Lesson 10: Comparing situations by examining ratios

Goals

- Choose and create diagrams to help compare two situations and explain whether they happen at the same rate.
- Justify that two situations do not happen at the same rate by finding a ratio to describe each situation where the two ratios share one value but not the other, i.e., $a : b$ and $a : c$, or $x : z$ and $y : z$.
- Recognise that a question asking whether two situations happen "at the same rate" is asking whether the ratios are equivalent.

Learning Targets

- I can decide whether or not two situations are happening at the same rate.
- I can explain what it means when two situations happen at the same rate.
- I know some examples of situations where things can happen at the same rate.

Lesson Narrative

In previous lessons, students learned that if two situations involve equivalent ratios, we can say that the situations are described by the **same rate**. In this lesson, students compare ratios to see if two situations in familiar contexts involve the same rate. The contexts and questions are:

- Two people run different distances in the same amount of time. Do they run at the same speed?
- Two people pay different amounts for different numbers of concert tickets. Do they pay the same cost per ticket?
- Two recipes for a drink are given. Do they taste the same?

In each case, the numbers are purposely chosen so that reasoning directly with equivalent ratios is a more appealing method than calculating how-many-per-one and then scaling. The reason for this is to reinforce the concept that equivalent ratios describe the same rate, before formally introducing the notion of unit rate and methods for calculating it. However, students can use any method. Regardless of their chosen approach, students need to be able to explain their reasoning in the context of the problem.

Addressing

- Understand the concept of a unit rate $\frac{a}{b}$ associated with a ratio $a : b$ with $b \neq 0$, and use rate language in the context of a ratio relationship. Expectations for unit rates in this grade are limited to non-complex fractions.

- Use ratio and rate reasoning to solve real-world and mathematical problems, e.g., by reasoning about tables of equivalent ratios, tape diagrams, double number line diagrams, or equations.

- Solve unit rate problems including those involving unit pricing and constant speed. For example, if it took 7 hours to mow 4 lawns, then at that rate, how many lawns could be mowed in 35 hours? At what rate were lawns being mowed?

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Stronger and Clearer Each Time
- Three Reads
- Think Pair Share

Student Learning Goals

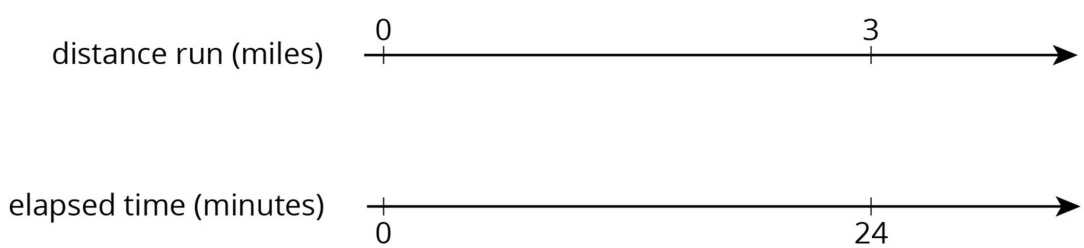
Let's use ratios to compare situations.

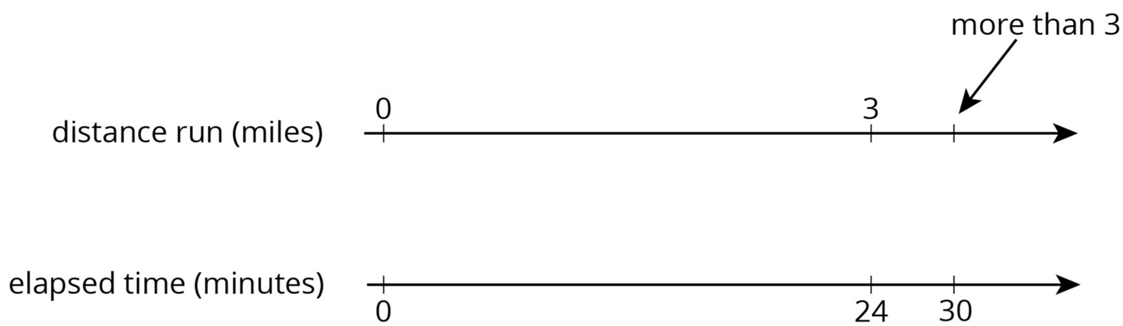
10.1 Treadmills

Warm Up: 10 minutes

In this activity, students encounter two distance-time ratios in which one quantity (distance) has the same value and the other quantity (time) has different values. Students interpret what the ratios mean in context, i.e., in terms of the speeds of two runners. There are several ways to reason about this with or without double number lines. Students may argue that since both runners ran for the same distance but Mai ran a shorter amount of time, she ran at a greater speed.

Students may also say that if Mai could run 3 miles in 24 minutes, at that speed, she would run more than 3 miles in 30 minutes. Since Jada only ran 3 miles in 30 minutes, Mai ran faster. The double number line that corresponds to these arguments may be as shown below.





As students work, monitor for the different ways they reason about the situation and identify a few students to share different approaches later.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Think Pair Share

Launch

Explain to students that a treadmill is an exercise machine for walking or running. Explain that while the runner does not actually go anywhere on a treadmill, a computer inside the treadmill keeps track of the distance travelled as if she were running outside.

If desired, this video shows a person starting a treadmill and walking at a constant speed for a few seconds.

Video 'Treadmill' available here: https://player.vimeo.com/video/304136549.

Students work on all parts of the activity silently and individually, then share their explanation with a partner.

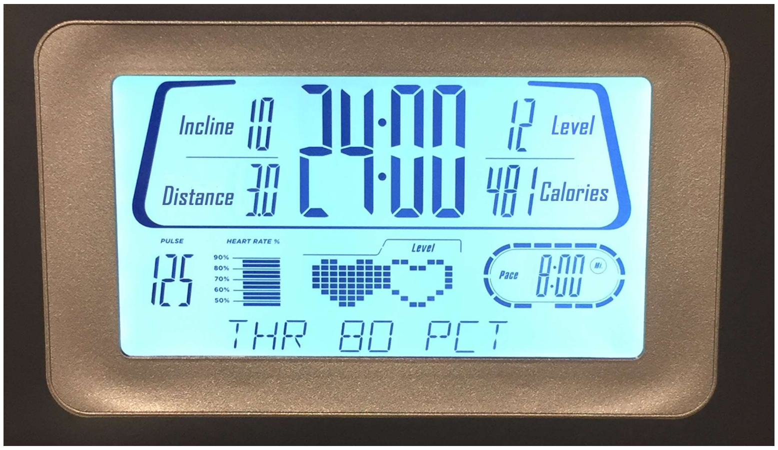
Anticipated Misconceptions

Because a person running on a treadmill does not actually go anywhere, it may be challenging to think about a distance covered. If this comes up, suggest that students think about running the given distances outside on a straight, flat road at a constant speed.

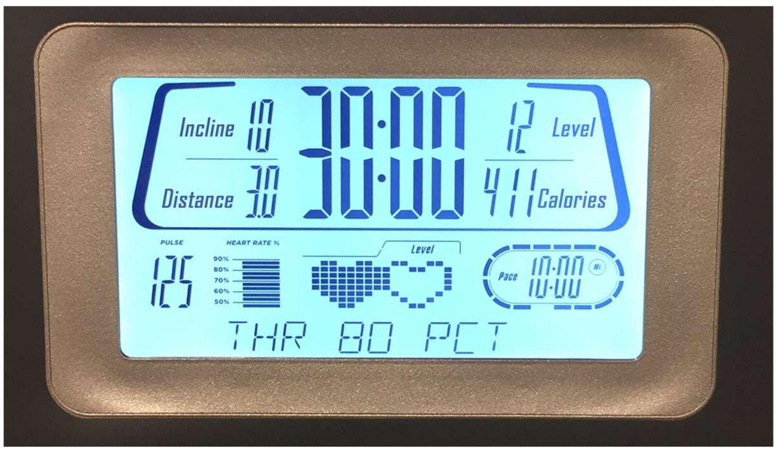
Student Task Statement

Mai and Jada each ran on a treadmill. The treadmill display shows the distance, in miles, each person ran and the amount of time it took them, in minutes and seconds.

Here is Mai's treadmill display:



Here is Jada's treadmill display:



1. What is the same about their workouts? What is different about their workouts?

2. If each person ran at a constant speed the entire time, who was running faster? Explain your reasoning.

Student Response

1. They both ran the same distance—3 miles. Also, the incline, level, and pulse are the same. The amount of time it took them is different—24 minutes versus 30 minutes. Also, the pace and calories differ.

2. Mai ran faster. She ran 3 miles in less time than it took Jada.

Activity Synthesis

Select a few students to share their reasoning about the speeds of the runners. If no students use a double number line to make an argument, illustrate one of their explanations using a double number line.

Remind students that even though a double number line is not always necessary, it can be a helpful tool to support arguments about ratios in different contexts.

10.2 Concert Tickets

10 minutes (there is a digital version of this activity)

Previously, students worked with ratios in which one quantity (distance run) had the same value and the other (time elapsed) did not. In the context of running, they concluded that the runners did not run at the same rate. Here, students work with two ratios in which neither quantity (number of tickets bought and money paid) has the same value, and decide if the two people in the situation bought tickets at the same rate. Students may approach the task in several ways. They may use a double number line to generate ratios that are equivalent to 47 : 3, representing the prices Diego would pay for different number of tickets. Once the price for 9 tickets is determined by scaling up the first ratio, they can compare it to the amount that Andre paid for the same number of tickets.

They may remember, without drawing number lines, that multiplying two values of a ratio by the same number produces a ratio that is equivalent. They may notice that multiplying 3 tickets by 3 results in 9 tickets, making the values for one quantity in the ratios match. They can then compare the two prices for 9 tickets. Alternatively, they may divide Andre's 9 tickets and £141 payment by 3 to get his price for 3 tickets.

Another approach is to calculate the price of 1 ticket, as students did in a previous lesson. To make this approach less attractive here, the numbers have been deliberately chosen so the price of a single ticket (£15$\frac{2}{3}$) is not a whole number.

As students work, monitor for those who reason correctly using the three approaches described above.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Three Reads
- Think Pair Share

Launch

Ask students what comes to mind when they hear the term "at the **same rate**"? Ask if they can think of any examples of situations that happen at the same rate. An example is two things travelling at the same speed. Any distance travelled will have the same associated time for things travelling at the same rate. Remind students that when situations happen "at the same rate," they can be described by ratios that are equivalent.

Give students quiet think time to complete the activity and then time to share their explanation with a partner.

If students have digital access, they can use an applet to explore the problem and justify their reasoning before sharing with a partner. If students have not used the number line applet in previous activities or need a refresher as to how to use it, demonstrate the treadmill problem with the applet.

Representation: Internalise Comprehension. Activate background knowledge about finding ratios that are equivalent. Allow students to use calculators to ensure inclusive participation in the activity.
Supports accessibility for: Memory; Conceptual processing

Reading: Three Reads. In this routine, students are supported in reading a mathematical text, situation, or word problem three times, each with a particular focus. During the first read, students focus on comprehending the situation; during the second read, students identify quantities; during the third read, students brainstorm possible strategies to answer the question. The question to be answered does not become a focus until the third read so that students can make sense of the whole context before rushing to a solution. The purpose of this routine is to support students' reading comprehension as they make sense of mathematical situations and information through conversation.
Design Principle(s): Support sense-making

How It Happens:

1. In the first read, students read the problem with the goal of comprehending the situation.

   Invite a student to read the problem aloud while everyone else reads with them and then ask, "What is this situation about?"

   Allow one minute to discuss with a partner, and then share with the whole class. A clear response would be: "Diego and Andre both bought tickets to a concert."

2. In the second read, students analyse the mathematical structure of the story by naming quantities.

   Invite students to read the problem aloud with their partner or select a different student to read to the class and then prompt students by asking, "What can be counted or measured in this situation? For now we don't need to focus on how many or how much of anything, but what can we count in this situation?" Give students one minute of quiet think time followed by another minute to share with their partner. Quantities may include: the number of tickets Diego bought, the number of tickets Andre bought, the amount Diego paid, the amount Andre paid, the rate of pounds per ticket that Diego paid, the rate of pounds per ticket that Andre paid.

   Call attention to the fact that whether we are talking about Andre or Diego, the two important quantities in this situation are: number of tickets, and amount paid in pounds.

3. In the third read, students brainstorm possible strategies to answer the question, "Did they pay at the same rate?"

Invite students to read the problem aloud with their partner or select a different student to read to the class. Instruct students to think of ways to approach the question without actually solving the problem.

Consider using these questions to prompt students: "How would you approach this question?," "What strategy or method would you try first?," and "Can you think of a different way to solve it?"

Give students one minute of quiet think time followed by another minute to discuss with their partner. Provide these sentence frames as partners discuss: "To compare the rates, I would use a double number line by….", "One way to approach the question would be to…."

Sample responses include: "I would figure out how much each person paid for one ticket", "I know that if I multiply 3 tickets by 3, I get 9, so I would see what happens when I multiply £47 by 3", "I would draw a diagram to figure out how much 3 groups of three tickets would cost at Diego's rate", and "I would use a double number line to scale up the rate for Diego's tickets (or scale down the rate for Andre's tickets) to see if they are the same rate." This will help students concentrate on making sense of the situation before rushing to a solution or method.

4. As partners are discussing their strategies, select 1–2 students to share their ideas with the whole class. As students are presenting ideas to the whole class, create a display that summarises ideas about the question.

   Listen for quantities that were mentioned during the second read, and take note of strategies for relating number of tickets to cost (amount paid).

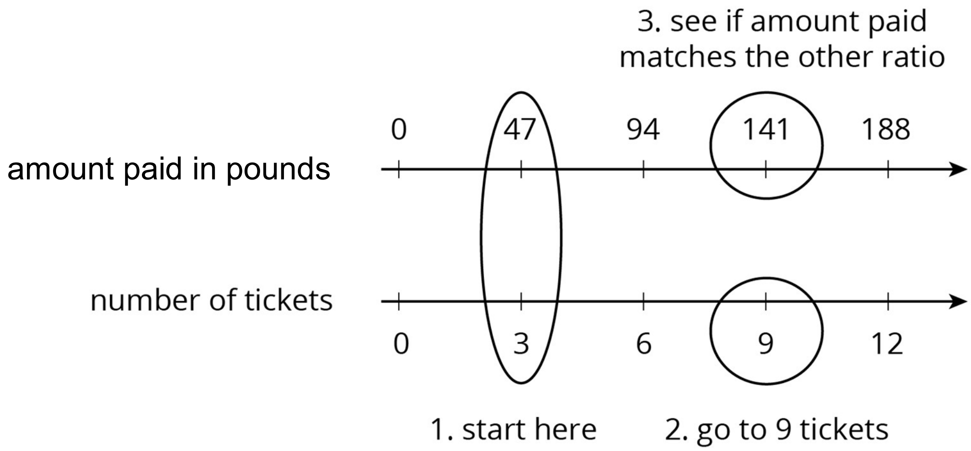
5. Post the summary where all students can use it as a reference, and suggest that students consider using the Three Reads routine for the next activity as well.

Student Task Statement

Diego paid £47 for 3 tickets to a concert. Andre paid £141 for 9 tickets to a concert. Did they pay at the **same rate**? Explain your reasoning.

Student Response

Yes, Andre paid at the same rate. Sample explanation: Since 9 is 3×3, multiply 47 by 3. $47 \times 3 = 141$. Diego would have paid £141 for 9 tickets if he paid at the same rate he did for 3 tickets. Since this is what Andre paid for 9 tickets, they paid at the same rate.



Activity Synthesis

The main strategy to highlight here is one that could tell us what Diego would pay for 9 tickets if he paid at the same rate as he did for 3 tickets. For 9 tickets, he would have paid £141, which is what Andre paid for 9 tickets and which tells us that they paid at the same rate.

Select 2–3 students to present their work to the class in the order of their methods:

- Correct use of a double number line to show that the given ratios are equivalent.
- Correct use of multiplication (or division) without using a double number line.
- (Optional) Correct use of unit price (i.e., by finding out 141 ÷ 9 and 47 ÷ 3). Though it's a less efficient approach here, the outcome also shows that the two people paid at the same rate.

Recap that using equivalent ratios to make one of the corresponding quantities the same can help us compare the other quantity and tell whether the situations involve the same rate.

10.3 Sparkling Orange Juice

15 minutes (there is a digital version of this activity)

Here, students compare the tastes of two sparkling orange juice mixtures, which involves reasoning about whether the two situations involve equivalent ratios. The problem is more challenging because no values of the quantities match or are multiples of one another. Instead of finding an equivalent ratio for one recipe so that it matches the other, students need to do so for *both* recipes.

To answer the question, students can either make the values of the sparkling water match and then compare the orange juice amounts, or make the orange juice amounts match and compare the values for sparkling water. Again, they may use a double number line, multiplication, and possibly finding how much of one quantity per 1 unit of the other quantity.

Instructional Routines

- Stronger and Clearer Each Time
- Think Pair Share

Launch

Introduce the task by saying that some people make sparking orange juice by mixing orange juice and sparkling water. Ask students to predict how the drink would taste if we mixed a *huge* amount of sparkling water with just a *little bit* of orange juice (it would not have a very orange-y flavour), or the other way around.

Explain that they will now compare the tastes of two sparkling orange juice recipes. Remind them that we previously learned that making larger or smaller batches of the same recipe does not change its taste.

Give students quiet think time to complete the activity and then time to share their explanation with a partner.

If students have digital access, they can use an applet to explore the problem and justify their reasoning before sharing with a partner.

Action and Expression: Internalise Executive Functions. Provide students with printed double number lines to represent Lin's and Noah's recipes.
Supports accessibility for: Language; Organisation Writing, Conversing: Stronger and Clearer Each Time. Use this routine to help students improve their writing, by providing them with multiple opportunities to clarify their explanations through conversation. At the appropriate time, give students time to meet with 2–3 partners, to share their response to the question, "How do the two mixtures compare in taste?" Students should first check to see if they agree with each other about how Lin and Noah's mixtures compare. Provide listeners with prompts for feedback that will help their partner add detail to strengthen and clarify their ideas. For example, "How did you use double number lines to solve this problem?" or "Can you say more about what each ratio means?" Give students with 3–4 minutes to revise their initial draft based on feedback from their peers. This will help strengthen students' understanding of how to determine whether two situations involve equivalent ratios.
Design Principle(s): Support sense-making; Optimise output (for explanation)

Anticipated Misconceptions

Some students may say that these two recipes would taste the same because they each use 1 more litre of sparkling water than orange juice (an additive comparison instead of a multiplicative comparison). Remind them of when we made batches of drink mix, and that mixtures have the same taste when mixed in equivalent ratios.

Student Task Statement

Lin and Noah each have their own recipe for making sparkling orange juice.

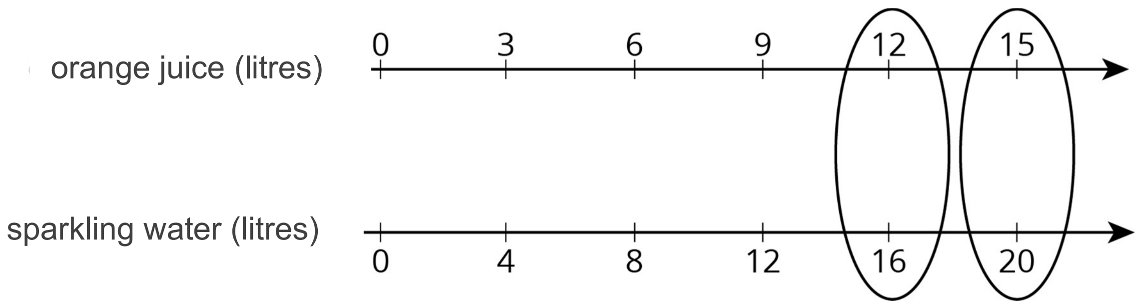
- Lin mixes 3 litres of orange juice with 4 litres of sparkling water.
- Noah mixes 4 litres of orange juice with 5 litres of sparkling water.

How do the two mixtures compare in taste? Explain your reasoning.

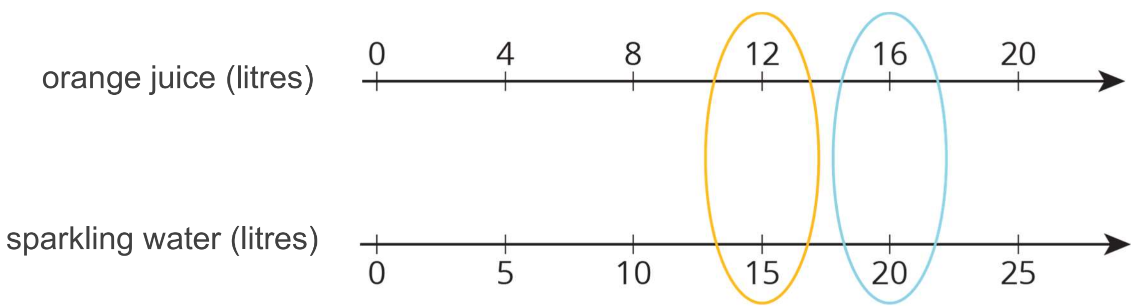
Student Response

Lin's mixture tastes a little more like sparkling water and Noah's mixture tastes a little more like orange juice.

Double number line for Lin's 3 : 4 recipe:



Double number line for Noah's 4 : 5 recipe:



With these double number line diagrams, we can see that the mixtures do not taste the same. For 12 litres of orange juice, the first recipe has 16 litres of sparkling water and the second recipe has 15 litres of sparkling water, so Noah's mixture has a stronger orange flavour. We can also see that for 20 litres of sparkling water, the first recipe has less orange juice than the second recipe, so Lin's mixture has a weaker orange flavour.

Are You Ready for More?

1. How can Lin make her sparkling orange juice taste the same as Noah's just by adding more of one ingredient? How much will she need?

2. How can Noah make his sparkling orange juice taste the same as Lin's just by adding more of one ingredient? How much will he need?

Student Response

1. If Lin adds $\frac{1}{5}$ litre of orange juice, then the ratio of juice to sparkling water will be $3\frac{1}{5}$: 4, which you can see is equivalent to 16 : 20 if you multiply by 5, which is equivalent to Noah's ratio.

2. If Noah adds $\frac{1}{3}$ litre of sparkling water, then the ratio of juice to sparkling water will be 4 : $5\frac{1}{3}$, which you can see is equivalent to 12 : 16 if you multiply by 3, which is equivalent to Lin's ratio.

Activity Synthesis

Display two double number line diagrams for all to see: one that represents batches of the 3 : 4 recipe and another that represents batches of the 4 : 5 recipe. Scale them up to show enough batches of each recipe to be able to make comparisons.

Ask students to explain how they can tell that the 4 : 5 recipe tastes more orange-y. Elicit both explanations: comparing the amount of sparkling water for the same amount of orange juice, and the other way around. Ensure students can articulate *why* each way of comparing means that the second recipe is more orange-y.

If any students calculated a unit rate for each recipe, you might consider inviting them to share, but help them be careful with their choice of words. It is important to say, for example, "In the first recipe, there is $\frac{3}{4}$ or 0.75 cup of orange juice per cup of sparkling water, but in the second recipe, there is $\frac{4}{5}$ or 0.8 cup of orange juice per cup of sparkling water." A student with this response would be comparing the number of cups of orange juice for every 1 cup of sparkling water in each mixture. (Note: if this approach comes up, consider taking this opportunity to discuss fraction comparison methods that students should know from earlier grades. In particular, $\frac{3}{4} = 1 - \frac{1}{4}$ is less than $\frac{4}{5} = 1 - \frac{1}{5}$ because $\frac{1}{5}$ is smaller than $\frac{1}{4}$.)

Lesson Synthesis

This lesson is all about figuring out whether two situations happen at the **same rate** by comparing one quantity when the other quantity is the same. In order to do that, it's helpful to generate equivalent ratios.

Briefly review the strategies used in the three activities in this lesson.

- How did we know that the people on the treadmill were not going the same speed? (They went different distances in the same amount of time.)
- How did we know the people paid the same rate for the concert tickets? (We figured out how much one person would have paid for 9 tickets at the same rate he paid for 3. We compared that to what the other person paid for 9 tickets.)

- How did we know that the sparkling orange juice recipes did not taste the same? (We made equivalent ratios so we could compare orange juice for the same amount of sparkling water or compare sparkling water for the same amount of orange juice.)
- How were all these problems alike? (We used equivalent ratios to make one part of the ratio the same and compared the other part.)

10.4 Comparing Runs

Cool Down: 5 minutes

Student Task Statement

Andre ran 2 kilometres in 15 minutes, and Jada ran 3 kilometres in 20 minutes. Both ran at a constant speed.

Did they run at the *same* speed? Explain your reasoning.

Student Response

They did not run at the same speed. There are many ways to justify this response. Here are some examples:

Andre would have run 6 kilometres in 45 minutes, and Jada would have run 6 kilometres in 40 minutes. Jada completes the 6 kilometres in less time, so she runs at a faster speed than Andre.

Andre would have run 8 kilometres in 60 minutes, and Jada would have run 9 kilometres in 60 minutes. Jada travels further in the same amount of time, so she runs at a faster speed than Andre.

These examples, while they also explain why Jada runs faster, also explain why the two runners did not run at the *same* speed.

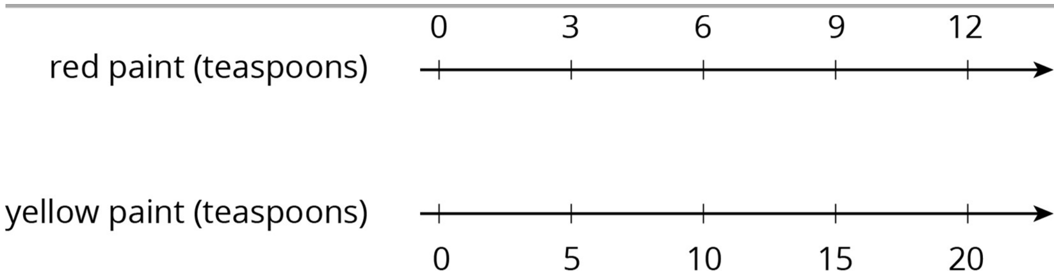
Student Lesson Summary

Sometimes we want to know whether two situations are described by the **same rate**. To do that, we can write an equivalent ratio for one or both situations so that one part of their ratios has the same value. Then we can compare the other part of the ratios.

For example, do these two paint mixtures make the same shade of orange?

- Kiran mixes 9 teaspoons of red paint with 15 teaspoons of yellow paint.
- Tyler mixes 7 teaspoons of red paint with 10 teaspoons of yellow paint.

Here is a double number line that represents Kiran's paint mixture. The ratio 9 : 15 is equivalent to the ratios 3 : 5 and 6 : 10.



For 10 teaspoons of yellow paint, Kiran would mix in 6 teaspoons of red paint. This is less red paint than Tyler mixes with 10 teaspoons of yellow paint. The ratios 6 : 10 and 7 : 10 are not equivalent, so these two paint mixtures would not be the same shade of orange.

When we talk about two things happening at the same rate, we mean that the ratios of the quantities in the two situations are equivalent. There is also something specific about the situation that is the same.

- If two ladybugs are moving at the same rate, then they are travelling at the *same constant speed*.
- If two bags of apples are selling for the same rate, then they have the *same unit price*.
- If we mix two kinds of juice at the same rate, then the mixtures have the *same taste*.
- If we mix two colours of paint at the same rate, then the mixtures have the *same shade*.

Glossary

- same rate

Lesson 10 Practice Problems

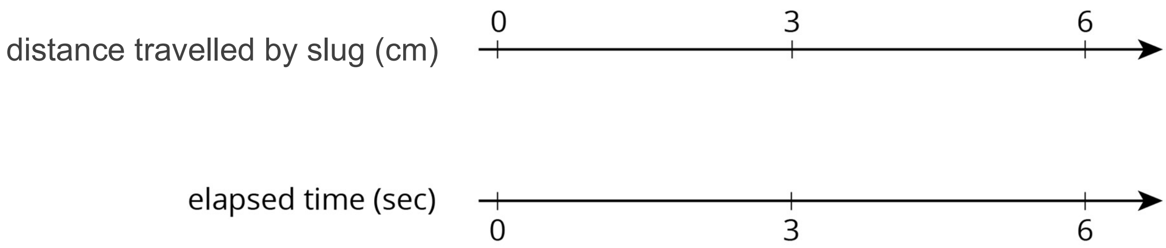
Problem 1 Statement

A slug travels 3 centimetres in 3 seconds. A snail travels 6 centimetres in 6 seconds. Both travel at constant speeds. Mai says, “The snail was travelling faster because it went a greater distance.” Do you agree with Mai? Explain or show your reasoning.

Solution

Answers vary. Sample responses:

- I disagree. The slug and the snail are both travelling 1 centimetre per second. They are travelling at the same speed.
- I disagree. The double number line for the slug shows that in 6 seconds it also travels 6 centimetres.



Problem 2 Statement

If you blend 2 scoops of chocolate ice cream with 1 cup of milk, you get a milkshake with a stronger chocolate flavour than if you blended 3 scoops of chocolate ice cream with 2 cups of milk. Explain or show why.

Solution

Answers vary. Sample responses:

- 3 scoops of chocolate ice cream with 2 cups of milk is 1.5 scoops of chocolate ice cream per cup of milk. This is less chocolate ice cream per cup of milk than in the first mixture (2 scoops of chocolate ice cream per cup of milk), so the first mixture has stronger chocolate flavour.
- 2 scoops of chocolate ice cream with 1 cup of milk will taste the same as 4 scoops of chocolate ice cream with 2 cups of milk. This mixture has an extra scoop of chocolate ice cream so will taste more chocolatey than 3 scoops of chocolate ice cream and 2 cups of milk.

Problem 3 Statement

There are 2 mixtures of light purple paint.

- Mixture A is made with 5 cups of purple paint and 2 cups of white paint.
- Mixture B is made with 15 cups of purple paint and 8 cups of white paint.

Which mixture is a lighter shade of purple? Explain your reasoning.

Solution

Mixture B is lighter. Explanations vary. Sample responses:

- Mixture A contains 2.5 cups of purple paint per cup of white paint. Mixture B contains only 1.875 cups of purple paint per cup of white paint. Less purple paint for the same amount of white paint will result in a lighter shade of purple.
- The ratio of purple paint to white paint in mixture A is 5 : 2. The ratio of purple paint to white paint in mixture B is 15 : 8. The amount of purple paint in

mixture B is 3 times the amount of mixture A, but the amount of white paint in B is 4 times the amount of A.

Problem 4 Statement

Tulip bulbs are on sale at store A, at 5 for £11.00, and the usual price at store B is 6 for £13. Is each store pricing tulip bulbs at the same rate? Explain how you know.

Solution

No. Explanations vary. Sample response: At store A, 30 bulbs would cost £66, but at store B, 30 bulbs would cost £65.

Problem 5 Statement

A plane travels at a constant speed. It takes 6 hours to travel 3 360 miles.

a. What is the plane's speed in miles per hour?

b. At this rate, how many miles can it travel in 10 hours?

Solution

a. 560 because $3\,360 \div 6 = 560$.

b. In 10 hours, it can travel 5 600 miles because $10 \times 560 = 5\,600$.

Problem 6 Statement

A pound of ground beef costs £5. At this rate, what is the cost of:

a. 3 pounds?

b. $\frac{1}{2}$ pound?

c. $\frac{1}{4}$ pound?

d. $\frac{3}{4}$ pound?

e. $3\frac{3}{4}$ pounds?

Solution

a. £15 (because $5 \times 3 = 15$)

b. £2.50 (because $\frac{1}{2} \times 5 = 2\frac{1}{2}$)

c. £1.25 (because $\frac{1}{4} \times 5 = 1\frac{1}{4}$)

d. £3.75 (three times the cost of $\frac{1}{4}$ pound)

e. £18.75 (the total cost of 3 pounds and $\frac{3}{4}$ pound)

Problem 7 Statement

In a triple batch of a spice mix, there are 6 teaspoons of garlic powder and 15 teaspoons of salt. Answer the following questions about the mix. If you get stuck, create a double number line.

a. How much garlic powder is used with 5 teaspoons of salt?

b. How much salt is used with 8 teaspoons of garlic powder?

c. If there are 14 teaspoons of spice mix, how much salt is in it?

d. How much more salt is there than garlic powder if 6 teaspoons of garlic powder are used?

Solution

a. 2 teaspoons

b. 20 teaspoons

c. 10 teaspoons

d. 9 teaspoons