \frac{\text{vertical distance}}{\text{horizontal distance}}$ would have us dividing by zero. For this reason, we do not define the gradient of a vertical line.

Activity Synthesis

Ask selected students to share how they found their lines with gradient 3. Sequence them so that students who use gradient triangles present their work first and students who count horizontal and vertical displacement (without drawing a triangle) present second. Help students see that the second method is the same as the first except that the gradient triangle connecting two points on the line is only "imagined" rather than drawn. Further, that it doesn't matter if you go up 3 and over 1, or up 6 and over 2, or up 9 and over 3 . . . you get a line with the same gradient. Encourage students to draw gradient triangles if it helps them to see and understand the underlying structure.

If it does not come up during student comments, make sure that students notice that the two lines of gradient 3 are parallel and so are the two lines with gradient $\frac{1}{2}$.

Also, point out that the lines of gradient 3 are "steeper" than the lines of gradient $\frac{1}{2}$. Hence the definition of gradient corresponds to our intuition in the sense that a larger gradient corresponds to a "steeper graph" (going from left to right).

It is likely that some students will draw lines with gradient -3 and -$\frac{1}{2}$. Display these alongside lines of gradient 3 or gradient $\frac{1}{2}$, and ask students to describe how they are alike and different. It is okay to let them know that the "uphill" lines (leaning to the right) are positive and "downhill" (leaning to the left) are negative. But you might also leave the question open, for now, about whether such lines have the same or different gradients, and how the gradient would be different. Later lessons will focus on distinguishing positive from negative gradients.

Engagement: Develop Effort and Persistence. Break the class into small discussion groups and then invite a representative from each group to report back to the whole class.
Supports accessibility for: Attention; Social-emotional skills Representing: Discussion Supports. Provide sentence frames to help students to formulate statements about what they notice about two lines with the same gradient, "Two lines with the same gradient are____ because…," and lines with different gradients, "The lines with gradients equal to 3 are____than lines with gradient 1/2, because...." This will help to encourage students to justify what they notice. Remind students to include words like "parallel" and "steep," along with informal language describing equal rates of change.
Design Principle(s): Optimise output (for explanation)

10.4 Different Gradients of Different Lines

Optional: 15 minutes

Earlier in this lesson, students have seen that the gradient of a line can be calculated using any (gradient) triangle, that is a right-angled triangle whose longest side is on the line and whose other two sides are horizontal and vertical. In this activity, students identify given lines with different gradients and draw a line with a particular gradient.

For lines D and E as well as the line that students draw, encourage students to draw in gradient triangles to help calculate the gradient. Monitor for students who draw gradient triangles of different sizes and invite them to share during the discussion to reinforce the fundamental idea for this lesson: different gradient triangles whose longest side lies on the same line give the same value for gradient.

Instructional Routines

- Collect and Display

Launch

Instruct students to work individually to match each gradient to a line and draw the line for the gradient that does not have a match. Remind students of the previous definition of gradient, referring to the display as necessary. Students still need a straightedge for drawing line F.

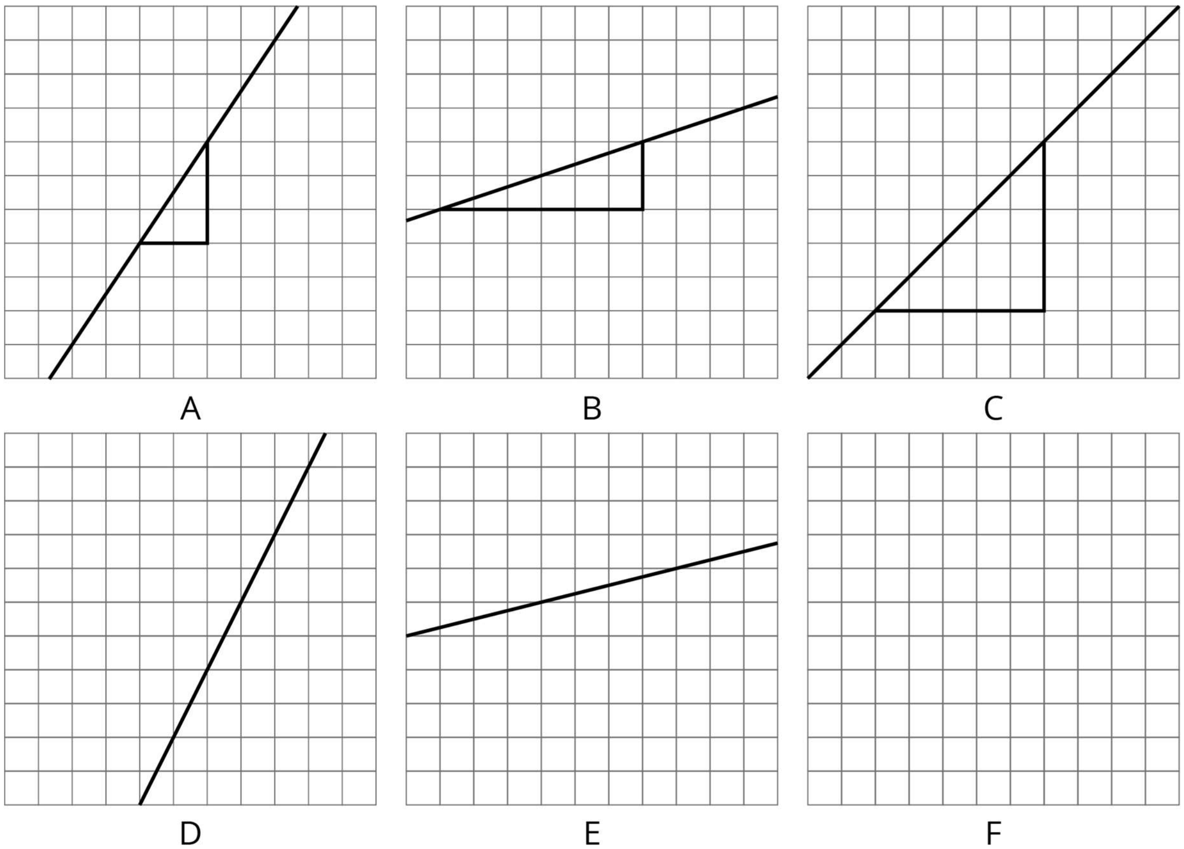
Representation: Develop Language and Symbols. Create a display of important terms and vocabulary. During the launch, take time to review terms students will need to access for this activity. Invite students to suggest language or diagrams to include that will support their understanding of: gradient, vertical distance, and horizontal distance.
Supports accessibility for: Conceptual processing; Language

Anticipated Misconceptions

For students who find it difficult to draw a triangle when one is not given, suggest that they examine two places on the line where the line crosses an intersection of grid lines.

Student Task Statement

Here are several lines.



1. Match each line shown with a gradient from this list: $\frac{1}{3}$, 2, 1, 0.25, $\frac{3}{2}$, $\frac{1}{2}$.

2. One of the given gradients does not have a line to match. Draw a line with this gradient on the empty grid (F).

Student Response

1. $\frac{3}{2}$

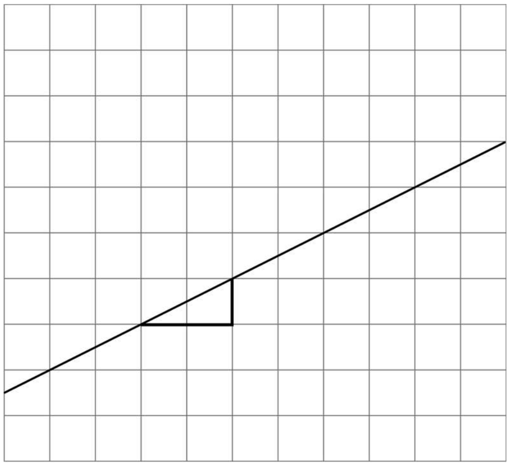
2. $\frac{1}{3}$

3. 1

4. 2

5. 0.25

6. $\frac{1}{2}$ (A valid response may not include a gradient triangle, or may include a gradient triangle that is similar but not congruent to the one shown.)



Activity Synthesis

Two important conclusions for students to understand are:

- Given a line on a grid, they can draw a right-angled triangle whose longest side is on the line, and then use the quotient of the vertical and horizontal sides to find the gradient.
- Given a gradient, they can draw a right-angled triangle using vertical and horizontal lengths corresponding to the gradient, and then extend the longest side of the right-angled triangle to create a line with that gradient.

When discussing line F, ask students to share how they drew their triangle. If possible, select students who drew their triangles correctly but at a different scale (for example, one student who used a triangle with a vertical length of 1 and a horizontal length of 2, and a different student who used a vertical length of 4 and a horizontal length of 8). Demonstrate (or get students who have drawn different triangles to do so) that the *quotient* of side lengths is the important feature, since any triangle drawn to match a given gradient will be similar to any other triangle drawn to match the same gradient.

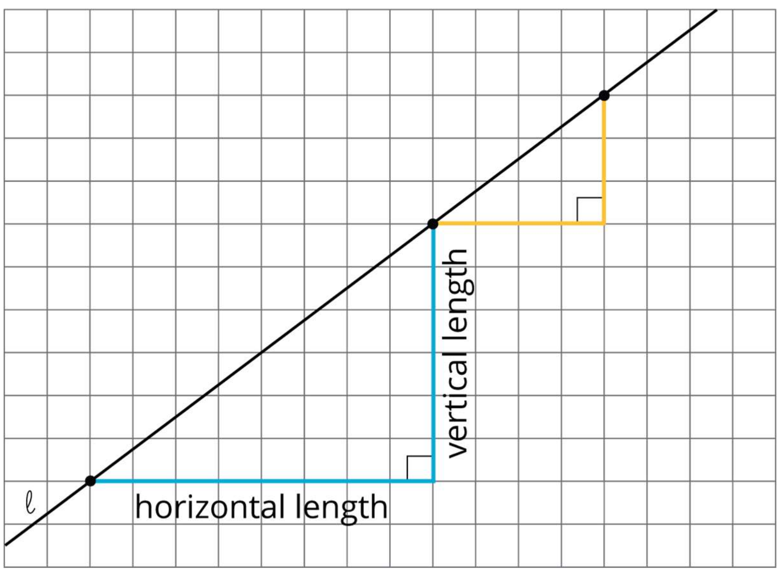
Conversing, Representing, Writing: Collect and Display. After students have had individual work time, ask students to work in pairs to explain to each other how they matched each graph to its gradient. While pairs are working, circulate and listen to student talk about how they matched each representation to another representation. Listen for how students created gradient triangles where no triangle already existed, and how they used the gradient to draw the line. Display the language collected visually for the whole class to use as a reference during further discussions throughout the unit. This allows students' own output to become a reference in developing mathematical language.
Design Principle(s): Support Sense-making; Maximise meta-awareness

Lesson Synthesis

In this lesson, we learned what the **gradient** of a line is. Discuss:

- "What is a gradient triangle for a line?" (A triangle whose long side is on the line and whose other sides are horizontal and vertical.)
- "How can you use a gradient triangle to find the gradient of a line?" (Divide the length of the vertical side by the length of the horizontal side.)
- "Does it matter which two points you use to create a gradient triangle? Why?" (No. Any two gradient triangles are similar. So the quotient of the two corresponding sides will give the same value.)
- "Why are any two gradient triangles similar?" (They are right-angled triangles whose other angles are corresponding angles for a transversal meeting parallel grid lines.)

Consider creating a permanent classroom display showing the definition of gradient.



Display the diagram along with the accompanying text: The gradient of the line is vertical length ÷ horizontal length. The gradient of line ℓ can be written $\frac{6}{8}$, $\frac{3}{4}$, 0.75, or any equivalent value.

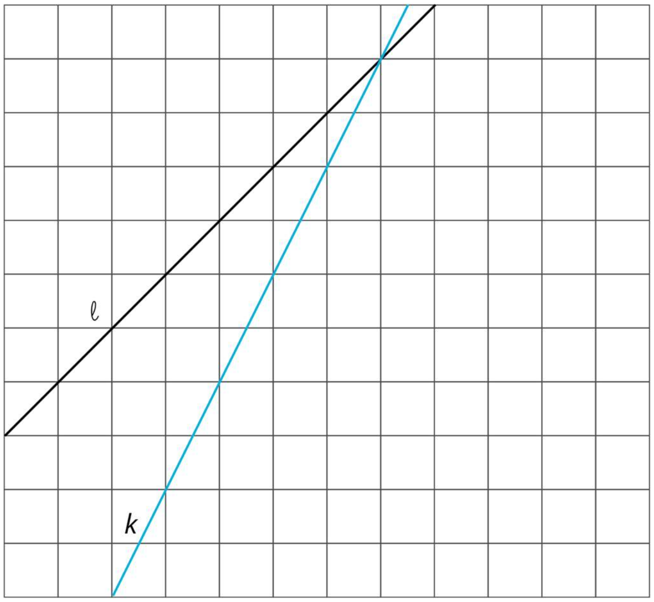
10.5 Finding Gradient and Graphing Lines

Cool Down: 5 minutes

Students identify lines with different gradients and then draw a line with specified gradient. They can use what they know about the meaning of gradient, drawing appropriate gradient triangles. They can also use the fact that a steeper line has a larger gradient.

Student Task Statement

Lines ℓ and k are graphed.



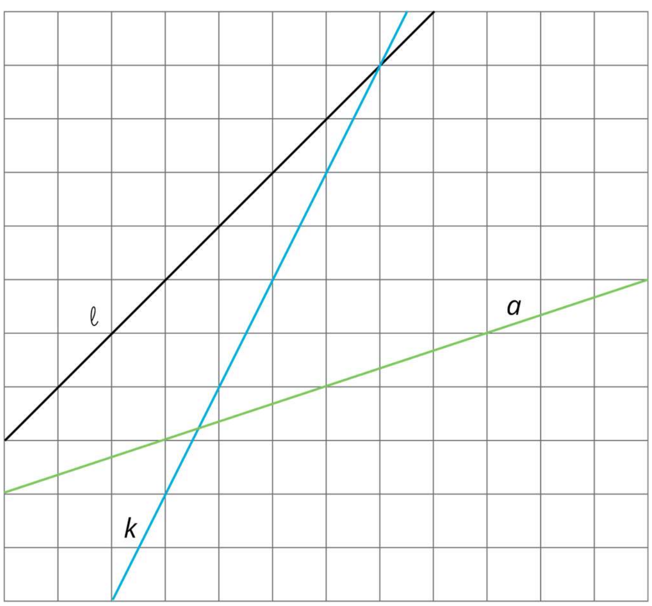
1. Which line has a gradient of 1, and which has a gradient of 2?

2. Use a ruler to help you graph a line whose gradient is $\frac{1}{3}$. Label this line a.

Student Response

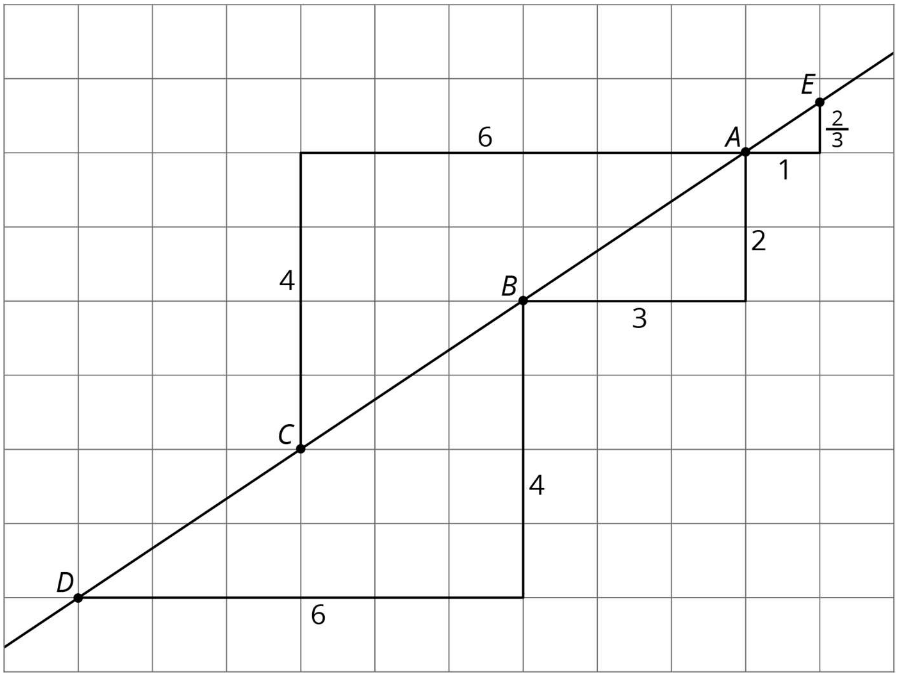
1. The gradient of line ℓ is 1; the gradient of line k is 2.

2. Answers vary. Possible response:



Student Lesson Summary

Here is a line drawn on a grid. There are also four right-angled triangles drawn. Do you notice anything the triangles have in common?



These four triangles are all examples of *gradient triangles*. One side of a gradient triangle is on the line, one side is vertical, and another side is horizontal. The **gradient** of the line is the quotient of the length of the vertical side and the length of the horizontal side of the gradient triangle. This number is the same for *all* gradient triangles for the same line because all gradient triangles for the same line are similar.

In this example, the gradient of the line is $\frac{2}{3}$, which is what all four triangles have in common. Here is how the gradient is calculated using the gradient triangles:

- Points A and B give $2 \div 3 = \frac{2}{3}$
- Points D and B give $4 \div 6 = \frac{2}{3}$
- Points A and C give $4 \div 6 = \frac{2}{3}$
- Points A and E give $\frac{2}{3} \div 1 = \frac{2}{3}$

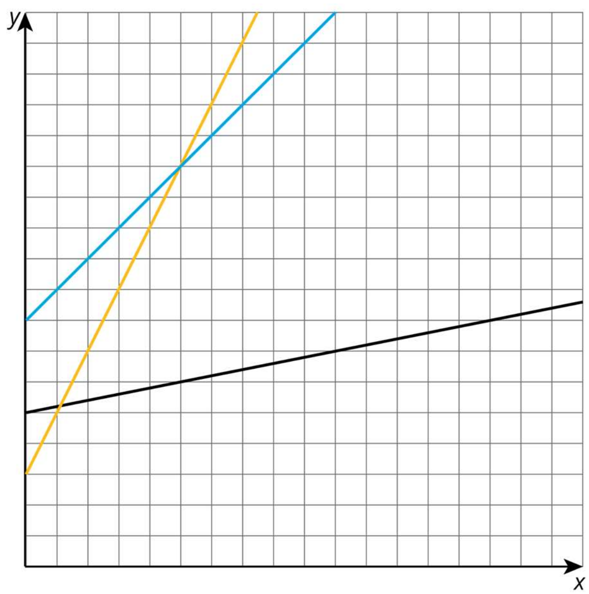
Glossary

- similar
- gradient

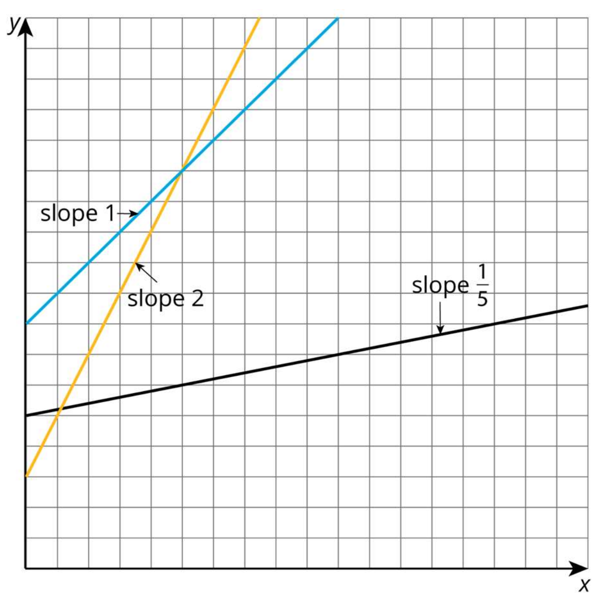
Lesson 10 Practice Problems

1. **Problem 1 Statement**

Of the three lines in the graph, one has gradient 1, one has gradient 2, and one has gradient $\frac{1}{5}$. Label each line with its gradient.

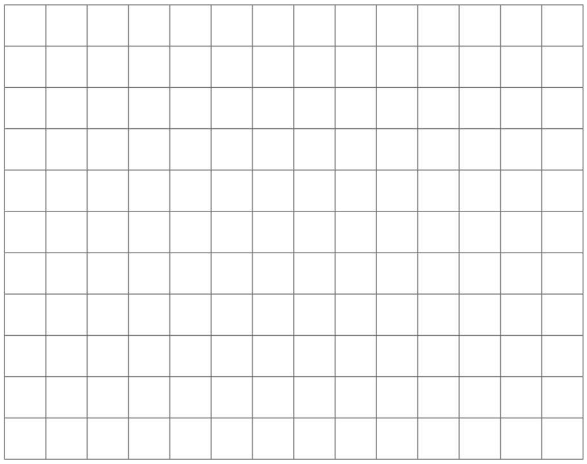


Solution



2. **Problem 2 Statement**

Draw three lines with gradient 2, and three lines with gradient $\frac{1}{3}$. What do you notice?

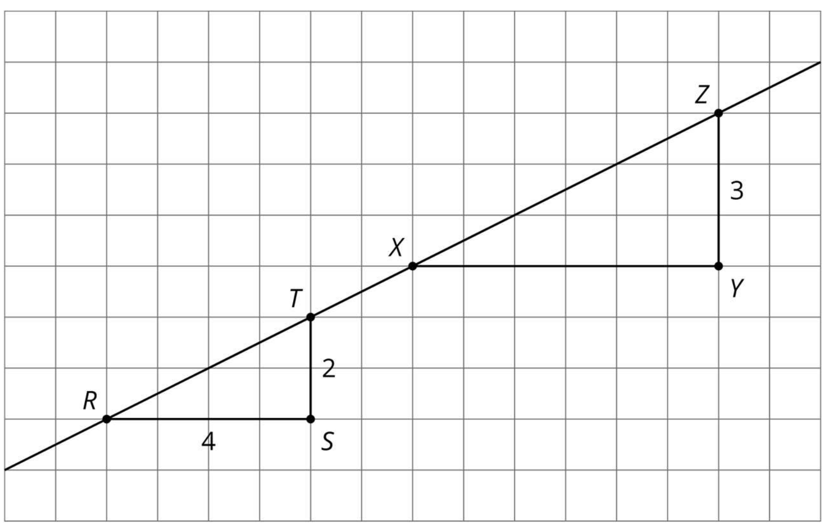


Solution

Answers vary. The three lines in each set should be parallel.

3. **Problem 3 Statement**

The diagram shows two right-angled triangles, each with its longest side on the same line.



a. Explain how you know the two triangles are similar.

b. How long is XY?

c. For each triangle, calculate (vertical side) ÷ (horizontal side).

d. What is the gradient of the line? Explain how you know.

Solution

a. Explanations vary. Sample explanation: translating R to X and enlarging shows there is a sequence of translations, rotations, reflections, and enlargements taking one triangle to the other.

b. 6 units

c. For both triangles, the result is $\frac{1}{2}$.

d. The gradient of the line is $\frac{1}{2}$. It is the quotient of the vertical side length of a gradient triangle and the horizontal side length of a gradient triangle. These all give the same value because the gradient triangles are similar.

4. **Problem 4 Statement**

Triangle A has side lengths 3, 4, and 5. Triangle B has side lengths 6, 7, and 8.

a. Explain how you know that triangle B is *not* similar to triangle A.

b. Give possible side lengths for triangle B so that it is similar to triangle A.

Solution

a. Explanations vary. Sample explanation: the shortest side in triangle B is twice as long, but the longest side is only 1.6 times as long. These different ratios mean the triangles cannot be similar.

b. Answers vary. Sample response: 6, 8, and 10