[MAA 5.20] INTEGRATION BY PARTS

SOLUTIONS

Compiled by: Christos Nikolaidis

O. Practice questions

1. We apply the formula $\int uv'\mathrm{d}x = uv - \int u'v\mathrm{d}x$

$$I_1 = \int 2x\mathrm{e}^x\mathrm{d}x = 2x\mathrm{e}^x - \int 2\mathrm{e}^x\mathrm{d}x = 2x\mathrm{e}^x - 2\mathrm{e}^x + c = (2x-2)\mathrm{e}^x + c$$

$u = 2x$	$\to$	$u' = 2$
$v' = \mathrm{e}^x$	$\to$	$v = \mathrm{e}^x$

For the remaining integrals we apply the method directly (without mentioning u and v')

$$I_2 = \int 2x\cos x\mathrm{d}x = 2x\sin x - \int 2\sin x\mathrm{d}x = 2x\sin x + 2\cos x + c$$

$$I_3 = \int 2x\sin x\mathrm{d}x = -2x\cos x + \int 2\cos x\mathrm{d}x = -2x\cos x + 2\sin x + c$$

$$I_4 = \int 2x\ln x\mathrm{d}x = x^2\ln x - \int x^2\frac{1}{x}\mathrm{d}x = x^2\ln x - \int x\mathrm{d}x = x^2\ln x - \frac{x^2}{2} + c$$

2. $I_1 = \int (3x^2+4x+1)\cos x\mathrm{d}x = (3x^2+4x+1)\sin x - \int (6x+4)\sin x\mathrm{d}x$

$$= (3x^2+4x+1)\sin x - \left[-(6x+4)\cos x + \int 6\cos x\mathrm{d}x\right]$$

$$= (3x^2+4x+1)\sin x + (6x+4)\cos x - 6\sin x + c$$

$$\left(= (3x^2+4x-5)\sin x + (6x+4)\cos x + c\right)$$

$$I_2 = \int (3x^2+4x+1)\ln x\mathrm{d}x = (x^3+2x^2+x)\ln x - \int (x^3+2x^2+x)\frac{1}{x}\mathrm{d}x$$

$$= (x^3+2x^2+x)\ln x - \int (x^2+2x+1)\mathrm{d}x$$

$$= (x^3+2x^2+x)\ln x - \frac{x^2}{3} - x^2 - x + c$$

3. **METHOD A**

$$I = \int \mathrm{e}^{2x}\cos 2x\mathrm{d}x = \frac{\mathrm{e}^{2x}}{2}\cos 2x + \int \frac{\mathrm{e}^{2x}}{2}2\sin 2x\mathrm{d}x = \frac{\mathrm{e}^{2x}}{2}\cos 2x + \int \mathrm{e}^{2x}\sin 2x\mathrm{d}x$$

$$= \frac{\mathrm{e}^{2x}}{2}\cos 2x + \left[\frac{\mathrm{e}^{2x}}{2}\sin 2x - \int \mathrm{e}^{2x}\cos 2x\mathrm{d}x\right]$$

$$= \frac{\mathrm{e}^{2x}}{2}\cos 2x + \frac{\mathrm{e}^{2x}}{2}\sin 2x - I$$

$$\Rightarrow 2I = \frac{\mathrm{e}^{2x}}{2}\cos 2x + \frac{\mathrm{e}^{2x}}{2}\sin 2x \Rightarrow I = \frac{\mathrm{e}^{2x}(\cos 2x + \sin 2x)}{4} + c$$

METHOD B We start with

$I = \int \mathrm{e}^{2x}\cos 2x\mathrm{d}x = \mathrm{e}^{2x}\frac{\sin 2x}{2} - \int \mathrm{e}^{2x}\sin 2x\mathrm{d}x = \ \ldots$ and find the same result.

4. Let $I_n = \int x^n e^x dx$.

(a) $I_0 = \int e^x dx = e^x + c$

(b) $I_n = x^n e^x - n\int x^{n-1} e^x dx$

$\Rightarrow I_n = x^n e^x - nI_{n-1}$

(c) $I_1 = xe^x - e^x + c = (x-1)e^x + c$,

$I_2 = x^2 e^x - 2(xe^x - e^x) + c = (x^2 - 2x + 2)e^x + c$

$I_3 = x^3 e^x - 3(x^2 - 2x + 2)e^x + c = (x^3 - 3x^2 + 6x - 6)e^x + c$

5. (a) (i) $2\sin^2 x + c$

(ii) $-2\cos^2 x + c$

(iii) $I = \int 2\sin 2x dx = -\cos 2x + c$

(iv) $I = \int 4\sin x \cos x dx = 4\sin x \sin x - \int 4\cos x \sin x dx = 4\sin^2 x - I$

$2I = 4\sin^2 x \Rightarrow I = 2\sin^2 x + c$

(b) The results may look different but, in fact, they differ by a constant (incorporated in c).

6. **METHOD A**

$\int (3x+2)e^x dx = (3x+2)e^x - \int 3e^x dx$

$= (3x+2)e^x - 3e^x + c = (3x-1)e^x + c$

$I = \left[(3x-1)e^x\right]_0^1 = (2e) - (-1) = 2e + 1$

METHOD B

$\int_0^1 (3x+2)e^x dx = \left[(3x+2)e^x\right]_0^1 - \int_0^1 3e^x dx$

$= 5e - 2 - \left[3e^x\right]_0^1 = 5e - 2 - (3e - 3)$

$= 2e + 1$

A. Exam style questions (SHORT)

7. For $\int \theta \cos\theta \, d\theta$

$u = \theta$ $du = d\theta$

$dv = \cos\theta d\theta$ $v = \sin\theta$

$\int \theta\cos\theta \, d\theta = \theta\sin\theta - \int \sin\theta \, d\theta = \theta\sin\theta + \cos\theta + c$

Therefore, $\int(\theta\cos\theta - \theta)d\theta = \theta\sin\theta + \cos\theta - \dfrac{\theta^2}{2} + c$

8. $\int \frac{\ln x}{\sqrt{x}} dx = 2x^{\frac{1}{2}} \ln x - 2\int x^{\frac{1}{2}} \times \frac{1}{x} dx = 2x^{\frac{1}{2}} \ln x - 2\int \frac{1}{\sqrt{x}} dx = 2x^{\frac{1}{2}} \ln x - 4x^{\frac{1}{2}} + C$

9.

$$\int e^x \cos x\,dx = e^x \cos x + \int e^x \sin x\,dx$$
$$= e^x \cos x + e^x \sin x - \int e^x \cos x\,dx$$
$$2\int e^x \cos x\,dx = e^x(\cos x + \sin x) + c$$
$$\int e^x \cos x\,dx = \frac{e^x}{2}(\cos x + \sin x) + k$$

10.

METHOD 1

$$\int e^{2x} \sin x\,dx = -e^{2x}\cos x + 2\int e^{2x}\cos x\,dx$$
$$= -e^{2x}\cos x + 2\left(e^{2x}\sin x - 2\int e^{2x}\sin x\,dx\right)$$
$$= -e^{2x}\cos x + 2e^{2x}\sin x - 4\int e^{2x}\sin x\,dx$$
$$\Rightarrow 5\int e^{2x}\sin x\,dx = e^{2x}(2\sin x - \cos x)$$
$$\int e^{2x}\sin x\,dx = \frac{e^{2x}}{5}(2\sin x - \cos x) + C$$

METHOD 2

$$\int e^{2x}\sin x\,dx = \frac{1}{2}e^{2x}\sin x - \frac{1}{2}\int e^{2x}\cos x\,dx$$
$$= \frac{1}{2}e^{2x}\sin x - \frac{1}{4}\int e^{2x}\cos x - \frac{1}{4}\int e^{2x}\sin x\,dx$$
$$\Rightarrow \quad \frac{5}{4}\int e^{2x}\sin x\,dx = \frac{1}{2}e^{2x}\sin x - \frac{1}{4}e^{2x}\cos x$$
$$\int e^{2x}\sin x\,dx = \frac{e^{2x}}{5}(2\sin x - \cos x) + C$$

11. **(a)** $f'(x) = \ln x + x\frac{1}{x} - 1 = \ln x$

(b) Using integration by parts

$$\int(\ln x)^2\,dx = (\ln x)^2 \bullet x - \int x \bullet \frac{2}{x}(\ln x)\,dx$$
$$= x(\ln x)^2 - 2\int(\ln x)\,dx$$
$$= x(\ln x)^2 - 2(x\ln x - x) + C$$
$$\left(= x(\ln x)^2 - 2x\ln x + 2x + C\right)$$

12. $$\int \arctan x\,dx = x\arctan x - \int\frac{x}{1+x^2}\,dx$$

$$= x\arctan x - \frac{1}{2}\int\frac{2x}{1+x^2}\,dx = x\arctan x - \frac{1}{2}\ln(1+x^2) + c$$

13. $$\int 2x\arctan x\,dx = x^2\arctan x - \int\frac{x^2}{1+x^2}\,dx$$

$$= x^2\arctan x - \int\frac{1+x^2-1}{1+x^2}\,dx = x^2\arctan x - x + \arctan x + c$$

14. $\int \frac{x^2}{e^{2x}}\,dx = \int x^2 e^{-2x}dx = \frac{x^2e^{-2x}}{-2} + \int xe^{-2x}dx$

$$= \frac{x^2e^{-2x}}{-2} + \frac{xe^{-2x}}{-2} + \frac{1}{2}\int e^{-2x}dx$$

$$= \frac{x^2e^{-2x}}{-2} + \frac{xe^{-2x}}{-2} - \frac{e^{-2x}}{4} + c$$

15. (a) $\int x^2 \ln x \, dx = \left[\frac{x^3}{3}\ln x\right] - \int \frac{x^3}{3}\frac{1}{x}dx$

$$= \frac{x^3}{3}\ln x - \frac{x^3}{9} + \mathrm{c}$$

(b) $\int_1^2 x^2 \ln x \, dx \quad = \quad \frac{8}{3}\ln 2 - \frac{7}{9}$

With GDC, directly = 1.07

16. Integration by parts

$$\int_1^e x^5 \ln x dx = \left[\frac{x^6}{6}\ln x\right]_1^e - \int_1^e \frac{x^6}{6}\frac{1}{x}dx$$

$$= \left[\frac{x^6}{6}\ln x\right]_1^e - \int_1^e \frac{x^5}{6}dx = \left[\frac{x^6}{6}\ln x\right]_1^e - \left[\frac{x^6}{36}\right]_1^e$$

$$= \left(\frac{e^6}{6}\right) - \frac{e^6}{36} + \frac{1}{36} \quad \left(= \frac{5e^6+1}{36}\right)$$

17. $\int x\sin 2x dx = -x\frac{\cos 2x}{2} + \int \frac{\cos 2x}{2}dx = -\frac{x\cos 2x}{2} + \frac{\sin 2x}{4}$

Hence

$$\int_0^{\frac{\pi}{6}} x\sin 2x dx = \left[\frac{\sin 2x}{4} - \frac{x\cos 2x}{2}\right]_0^{\frac{\pi}{6}} = \left(\frac{\sqrt{3}}{8} - \frac{\pi}{24}\right) - 0 =$$

$$= \frac{\sqrt{3}}{8} - \frac{\pi}{24}$$

18.

$$\int_0^a \arcsin x \, dx = \ x\arcsin x \ {}_0^a - \int_0^a \frac{x}{\sqrt{1-x^2}}dx$$

$$= \ a\arcsin a - 0 \ + \left[\sqrt{1-x^2}\right]_0^a$$

$$= a\arcsin a + \sqrt{1-a^2} - 1$$

B. Exam style questions (LONG)

19. (a) Integration by parts

$$I_n = \int x^n \ln x \mathrm{d}x = \frac{x^{n+1}}{n+1}\ln x - \frac{1}{n+1}\int x^n \mathrm{d}x$$

$$= \frac{x^{n+1}}{n+1}\ln x - \frac{x^{n+1}}{(n+1)^2} + c$$

(b) Integration by substitution $u = \ln x$

$$J_n = \int \frac{(\ln x)^n}{x}\mathrm{d}x = \frac{(\ln x)^{n+1}}{n+1} + c$$

(c) (i) $\int \sqrt{x}\ln x \mathrm{d}x = I_{1/2} = \frac{2}{3}x^{3/2}\ln x - \frac{4}{9}x^{3/2} + c$

(ii) $\int \frac{\sqrt{\ln x}}{x}\mathrm{d}x = J_{1/2} = \frac{2}{3}(\ln x)^{3/2} + c$

(iii) $\int \left(\frac{\ln x}{x^2} + \frac{(\ln x)^2}{x}\right)\mathrm{d}x = I_{-2} + J_2 = \frac{x^{-1}}{-1}\ln x - \frac{x^{-1}}{(-1)^2} + \frac{(\ln x)^3}{3} + c$

$$= -\frac{\ln x}{x} - \frac{1}{x} + \frac{(\ln x)^3}{3} + c$$

20. (a) $A_0 = \int \sin x \mathrm{d}x = -\cos x + c$ and $B_0 = \int \cos x \mathrm{d}x = \sin x + c$

(b) $A_n = \int x^n \sin x \mathrm{d}x = -x^n \cos x + n\int x^{n-1}\cos x \mathrm{d}x = -x^n \cos x + nB_{n-1}$

(c) $B_n = \int x^n \cos x \mathrm{d}x = x^n \sin x + n\int x^{n-1}\sin x \mathrm{d}x = x^n \sin x + nA_{n-1}$

(d) (i) $A_n = -x^n \cos x + nB_{n-1}$

$= -x^n \cos x + n[x^{n-1}\sin x + (n-1)A_{n-2}]$

$= -x^n \cos x + nx^{n-1}\sin x + n(n-1)A_{n-2}$

(ii) $B_n = x^n \sin x + nA_{n-1}$

$= x^n \sin x + n[-x^{n-1}\cos x + (n-1)B_{n-2}]$

$= x^n \sin x - nx^{n-1}\cos x + n(n-1)B_{n-2}$

(e) $A_1 = -x\cos x + \sin x + c$,

$A_2 = -x^2 \cos x + 2x\sin x + 2A_0$

$= -x^2 \cos x + 2x\sin x - 2\cos x + c$

$A_3 = -x^3 \cos x + 3x^2 \sin x + 6A_1$

$= -x^3 \cos x + 3x^2 \sin x + 6[-x\cos x + \sin x] + c$

$= -x^3 \cos x + 3x^2 \sin x - 6x\cos x + 6\sin x + c$

21. (a) $I_0 = \int \mathrm{d}x = x + c$ and $I_1 = \int \sin x \mathrm{d}x = -\cos x + c$

(b) $I_n = \int \sin^{n-1} x \sin x \mathrm{d}x = -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x \cos^2 x \mathrm{d}x$

$$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x(1-\sin^2 x)\mathrm{d}x$$

$$= -\sin^{n-1} x \cos x + (n-1)I_{n-2} - (n-1)I_n$$

$$\Rightarrow \quad nI_n = -\sin^{n-1} x \cos x + (n-1)I_{n-2}$$

Hence, $I_n = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n} I_{n-2}$

(c)
$$I_2 = -\frac{1}{2}\sin x \cos x + \frac{1}{2}I_0 = -\frac{1}{2}\sin x \cos x + \frac{x}{2} + c$$

$$I_3 = -\frac{1}{3}\sin^2 x \cos x + \frac{2}{3}I_1 = -\frac{1}{3}\sin^2 x \cos x - \frac{2}{3}\cos x + c$$

$$I_4 = -\frac{1}{4}\sin^3 x \cos x + \frac{3}{4}I_2 = -\frac{1}{4}\sin^3 x \cos x - \frac{3}{8}\sin x \cos x + \frac{3x}{8} + c$$

$$I_5 = -\frac{1}{5}\sin^4 x \cos x + \frac{4}{5}I_3 = -\frac{1}{5}\sin^4 x \cos x - \frac{4}{15}\sin^2 x \cos x - \frac{8}{15}\cos x + c$$

(d)
$$I_2 = \int \sin^2 x \mathrm{d}x = \int \frac{1-\cos 2x}{2}\mathrm{d}x = \frac{x}{2} - \frac{\sin 2x}{4} + c$$

$$I_4 = \int \sin^4 x \mathrm{d}x = \int \left(\frac{1-\cos 2x}{2}\right)^2 \mathrm{d}x = \int \frac{1-2\cos 2x + \cos^2 2x}{4}\mathrm{d}x$$

$$= \frac{x}{4} - \frac{\sin 2x}{4} + \frac{x}{8} + \frac{\sin 4x}{32} = \frac{3x}{8} - \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + c$$

(e)
$$u = \cos x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = -\sin x \Rightarrow \mathrm{d}x = -\frac{\mathrm{d}u}{\sin x}$$

$$I_3 = \int \sin^3 x \mathrm{d}x = -\int \sin^2 x \mathrm{d}u = \int (u^2-1)\mathrm{d}u = \frac{u^3}{3} - u + c = \frac{1}{3}\cos^3 x - \cos x + c$$

$$I_5 = \int \sin^5 x \mathrm{d}x = -\int \sin^4 x \mathrm{d}u = -\int (1-u^2)^2 \mathrm{d}u = -\int \left(u^4 - 2u^2 + 1\right)\mathrm{d}u$$

$$= -\frac{u^5}{5} + 2\frac{u^3}{3} - u + c = -\frac{1}{5}\cos^5 x + \frac{2}{3}\cos^3 x - \cos x + c$$