Lesson 13: Congruence

Goals

- Determine whether shapes are congruent by measuring corresponding points.
- Draw and label corresponding points on congruent shapes.
- Justify (orally and in writing) that congruent shapes have equal corresponding distances between pairs of points.

Learning Targets

- I can use distances between points to decide if two shapes are congruent.

Lesson Narrative

So far, we have mainly looked at congruence for polygons. Polygons are special because they are determined by line segments. These line segments give polygons easily defined distances and angles to measure and compare. For a more complex shape with curved sides, the situation is a little different (unless the shape has special properties such as being a circle). The focus here is on the fact that the distance between *any* pair of corresponding points of congruent shapes must be the same. Because there are too many pairs of points to consider, this is mainly a criterion for showing that two shapes are *not* congruent: that is, if there is a pair of points on one shape that are a different distance apart than the corresponding points on another shape, then those shapes are *not* congruent.

For congruent shapes built out of several different parts (for example, a collection of circles) the distances between *all* pairs of points must be the same. It is not enough that the constituent parts (circles for example) be congruent: they must also be in the same configuration, the same distance apart. This follows from the definition of congruence: translations, rotations and reflections do not change distances between points, so if shape 1 is congruent to shape 2 then the distance between *any* pair of points in shape 1 is equal to the distance between the corresponding pair of points in shape 2.

Addressing

- Lines are taken to lines, and line segments to line segments of the same length.
- Understand that a two-dimensional shape is congruent to another if the second can be obtained from the first by a sequence of rotations, reflections, and translations; given two congruent shapes, describe a sequence that exhibits the congruence between them.

Instructional Routines

- Co-Craft Questions
- Compare and Connect
- Discussion Supports

- Think Pair Share

Required Materials

Geometry toolkits

tracing paper, graph paper, coloured pencils, scissors, and an index card to use as a straightedge or to mark right angles, plus a ruler and protractor. Clear protractors with no holes and with radial lines printed on them are recommended.

Student Learning Goals

Let's find ways to test congruence of interesting shapes.

13.1 Not Just the Vertices

Warm Up: 5 minutes

Polygons are special shapes because once we know the vertices, listed in order, we can join them by line segments to produce the polygon. This is important when performing translations, rotations and reflections. Because translations, rotations and reflections take line segments to line segments, once we track where the vertices of a polygon go, we can join them in the correct order with line segments to find the image of the polygon.

In this warm-up, students begin to explore this structure, finding corresponding points in congruent polygons which are *not* vertices.

Launch

Give 3 minutes quiet think time followed by 2 minutes for a whole-class discussion.

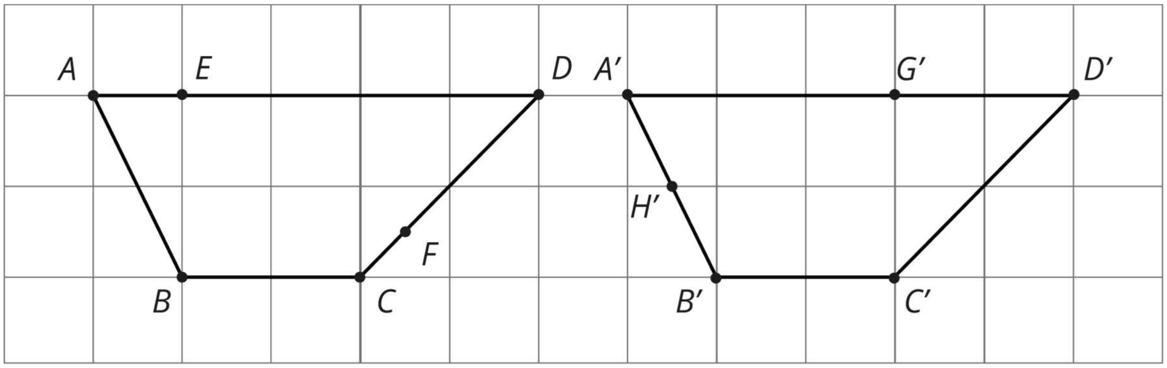
Anticipated Misconceptions

Students may struggle to find corresponding points that are not vertices. Suggest that they use tracing paper or the structure of the grid to help identify these corresponding points.

Student Task Statement

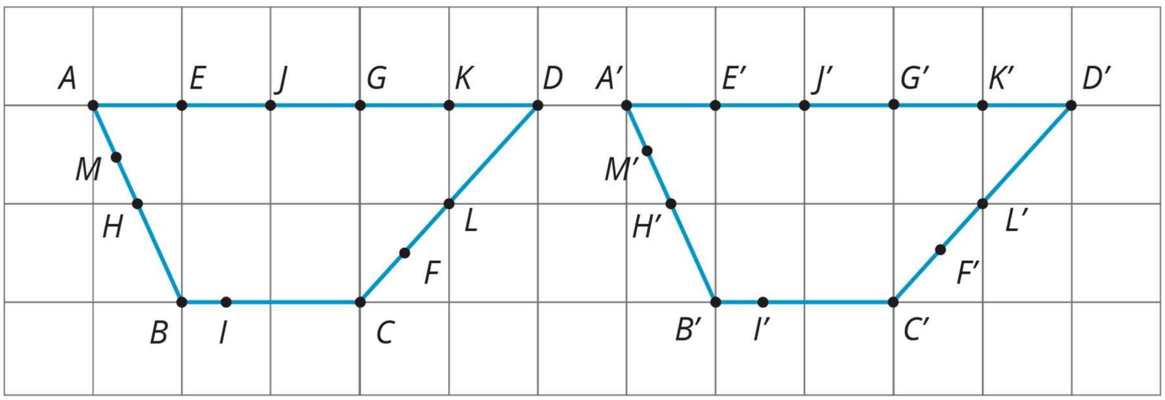
Trapeziums $ABCD$ and $A'B'C'D'$ are congruent.

- Draw and label the points on $A'B'C'D'$ that correspond to E and F.
- Draw and label the points on $ABCD$ that correspond to G' and H'.
- Draw and label at least three more pairs of corresponding points.



Student Response

Answers vary. Here are some possibilities:



Activity Synthesis

Remind students that when two shapes are congruent, every point on one shape has a corresponding point on the other shape.

Ask students what methods they used to find their corresponding points. Possible answers include:

- Using the grid and the corresponding vertices to keep track of distances and place the corresponding point in the right place
- Using tracing paper and translations, rotations or reflections taking one polygon to the other

13.2 Congruent Ovals

10 minutes

This activity begins a sequence which looks at shapes that are not polygons. From the point of view of congruence, polygons are special shapes because they are completely determined by the set of vertices. For curved shapes, we usually cannot check that they are congruent by examining a few privileged points, like the vertices of polygons. We *can* ascertain that they are not congruent by identifying a feature of one shape not shared by the other (for example, this oval is 3 units wide while this one is only $2\frac{1}{2}$ units wide). But to show that two curved shapes *are* congruent, we need to apply the definition of congruence and try to move one shape so that it matches up exactly with the other after some translations, rotations, or reflections.

In this activity, students begin to explore the subtleties of congruence for curved shapes. Make sure that students provide a solid mathematical argument for the shapes which are congruent, beyond saying that they look the same. Providing a viable argument requires careful thinking about the meaning of congruence and the structure of the shapes. Monitor for groups who use precise language of transformations as they attempt to move one traced oval to match up perfectly with another. Invite them to share their reasoning during the discussion. Also monitor for arguments based on measurement for why neither of the upper ovals can be congruent to either of the lower ones.

Instructional Routines

- Co-Craft Questions
- Think Pair Share

Launch

Arrange students in groups of 2. Provide access to geometry toolkits. Give students 3 minutes of quiet work time, then invite them to share their reasoning with a partner, followed by a whole-class discussion.

Engagement: Develop Effort and Persistence. Connect a new concept to one with which students have experienced success. For example, review the criteria used to determine congruence for polygons so that students can transfer these strategies in determining congruence for curved shapes.
Supports accessibility for: Social-emotional skills; Conceptual processing

Speaking: Co-Craft Questions. In this routine, students are given a context or situation, often in the form of a problem stem with or without numerical values. Students develop mathematical questions that can be asked about the situation. A typical prompt is: "What mathematical questions could you ask about this situation?" The purpose of this routine is to allow students to make sense of a context before feeling pressure to produce answers, and to develop students' awareness of the language used in mathematics problems.
Design Principle(s): Cultivating Conversations; Maximise meta-awareness

How It Happens:

1. Display the four images of the ovals without the directions.

   Ask students, "What mathematical questions could you ask about this situation?"

2. Give students 1 minute of individual time to jot some notes, and then 3 minutes to share ideas with a partner.

As pairs discuss, support students in using conversation skills to generate and refine their questions collaboratively by seeking clarity, referring to students' written notes, and revoicing oral responses, as necessary. Listen for how students use language about transformations and/or refer to measurements of the shapes when talking about the curved polygons.

3. Ask each pair of students to contribute one written question to a poster, the whiteboard, or digital projection. Call on 2–3 pairs of students to present their question to the whole class, and invite the class to make comparisons among the questions shared and their own questions.

   Listen for questions comparing different features of the ovals to determine congruence, especially those that use distances between corresponding points. Revoice student ideas with an emphasis on measurements wherever it serves to clarify a question.

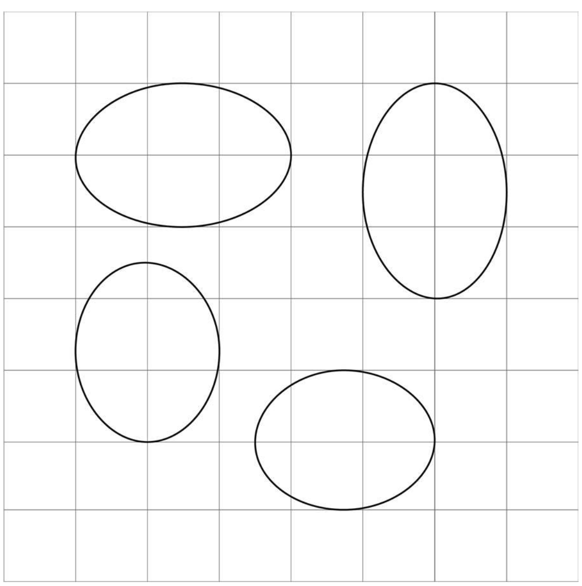
4. Reveal the question, "Are any of the ovals congruent to one another?" and give students a couple of minutes to compare it to their own question and those of their classmates. Identify similarities and differences.

   Consider providing these prompts: "Which of your questions is most similar to/different from the one provided? Why?", "Is there a main mathematical concept that is present in both your questions and the one provided? If so, describe it.", and "How do your questions relate to one of the lesson goals of measuring corresponding points to determine congruence?"

5. Invite students to choose one question to answer (from the class or from the curriculum), and then have students move on to the Activity Synthesis.

Student Task Statement

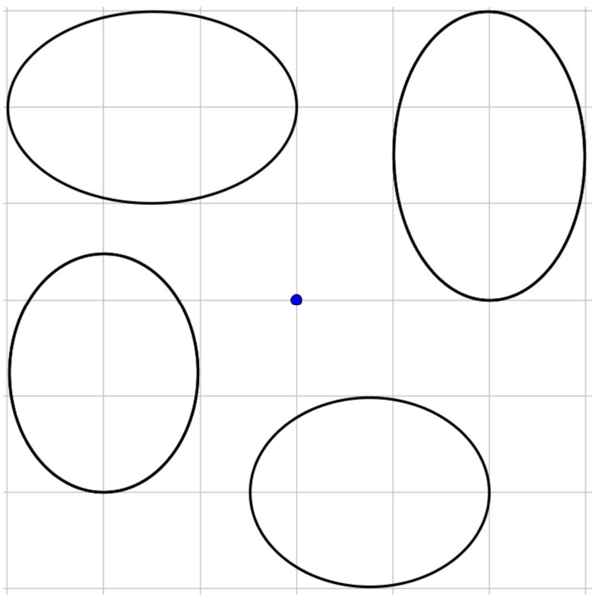
Are any of the ovals congruent to one another? Explain how you know.



Student Response

Sample response: All four shapes are ovals. For the top two shapes, they can be surrounded by rectangles that measure 3 units by 2 units. For the bottom two shapes, they measure about $2\frac{1}{2}$ (possibly a little less) units by 2 units. This means that the top shapes are not congruent to the bottom shapes.

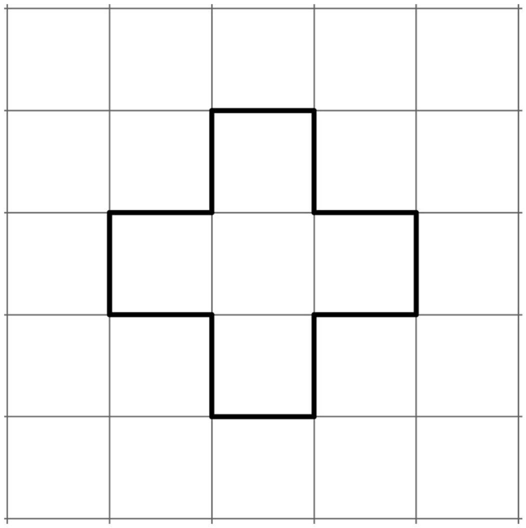
The top two shapes and the bottom two shapes have the same width and height (though the orientation is different). More is needed, however, to determine whether or not the two upper (and two lower) ovals are congruent. For this, we can trace one of the upper two ovals on tracing paper and check that it can be placed on top of the other and similarly for the lower pair of ovals. This can be done with, for example, a 90 degree clockwise rotation with centre at the point shown here:



Are You Ready for More?

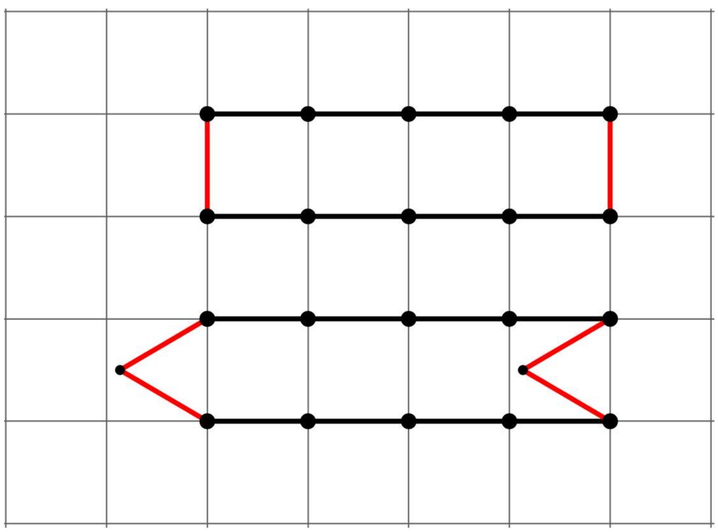
You can use 12 toothpicks to create a polygon with an area of five square toothpicks, like this:

Can you use exactly 12 toothpicks to create a polygon with an area of four square toothpicks?



Student Response

There is more than one solution, but here is one approach. The first shape has a perimeter of 10 units and an area of 4 square units. To get two more toothpicks in without changing the area, an indentation can be put on one side which is balanced by the part that "sticks out" on the other side:



Activity Synthesis

Invite groups to explain how they determined that the upper ovals are not congruent to the lower ones, with at least one explanation focusing on differing measurable attributes (for example length and width). Also invite previously selected groups to show how they demonstrated that the two upper (and two lower) ovals are congruent, focusing on the precise language of transformations.

Emphasise that showing that two oval shapes are congruent *requires* using the definition of congruence: is it possible to move one shape so that it matches up perfectly with the other using only translations, rotations and reflections? Experimentation with transformations is essential when showing that two of the ovals match up because, unlike polygons, these shapes are not determined by a finite list of vertices and side lengths.

Students have seen that rectangles that have the same side lengths are congruent and will later find criteria for determining when two triangles are congruent. For more complex curved shapes, the definition of congruence is required.

13.3 Corresponding Points in Congruent Shapes

15 minutes

Corresponding sides of congruent polygons have the same length. For shapes like ovals, examined in the previous activity, there are no "sides." However, if points A and B on one shape correspond to points A' and B' on a congruent shape, then the length of line segment AB is equal to the length of line segment $A'B'$ because translations, rotations, and reflections do not change distances between points. Students have seen and worked with this idea in the context of polygons and their sides. This remains true for other shapes as well.

Because translations, rotations and reflections do not change distances between points, corresponding points on congruent shapes (even oddly shaped shapes!) are the same distance apart. This is one more good example as the fundamental mathematical property of translations, rotations and reflections is that they do not change distances between corresponding points: this idea holds for *any* points on *any* congruent shapes.

There are two likely strategies for identifying corresponding points on the two corresponding shapes:

- Looking for corresponding parts of the shapes such as the line segments
- Performing translations, rotations and reflections with tracing paper to match the shapes up

Both are important. Watch for students using each technique and invite them to share during the discussion.

Instructional Routines

- Compare and Connect

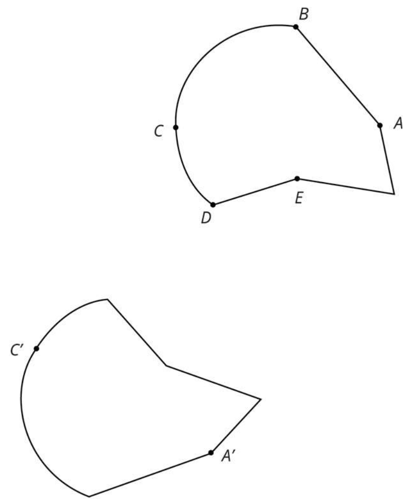
Launch

Keep students in the same groups. Allow for 5 minutes of quiet work time followed by sharing with a partner and a whole-class discussion. Provide access to geometry toolkits (rulers are needed for this activity).

Action and Expression: Internalise Executive Functions. Provide students with the images on grid or graph paper to assist in labelling corresponding parts and taking measurements of line segments.
Supports accessibility for: Language; Organisation

Student Task Statement

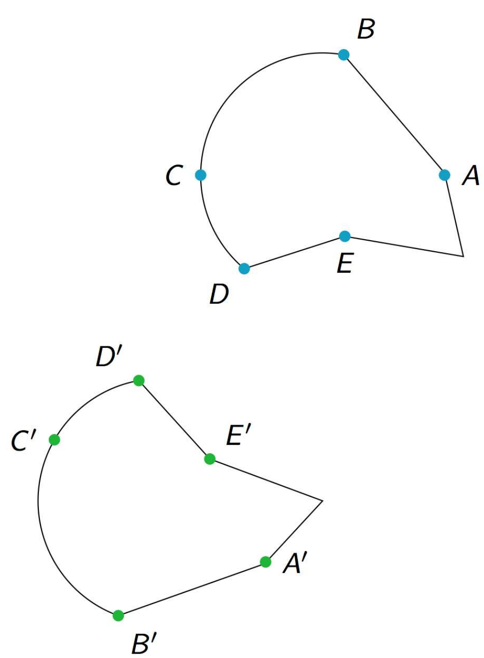
Here are two congruent shapes with some corresponding points labelled.



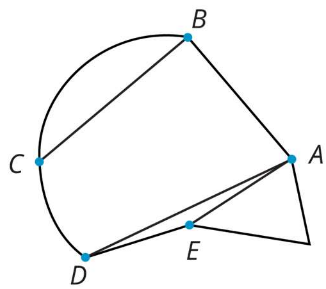
1. Draw the points corresponding to B, D, and E, and label them B', D', and E'.
2. Draw line segments AD and $A'D'$ and measure them. Do the same for line segments BC and $B'C'$ and for line segments AE and $A'E'$. What do you notice?
3. Do you think there could be a pair of corresponding line segments with different lengths? Explain.

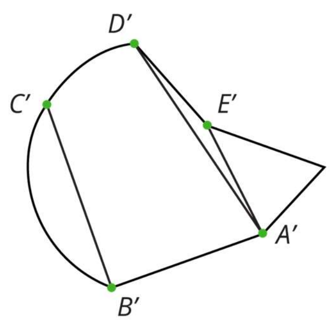
Student Response

1.



2. The lengths are the same. The translations, rotations and reflections of the plane used to show the congruence of these shapes do not change distances between points. So, the distance between A and D, for example, is the same as the distance between A' and D'. The same is true for the other pairs of corresponding points. The line segments connecting these points are all shown here:





3. Sample response: No, translations, rotations and reflections do not change distances between points. Corresponding line segments in the two congruent shapes *must* have the same length.

Activity Synthesis

Ask selected students to show how they determined the points corresponding to B, D, and E, highlighting different strategies (identifying key features of the shapes and performing translations, rotations and reflections). Ask students if these strategies would work for finding C' if it had not been marked. Performing translations, rotations and reflections matches the shapes up perfectly, and so this method allows us to find the corresponding point for *any* point on the shape. Identifying key features only works for points such as A, B, D, and E, which are essentially like vertices and can be identified by which parts of the shapes are "joined" at that point.

While it is challenging to test "by eye" whether or not complex shapes like these are congruent, the mathematical meaning of the word "congruent" is the same as with polygons: two shapes are congruent when there is a sequence of translations, reflections, and rotations that match up one shape exactly with the other. Because translations, reflections, and rotations do not change distances between points, *any* pair of corresponding line segments in congruent shapes will have the same length.

If time allows, have students use tracing paper to make a new shape that is either congruent to the shape in the activity or *slightly* different. Display several for all to see and poll the class to see if students think the shape is congruent or not. Check to see how the class did by lining up the new shape with one of the originals. Work with these complex shapes is important because we tend to rely heavily on visual intuition to check whether or not two polygons are congruent. This intuition is usually reliable unless the polygons are complex or have very subtle differences that cannot be easily seen. The meaning of congruence in terms of translations, rotations and reflections and our visual intuition of congruence can effectively reinforce one another:

- If shapes look congruent, then we can use this intuition to find the right transformations of the plane to demonstrate that they are congruent.
- Through experimenting with translations, rotations and reflections, we increase our visual intuition about which shapes are congruent.

Representing, Conversing, Listening: Compare and Connect. As students work, look for students who perform translations, rotations and reflections with tracing paper to test congruence of two shapes. Call students' attention to the different ways they match up shapes to identify corresponding points, and to the different ways these operations are made visible in each representation (e.g., lengths of line segments that are equal, translations, rotations and reflections do not change distances between points). Emphasise and amplify the mathematical language students use when determining if two shapes are congruent.
Design Principle(s): Maximise meta-awareness; Support sense-making

13.4 Astonished Faces

Optional: 10 minutes

Up to this point, students have mostly worked with simple shapes, that is, shapes which are individual and complete and not naturally divided into parts. In this activity, students examine two shapes made out of disjoint pieces: the outline of a face, two eyes, and a mouth. The disjoint pieces are congruent to one another taken in isolation. But they are not congruent taken as a collective whole because the relative positions of the eyes, mouth, and face outline are different in the two images.

This activity reinforces a goal of the previous activity. In order for two shapes to be congruent, *all* pairs of corresponding points in the two images must be the same distance apart. For the example studied here, this is true for many pairs of points. For example, if we choose a pair of points in one mouth and the corresponding pair of points in the other mouth, they will be the same distance from one another. The same is true if we choose a pair of points in the eye of one shape and a corresponding pair of points in an eye of the other shape. But if we choose one point in each eye or one point in the mouth and one point in an eye, the distances change. Strategically selecting corresponding points whose distance is not the same in the two shapes requires a solid understanding of the meaning of congruence and corresponding parts.

Monitor for students who:

- Identify that the parts (face outline, eyes, mouth) of the two faces are congruent.
- Identify that the relative position of the parts in the two faces are not the same.

Choose students to share these observations at the beginning of the discussion.

Instructional Routines

- Discussion Supports

Launch

Arrange students in groups of 2. Provide access to geometry toolkits. Give 5 minutes quiet work time followed by time to share their results and reasoning with their partner, then a whole-class discussion.

Supply tracing paper if desired. However, in this case, students should be encouraged to look for other ways to know the shapes are not congruent beyond saying that they do not match up when students try to place one on top of the other.

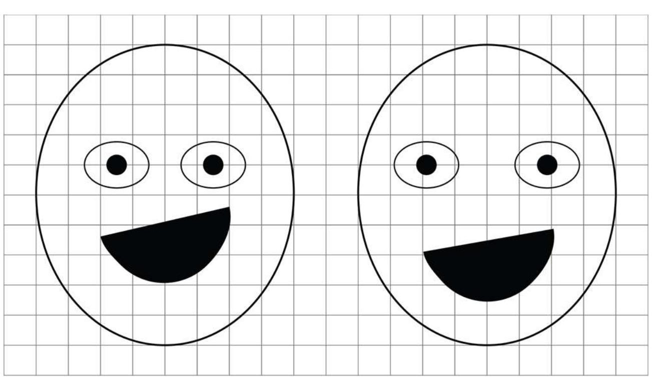
Action and Expression: Internalise Executive Functions. Chunk this task into more manageable parts to support students who benefit from support with organisational skills in problem solving. For example, present one section of the face at a time and monitor students to ensure they are making progress throughout the activity.
Supports accessibility for: Organisation; Attention

Anticipated Misconceptions

Students may think the two faces are congruent if all of the pieces of the faces match up, however the translations for each to match up are different. This may happen when students use tracing paper to test each individual piece. Ask students to find the distance between a pair of corresponding points that is not the same in the two faces. For example, ask them to measure the distance between a point on the left eye and point on the right eye and the corresponding distance in the second face.

Student Task Statement

Are these faces congruent? Explain your reasoning.



Student Response

No. Sample response: the faces are *not* congruent although the parts of the faces (outline, eyes, mouth) are congruent taken one by one. The two oval outlines of the faces sit in 8 unit by 10 unit rectangles and they are congruent as can be seen by translating the left face outline 10 units to the right (or reflecting over the centre vertical line on the grid). All 4 eyes in the two faces are also congruent: this can be shown using horizontal translations. The two mouths are congruent, also using a translation, but this time the translation has a horizontal component (10 units to the right) and a vertical component (about a half of a unit down).

While the individual parts are congruent, the faces as wholes are not. In the shape on the left, the eyes are one unit apart. For the shape on the right they are almost 2 units apart. The mouth on the left is about one unit below each eye while the mouth on the right is more than one unit below each eye. The mouth on the right is also closer to the outline of the face than the mouth on the left. In each case, there is a pair of corresponding points in the two shapes whose distances are different. This means that the shapes are not congruent.

Activity Synthesis

Ask selected students if the face outlines are congruent. What about the eyes? The mouths? Have them share their method for checking, emphasising precise language that uses translations, rotations and reflections (translations are all that is needed).

Ask selected students if the two faces, taken as a whole, are congruent. What corresponding points were they able to identify that were not the same distance apart in the two faces? Distance between the eyes, from the eyes to the mouth, from the eyes to the face outline, or from the mouth to face outline are all valid responses.

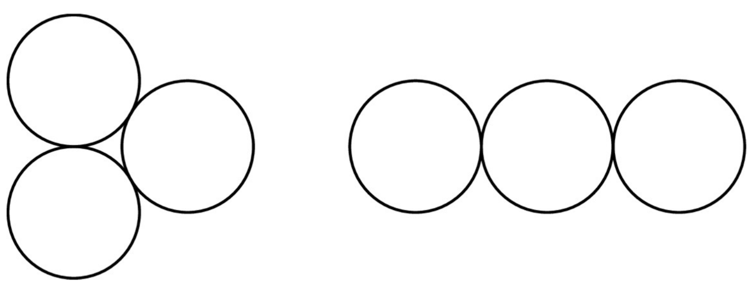
Even though the individual parts of the two faces are congruent, the two faces are *not* congruent. We could find one translation that takes the outline of one face to the outline of the other and similarly we could find a translation taking the left eye, right eye, and mouth of one shape to the left eye, right eye, and mouth of the other. But these translations are *different*. In order for two shapes to be congruent, there must be *one sequence of transformations* that match all parts of one shape perfectly with the other.

Speaking: Discussion Supports. Amplify students' uses of mathematical language to communicate about translations, rotations and reflections. As pairs share their results and reasoning, revoice their ideas using the terms “congruent shapes,” “vertical component,” and “horizontal component.” Then, invite students to use the terms when describing their results and strategies for determining congruence. Some students may benefit from chorally repeating the phrases that include the terms “congruent shapes,” “vertical component,” and “horizontal component” in context.
Design Principle(s): Optimise output (for explanation)

Lesson Synthesis

This lesson wraps up work on congruence. Important points to highlight include:

- Two shapes are congruent when there is a sequence of translations, rotations, and reflections matching up one shape with the other
- To show that two shapes are *not* congruent it is enough to find corresponding points on the shapes which are not the same distance apart, or corresponding angles that have different sizes
- The distance between pairs of corresponding points in congruent shapes is the same (this says that corresponding side lengths on polygons have the same length but it applies to curved shapes also or to any pair of points, not necessarily vertices, on polygons)
- Some shapes are made up of several parts. For example, these two designs are each made up of three circles:



All six of the circles are congruent (as we could check using tracing paper). But in the left design, each circle touches both of the other two, but this is not true in the design on the right. The distances between any two circle centres in one design will be different than the distances between any two circle centres in the other design.

13.5 Explaining Congruence

Cool Down: 5 minutes

Students decide whether or not two ovals are congruent. These particular ovals are visually quite distinct so expect students to use one of these methods:

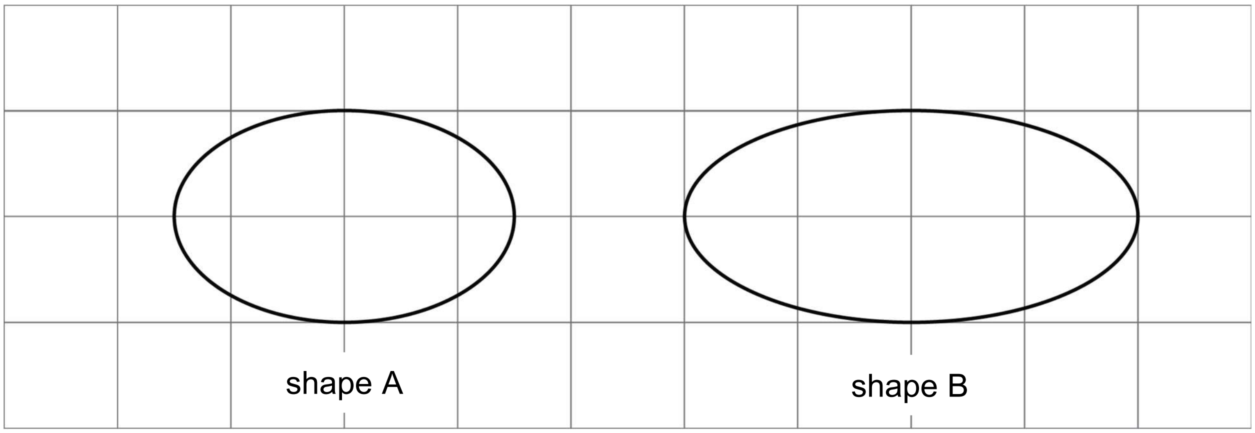
- Identify a distance on one oval that is different than the corresponding distance on the other oval.
- Trace one of the ovals and observe that it does not match up with the other one.

Launch

Make tracing paper available.

Student Task Statement

Are shapes A and B congruent? Explain your reasoning.

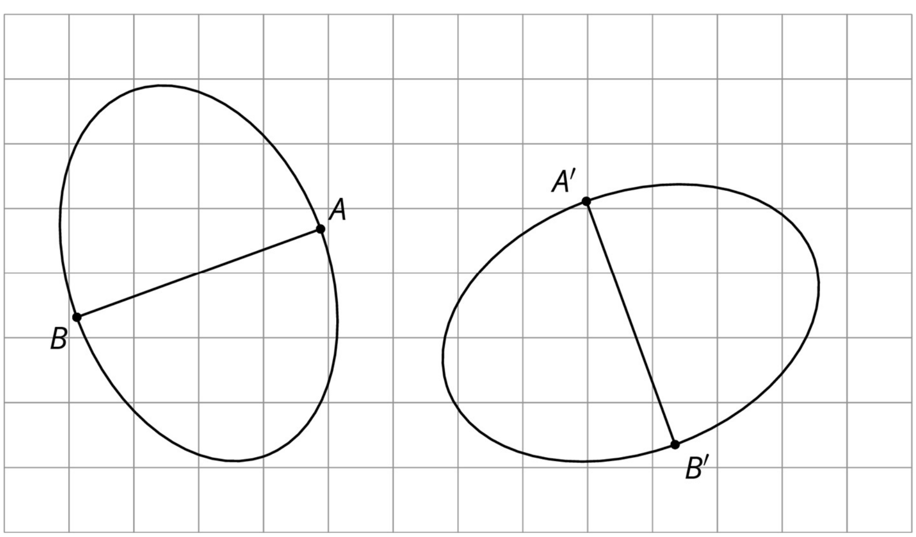


Student Response

Answers vary. Sample response: These shapes are not congruent because if they were, the longest horizontal distances between two points would be the same. However, for A it is less than 4 units, and for B it is about 4 units.

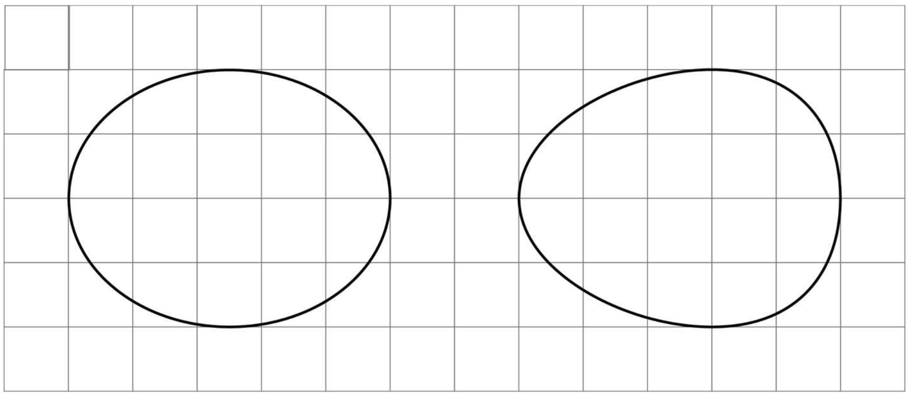
Student Lesson Summary

To show two shapes are congruent, you align one with the other by a sequence of translations, rotations and reflections. This is true even for shapes with curved sides. Distances between corresponding points on congruent shapes are always equal, even for curved shapes. For example, corresponding line segments AB and $A'B'$ on these congruent ovals have the same length:

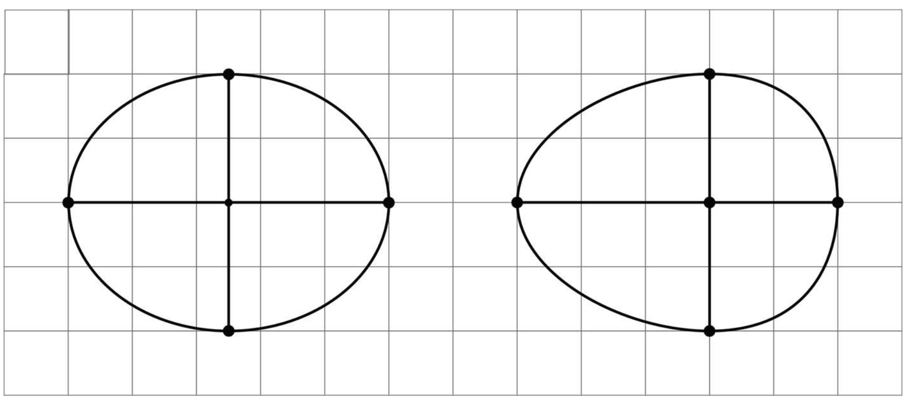


To show two shapes are not congruent, you can find parts of the shapes that should correspond but that have different measurements.

For example, these two ovals don't look congruent.



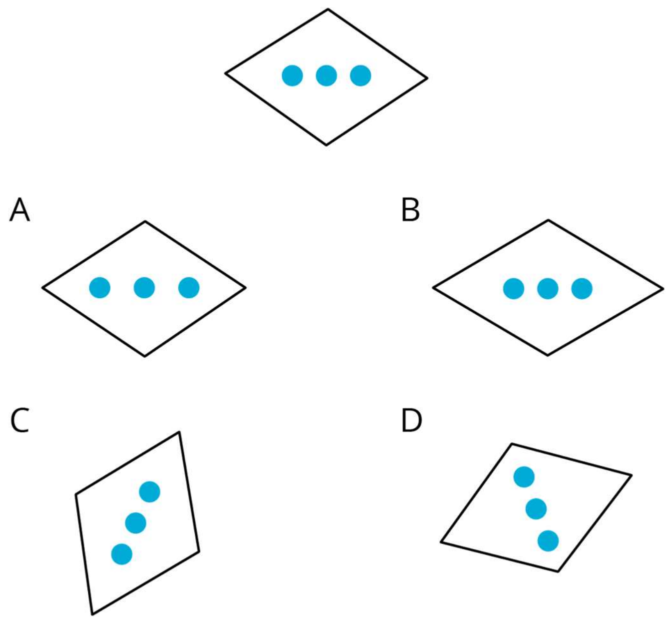
On both, the longest distance is 5 units across, and the longest distance from top to bottom is 4 units. The line segment from the highest to lowest point is in the middle of the left oval, but in the right oval, it's 2 units from the right end and 3 units from the left end. This proves they are not congruent.



Lesson 13 Practice Problems

1. **Problem 1 Statement**

   Which of these four shapes are congruent to the top shape?



   a. A

   b. B

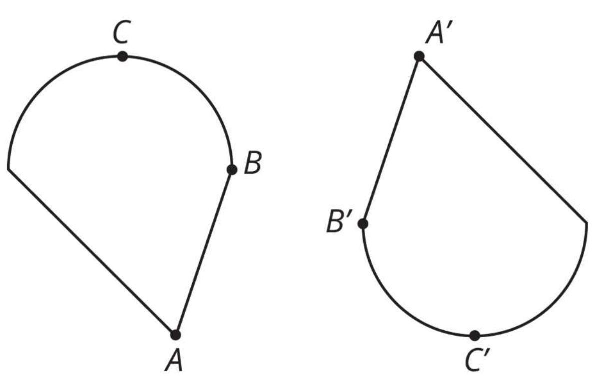
   c. C

   d. D

   Solution C

2. **Problem 2 Statement**

   These two shapes are congruent, with corresponding points marked.



   a. Are angles ABC and $A'B'C'$ congruent? Explain your reasoning.

b. Measure angles ABC and $A'B'C'$ to check your answer.

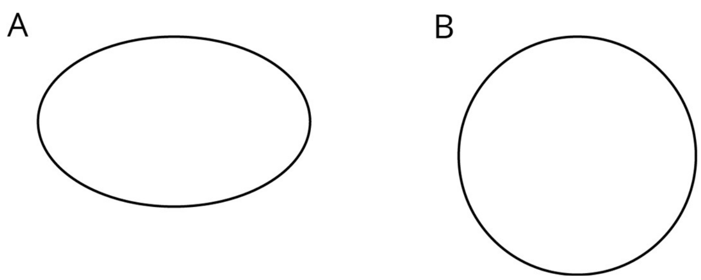
Solution

a. Yes they are angles made by corresponding points on congruent shapes so they are congruent.

b. Both angles measure about 110 degrees.

3. **Problem 3 Statement**

Here are two shapes.



Show, using measurement, that these two shapes are *not* congruent.

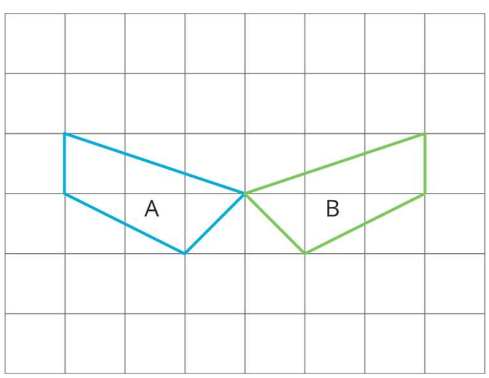
Solution

Answers vary. Sample response: The rightmost and leftmost points on shape A are further apart than any pair of points on shape B. So these two points cannot correspond to any pair of points on shape B and the two shapes are not congruent.

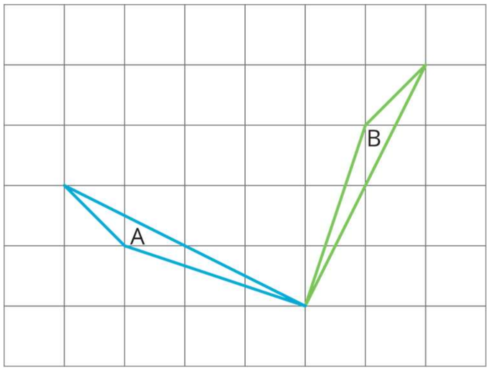
4. **Problem 4 Statement**

Each picture shows two polygons, one labelled polygon A and one labelled polygon B. Describe how to move polygon A into the position of polygon B using a transformation.

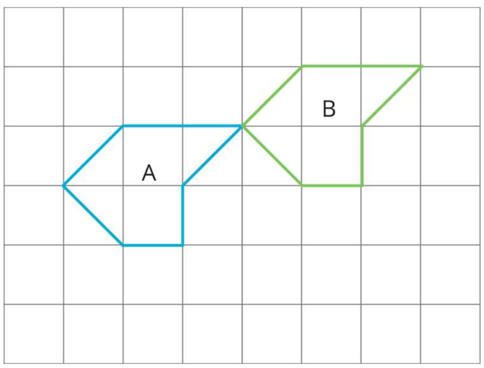
a.



b.



c.



Solution

a. Flip A over the vertical line through the vertex shared by A and B.

b. Rotate A in a clockwise direction around the vertex shared by the two polygons.

c. Translate A up and to the right. It needs to go up one unit and right 3 units.