[MAA 5.11] DEFINITE INTEGRALS – AREAS

SOLUTIONS

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DEFINITE INTEGRALS – PROPERTIES

O. Practice questions

1.

$\int_0^1 (2x+3)dx$	$= 4$
$\int_1^2 (2x+3)dx$	$= 6$
$\int_0^2 (2x+3)dx$	$= 10$
$\int_{-2}^2 (2x+3)dx$	$= 12$
$\int_0^1 (e^x+2)dx$	$= e+1$
$\int_0^\pi (\sin x+\cos x)dx$	$=2$

$\int_1^e \frac{7}{x}dx$	$= 7$
$\int_0^1 e^{2x+3}dx$	$\frac{e^5-e^3}{2}$
$\int_0^4 \frac{1}{x+1}dx$	$= \ln 5$
$\int_0^{10} xdx$	$=50$
$\int_0^{10} 5dx$	$=50$
$\int_4^{10} dx$	$=6$

2. (a) $f'(x) = \ln x$ (b) 3ln3-2

3.

$\int_5^7 3f(x)dx$	24
$\int_7^5 f(x)dx$	-2
$\int_5^7 (f(x)+1)dx$	10
$\int_5^7 (f(x)+x)dx$	20
$\int_5^7 [f(x)-4g(x)]dx$	0
$\int_5^8 f(x)dx - \int_7^8 f(x)dx$	8
$3\int_5^6 f(x)dx + \int_6^7 3f(x)dx$	24
$\int_8^{10} f(x-3)dx$	8
$\int_{2.5}^{3.5} f(2x)dx$	4

4. (a) $\int \frac{1}{2x+3}\mathrm{d}x = \frac{1}{2}\ln(2x+3)+C$

(b) $\int_0^3 \frac{1}{2x+3}\mathrm{d}x = \left[\frac{1}{2}\ln(2x+3)\right]_0^3 = \frac{1}{2}\ln 9 - \frac{1}{2}\ln 3 = \frac{1}{2}\ln 3 = \ln\sqrt{3}$

Thus $P = 3$

(c) $\frac{1}{2}\ln(2m+3) - \frac{1}{2}\ln 3 = 1 \Leftrightarrow \ln(2m+3) - \ln 3 = 2$

$\Leftrightarrow \ln\frac{2m+3}{3} = 2 \Leftrightarrow \frac{2m+3}{3} = \mathrm{e}^2 \Leftrightarrow 2m+3 = 3\mathrm{e}^2 \Leftrightarrow m = \frac{3\mathrm{e}^2-3}{2}$

A. Exam style questions (SHORT)

5. $\int_3^k \frac{1}{x-2}\,dx = [\ln(x-2)]_3^k = \ln(k-2) - \ln 1$

$\ln(k-2) - \ln 1 = \ln 7 \Rightarrow k - 2 = 7$, thus $k = 9$

6. $\int_1^k \left(1+\frac{1}{x^2}\right)dx = \frac{3}{2} \quad \left[x - \frac{1}{x}\right]_1^k = \frac{3}{2}$

$k - \frac{1}{k} = \frac{3}{2} \Rightarrow 2k^2 - 3k - 2 = 0 \quad \Rightarrow (2k+1)(k-2) = 0$

$k = 2$ since $k > 1$

7. $\int_0^a \cos^2 x\,dx = 0.740 \Rightarrow a = 1.047$ (using a graphic display calculator)

8. (a) $\frac{1}{2} \times 10 = 5$

(b) $\int_1^3 g(x)dx + \int_1^3 4dx = 10 + [4x]_1^3 = 10 + 8 = 18$

9. (a) (i) 16

(ii) $\int_0^3 f(x)\,dx + \int_0^3 2\,dx = 14$

(b) $\int_c^d f(x-2)\,dx = 8$

$c = 2, d = 5$

10. (a) 10

(b) $\int_1^3 3x^2 + f(x)dx = \int_1^3 3x^2 dx + \int_1^3 f(x)dx$

$\int_1^3 3x^2 dx = \left[x^3\right]_1^3 = 27 - 1 = 26$

Thus $\int_1^3 3x^2 + f(x)dx = 26 + 5 = 31$

11. (a) $\frac{d}{dx}\left(f(x) + g(x)\right) = f'(x) + g'(x) \quad \left(= f'(4) + g'(4)\right) = 7 + 4 = 11$

(b) $\int_1^3 \left(g'(x) + 6\right)dx = \left[g(x)\right]_1^3 + \left[6x\right]_1^3 = \left(g(3) - g(1)\right) + \left(18 - 6\right) \left(= (2-1) + 12\right) = 13$

12. (a) $\int_1^5 3f(x)dx = 3\int_1^5 f(x)dx = 12$

Thus $\int_1^5 f(x)\,dx = 4$

Since $\int_5^1 f(x)dx = -\int_1^5 f(x)dx$, $\int_5^1 f(x)dx = -4$

(b) $I = \int_1^2 (x + f(x))dx + \int_2^5 (x + f(x))dx = \int_1^5 (x + f(x))dx$

$\int_1^5 x dx + \int_1^5 f(x)dx$

$\int_1^5 x dx = \left[\frac{x^2}{2}\right]_1^5 = \frac{25}{2} - \frac{1}{2} = 12$

$I = 4 + 12 = 16$

AREAS

O. Practice questions

13. (a) (i) $\int_0^{\frac{\pi}{2}} \cos x dx = 1$, `(ii) $\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos x dx = -2$ `(iii) $\int_0^{\frac{3\pi}{2}} \cos x dx = -1$

(b) (i) 1 (ii) 2` (iii) 3

14. (a) $\frac{32}{3}(\sqrt{2}-1) \cong 4.42$

(b) 10.7

15.

Region enclosed by	Expression for the area	Area
$f(x) = \cos(x^2)$ and $g(x) = e^x$, for $-1.5 \le x \le 0.5$.	$\int_{-1.11}^{0} \cos x^2 - e^x \, dx$	0.282
$y = \sin x$ and $y = x^2 - 2x + 1.5$, for $0 \le x \le \pi$.	$\int_{0.6617}^{1.7010} (\sin x - (x^2 - 2x + 1.5))\, dx$	0.271
$y = \ln x$ and $y = e^x - e$, for $x > 0$.	$\int_{0.233}^{1} (\ln x - e^x + e)\, dx$	0.201
$y = \frac{2}{1+x^2}$ and $y = e^{x/3}$, for $-3 \le x \le 3$.	$\int_{-1.5247}^{0.74757} \left(\frac{2}{1+x^2} - e^{\frac{x}{3}}\right) dx=$	1.22
$f(x) = 4 - x^2$ and $g(x) = (x+1)\cos x$	$\int_{-1.92}^{2.72} (4 - x^2 - (x+1)\cos x) dx$	9.41
$y = e^{-x} - x + 1$ and the coordinate axes	$\int_{0}^{1.278} (e^{-x} - x + 1) dx$	1.18
$f: x \mapsto \frac{\sin x}{x}$, and x-axis for $\pi \le x \le 3\pi$	$\int_{\pi}^{3\pi} \left\lvert \frac{\sin x}{x} \right\rvert dx$	0.690
$y = x^3 - 3x^2 - 9x + 27$ and the line $y = x + 3$	$\int_{-3}^{4} \lvert y_1 - y_2 \rvert dx$	101.75

A. Exam style questions (SHORT)

16. (a) $f'(x) = 3\cos(x+2)$

(b) $\int f(x) dx = \int (1 + 3\sin(x+2)) dx = x - 3\cos(x+2) + c$

(c) (i) $1 + 3\sin(x + 2) = 0$, Required value of $a = 1.48$

(ii) 3.06

17. (a) $\int_0^1 12x^2(1-x) dx$

(b) $12\int_0^1 (x^2 - x^3)\, dx = 12\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = 12\left(\frac{1}{3} - \frac{1}{4}\right) = 1$

18. Area = $\int_0^{\frac{3\pi}{4}} \sin x \, dx = [-\cos x]_0^{\frac{3\pi}{4}} = \left(-\cos\frac{3\pi}{4}\right) - (-\cos 0) = -\left(-\frac{\sqrt{2}}{2}\right) - (-1) = 1 + \frac{\sqrt{2}}{2}$

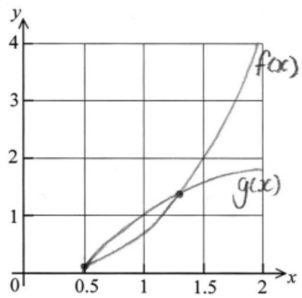
19. (a) $p = 3$

(b) Area $= \int_0^{\frac{\pi}{2}} 3\cos x \mathrm{d}x = [3\sin x]_0^{\frac{\pi}{2}} = 3$ square units

20. (i) $2x + \ln(x-1) + c$

(ii) $A = \int_2^4 f(x)\mathrm{d}x = \int_2^4 \left(2 + \frac{1}{x-1}\right)\mathrm{d}x = [2x + \ln(x-1)]_2^4 = (8 + \ln 3) - (4 + \ln 1) = 4 + \ln 3$

21. (a)



(b) $A = \int_a^b g(x) - f(x)\ \mathrm{d}x$ (using an appropriate definite integral)

$a = 0.50546...,\ b = 1.227...$

$A = 0.201$

22. (a) $x = 1,\ x = \pi$

(b) Attempting to find the area of two regions

$B = \int_{0.5}^{1} \sin x \ln x \,\mathrm{d}x + \int_{\pi}^{3.5} \sin x \ln x \,\mathrm{d}x$

$= -(0.09310... + 0.07736...)$

$B = 0.1704...$

$A = \int_1^{\pi} \sin x \ln x \,\mathrm{d}x = 0.8809...$

$0.8809 = k \times 0.1704$

$k = 5.17$

23.

$2 + x - x^2 = 2 - 3x + x^2$

$\Rightarrow 2x^2 - 4x = 0$

$\Rightarrow 2x(x-2) = 0$

$\Rightarrow x = 0,\ x = 2$

$\therefore \mathrm{A} = \int_0^2 \left((2 + x - x^2) - (2 - 3x + x^2)\right)\mathrm{d}x$

$= \int_0^2 (4x - 2x^2)\mathrm{d}x$ or equivalent

$= \left[2x^2 - \frac{2x^3}{3}\right]_0^2$

$= \frac{8}{3} \left(= 2\frac{2}{3}\right)$

24. $x^2 = 2a^2 - x^2 \Leftrightarrow 2x^2 = 2a^2 \Leftrightarrow x^2 = a^2 \Leftrightarrow x = \pm a$

$\int_{-a}^{a} (2a^2 - x^2 - x^2)\mathrm{d}x = 2\int_0^a (2a^2 - 2x^2)\mathrm{d}x = 4\int_0^a (a^2 - x^2)\mathrm{d}x$

$= 4\left[a^2 x - \frac{x^3}{3}\right]_0^a = 4\left(a^3 - \frac{a^3}{3}\right) = \frac{8a^3}{3}$

25. Area = $\int_0^k \sin 2x\,dx = 0.85$

$\int_0^k \sin 2x\,dx = \left[\frac{-1}{2}\cos 2x\right]_0^k = \frac{-1}{2}\cos 2k + 0.5$

Equation $\frac{-1}{2}\cos 2k + 0.5 = 0.85$ ($\Leftrightarrow \cos 2k = -0.7$)

THEN $k = 1.17$

OR directly by using SolveN : $\int_0^k \sin 2x\,dx = 0.85 \Rightarrow k = 1.17$

26. $\int_1^a \frac{1}{x}\,dx = 2 \Rightarrow [\ln x]_1^a = 2 \Rightarrow \ln a = 2 \Rightarrow a = e^2$

27. (a) 2.31

(b) (i) 1.02 (ii) 2.59

© $\int_p^q f(x)dx = 9.96$

split into two regions, make the area below the x-axis positive

28. (a) $\cos x + \sin x = 0 \Rightarrow \cos x = -\sin x \Rightarrow \tan x = -1$

$x = \frac{3\pi}{4}$

(b) Required area = $\int_0^{\frac{3\pi}{4}} e^x (\cos x + \sin x)\,dx = 7.46$ sq units

29. (a) π (3.14)

(b) (i) $\int_0^{\pi} e^x \sin x\,dx$

(ii) Area = 12.1

30. (a) (i) $f'(x) = \frac{-3(x^2-1)-(-3x)(2x)}{(x^2-1)^2} = \frac{-3x^2+3+6x^2}{(x^2-1)^2} = \frac{3x^2+3}{(x^2-1)^2} = \frac{3(x^2+1)}{(x^2-1)^2}$

(b) Area = $\int_0^a g(x)\,dx = \int_0^a \frac{3x^2+3}{(x^2-1)^2}\,dx = \left[\frac{-3x}{x^2-1}\right]_0^a = -\frac{3a}{a^2-1}$

Area = 2 $\Leftrightarrow -\frac{3a}{a^2-1} = 2 \Leftrightarrow 2a^2 + 3a - 2 = 0 \Leftrightarrow a = \frac{1}{2}$ $a = -2$ (rejected)

31. Area under parabola = $2\int_0^a (a^2 - x^2)\,dx = 2\left[a^2x - \frac{1}{3}x^3\right]_0^a = \frac{4}{3}a^3$

Since $PQ = 2a$, the dimensions of the rectangle are $2a \times \frac{2}{3}a^2$.

32. (a) At A, $x = 0.753$, At B, $x = 2.45$

(b) Area $\int_{0.753}^{2.45} y\,dx = 1.78$

33. (a) $f(x) = 0 \Leftrightarrow a = -1.73, b = 1.73$ ($a = -\sqrt{3}, b = \sqrt{3}$)

(b) **EITHER** setting $f'(x) = 0$, **OR** directly by GDC $c = 1.15$

(c) finding 2 areas

$-\int_{-1.73...}^{0} f(x)dx + \int_0^{1.149...} f(x)dx$

area = 2.07 (accept 2.06)

B. Exam style questions (LONG)

34. (a) $y = \ln x \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{x}$

when $x = e$, $\dfrac{dy}{dx} = \dfrac{1}{e}$

tangent line: $y - 1 = \left(\dfrac{1}{e}\right)(x - e) \Rightarrow y = \dfrac{x}{e}$

$x = 0 \Rightarrow y = \dfrac{0}{e} = 0$ Thud (0, 0) is on line

(b) $\dfrac{d}{dx}(x \ln x - x) = (1) \times \ln x + x \times \left(\dfrac{1}{x}\right) - 1 = \ln x$

(c) Area = area of triangle – area under curve

$= \left(\dfrac{1}{2} \times e \times 1\right) - \int_1^e \ln x dx = \dfrac{e}{2} - [x \ln x - x]_1^e = \dfrac{e}{2} - \{(e\ln e - 1 \ln 1) - (e - 1)\}$

$= \dfrac{e}{2} - \{e - 0 - e + 1\} = \dfrac{1}{2}e - 1.$

35. (a) $f(1) = 2$

$f'(x) = 4x,\ f'(1) = 4$

T: $y - 2 = 4(x - 1) \Rightarrow y = 4x - 2$

(b) $4x - 2 = 0 \Leftrightarrow x = \dfrac{1}{2}$

(c) **METHOD 1 (using only integrals)**

(i) Area $= \int_0^{0.5} 2x^2 dx + \int_{0.5}^{1} f(x) - (4x - 2) dx$

(ii) $\int 2x^2 dx = \dfrac{2x^3}{3}, \int (4x - 2) dx = 2x^2 - 2x$

Area $= \dfrac{1}{12} + \dfrac{2}{3} - 2 + 2 - \left(\dfrac{1}{12} - \dfrac{1}{2} + 1\right) = \dfrac{1}{6}$

METHOD 2 (using integral and triangle)

(i) Area $= \int_0^1 f(x) dx -$ area of triangle $= \int_0^1 f(x) dx - \dfrac{1}{2}$

(ii) $\int 2x^2 dx = \dfrac{2x^3}{3}$

Area $= \dfrac{2}{3} - \dfrac{1}{2} = \dfrac{1}{6}$

36. (a) $f'(x) = \dfrac{1}{2\sqrt{x}}$ $\quad f'(4) = \dfrac{1}{4}$

gradient of normal = – 4

normal; $y - 2 = -4(x - 4) \Rightarrow y = -4x + 18$

(b) $-4x + 18 = 0 \Rightarrow x = \dfrac{18}{4}\left(= \dfrac{9}{2}\right)$

(c) area of $R = \int_0^4 \sqrt{x} dx + \int_4^{4.5} (-4x + 18) dx$. **OR** $\int_0^4 \sqrt{x} dx + \dfrac{1}{2} \times 0.5 \times 2$ (triangle)

(d) $R = \left[\dfrac{2}{3}x^{\frac{3}{2}}\right]_0^4 + 0.5 = \dfrac{2}{3}4^{\frac{3}{2}} + \dfrac{1}{2} = \dfrac{16}{3} + \dfrac{1}{2} = \dfrac{35}{6}$

37. (a) (i) intersection points $x = 3.77$, $x = 8.30$

$$\int_{3.77}^{8.30} g(x)\,\mathrm{d}x - \int_{3.77}^{8.30} f(x)\,\mathrm{d}x = \int_{3.77}^{8.30} \big((-4\cos(0.5x)+2)-(\ln(3x-2)+1)\big)\,\mathrm{d}x$$

(ii) $A = 6.46$

(b) (i) $f'(x) = \dfrac{3}{3x-2}$

(ii) $g'(x) = 2\sin(0.5x)$

(c) $f'(x) = g'(x)$

$x = 1.43$, $x = 6.10$

38. (a) $y = x(x-4)^2$

(i) $y = 0 \Leftrightarrow x = 0$ or $x = 4$

(ii) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 1(x-4)^2 + x \times 2(x-4) = (x-4)(x-4+2x) = (x-4)(3x-4)$

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow x = 4$ or $x = \dfrac{4}{3}$

$$\left.\begin{array}{l} x = 1 \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x} = (-3)(-1) = 3 > 0 \\ x = 2 \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x} = (-2)(2) = -4 < 0 \end{array}\right\} \Rightarrow \frac{4}{3} \text{ is a } \textbf{maximum}$$

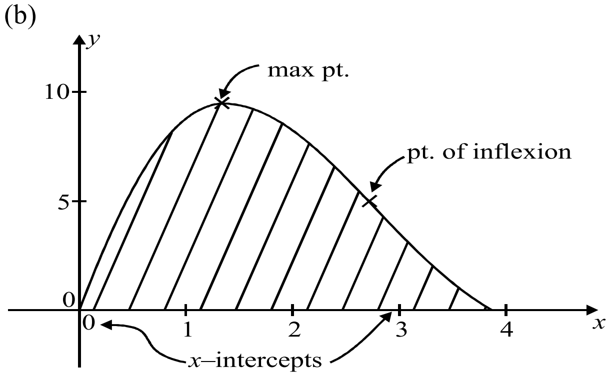
Note: *A second derivative test may be used*

$x = \dfrac{4}{3} \Rightarrow y = \dfrac{256}{27}$ Thus $\left(\dfrac{4}{3}, \dfrac{256}{27}\right)$

(iii) $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = \dfrac{\mathrm{d}}{\mathrm{d}x}\big((x-4)(3x-4)\big) = \dfrac{\mathrm{d}}{\mathrm{d}x}(3x^2 - 16x + 16) = 6x - 16$

$\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0 \Leftrightarrow 6x - 16 = 0 \Leftrightarrow x = \dfrac{8}{3}$

(b)

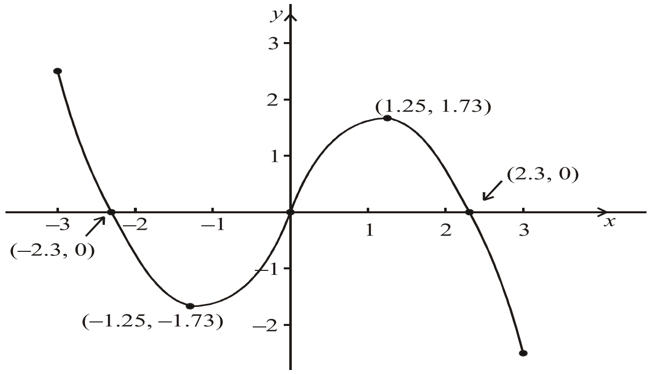


(c) (i) See diagram above

(ii) $0 < y < 10$ for $0 \le x \le 4$

So $\int_0^4 0\,\mathrm{d}x < \int_0^4 y\,\mathrm{d}x < \int_0^4 10\,\mathrm{d}x \Rightarrow 0 < \int_0^4 y\,\mathrm{d}x < 40$

39. (a)

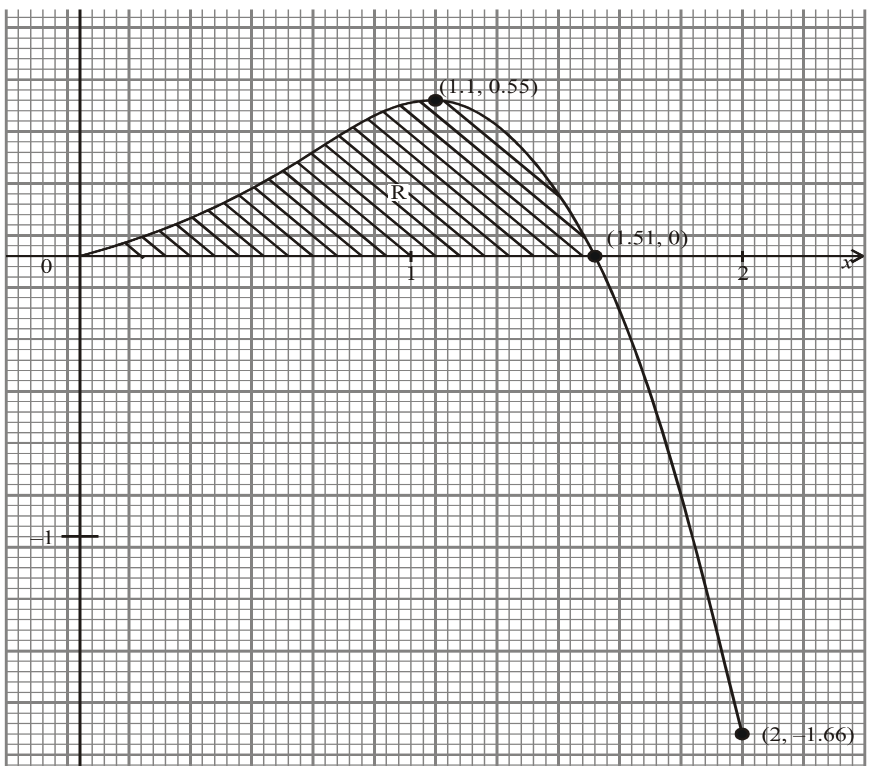


(b) $x = 2.31$

(c) $\int (\pi \sin x - x)\mathrm{d}x = -\pi \cos x - \frac{x^2}{2} + C$

Required area = $\int_0^1 (\pi \sin x - x)\mathrm{d}x = 0.944$

40. (a)(i)&(c)(i)



(ii) Approximate positions are
positive x-intercept (1.57,0), max point (1.1,0.55), end points (0, 0) and (2,–1.66)

(b) $x^2 \cos x = 0$ $\quad x \neq 0 \Rightarrow \cos x = 0 \Rightarrow x = \frac{\pi}{2}$

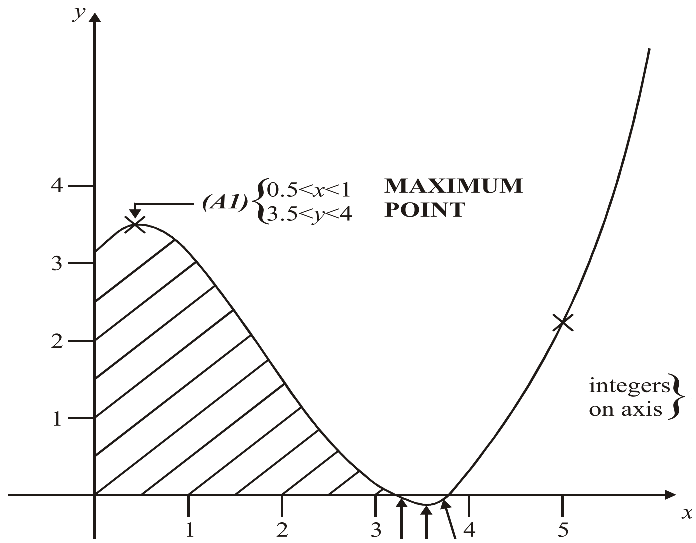
(c) (i) see graph (ii) $\int_0^{\pi/2} x^2 \cos x \, \mathrm{d}x$

(d) Integral = 0.467 by GDC **OR**

Integral = $\left[x^2 \sin x + 2x\cos x - 2\sin x\right]_0^{\pi/2} = \left[\frac{\pi^2}{4}(1) + 2\left(\frac{\pi}{2}\right)(0) - 2(1)\right] - [0 + 0 - 0]$

$= \frac{\pi}{2} - 2$ (exact) **or** 0.467 (3 s.f.)

41. (a)



(b) π is a solution if and only if $\pi + \pi\cos\pi = 0$.
Now $\pi + \pi\cos\pi = \pi + \pi(-1) = 0$

(c) $x = 3.696\ 722\ 9... = 3.69672$ (6s.f.)

(d) See graph: $\int_0^{\pi}(\pi + x\cos x)\mathrm{d}x$

(e) $\int_0^{\pi}(\pi + x\cos x)\mathrm{d}x = 7.86960$ (6 s.f.)

42. (a) $a = -3$, $b = 5$

(b) (i) $f'(x) = -3x^2 + 4x + 15$

$-3x^2 + 4x + 15 = 0 \Leftrightarrow x = -\frac{5}{3}$ or $x = 3$

(iii) $x = 3 \Rightarrow f(3) = -27 + 18 + 45 = 36$

(c) (i) At $x = 0$, $f'(0) = 15$
Line through (0, 0) of gradient 15 $\Rightarrow y = 15x$

(ii) $-x^3 + 2x^2 + 15x = 15x \Leftrightarrow -x^3 + 2x^2 = 0 \Leftrightarrow -x^2(x-2) = 0 \Leftrightarrow x = 2$

OR by GDC $x = 2$

(d) Area =115 (3 s.f.)

43. (a) (i) $f'(x) = \cos x\,(2\sin x\cos x) - \sin x\,(\sin x)^2 = \sin x\{2\cos^2 x - \sin^2 x\}$

(ii) $f'(x) = 0 \Rightarrow \sin x\{2\cos^2 x - \sin^2 x\} = 0 \Rightarrow \sin x\{3\cos^2 x - 1\} = 0$

$\Rightarrow 3\cos^2 x - 1 = 0 \Rightarrow \cos x = \pm\sqrt{\left(\frac{1}{3}\right)}$

At A, $f(x) > 0$, hence $\cos x = \sqrt{\left(\frac{1}{3}\right)}$

(iii) $f(x) = (\sin x)^2\cos x = (1-\cos^2 x)\cos x = \left(1-\frac{1}{3}\right)\frac{1}{\sqrt{3}} = \frac{2}{3\sqrt{3}}$ $\left(=\frac{2}{9}\sqrt{3}\right)$

(b) $x = \frac{\pi}{2}$

(c) (i) $\int (\cos x)(\sin x)^2 \, dx = \frac{1}{3}\sin^3 x + c$

(ii) Area = $\int_0^{\pi/2} (\cos x)(\sin x)^2 \, dx = \frac{1}{3}\left\{\left(\sin\frac{\pi}{2}\right)^3 - (\sin 0)^3\right\} = \frac{1}{3}$

(f) (i) $f'(x) = 2\sin x \cos^2 x - \sin^3 x$

$f''(x) = 2\cos x \cos^2 x + 2\sin x(-2\cos x \sin x) - 3\sin^2 x \cos x$

$= 2\cos^3 x - 4\cos x \sin^2 x - 3\sin^2 x \cos x$

$= 2\cos^3 x - 7\cos x \sin^2 x = 2\cos^3 x - 7\cos x(1 - \cos^2 x)$

$= 9(\cos x)^3 - 7\cos x$

(ii) At C $f''(x) = 0 \Leftrightarrow 9\cos^3 x - 7\cos x = 0 \Leftrightarrow \cos x(9\cos^2 x - 7) = 0$

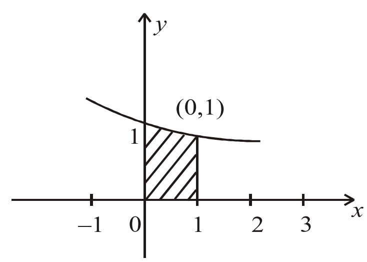
$\Rightarrow x = \frac{\pi}{2}$ (reject) **or** $\cos^2 x = \frac{7}{9}$

$\cos x = \frac{\sqrt{7}}{3}$

44. (a) $\int_0^1 e^{-kx} dx = \left[-\frac{1}{k}e^{-kx}\right]_0^1 = -\frac{1}{k}(e^{-k} - e^0) = -\frac{1}{k}(e^{-k} - 1) = -\frac{1}{k}(1 - e^{-k})$

(b) $k = 0.5$

(i)



(ii) Shading (see graph)

(iii) Area = $\int_0^1 e^{-kx} dx = \frac{1}{0.5}(1 - e^{05})$ **OR** Area = 0.787 (3 s.f.)

(c) (i) $\frac{dy}{dx} = -ke^{-kx}$

(ii) $x = 1$ $y = 0.8$ $\Rightarrow 0.8 = e^{-k} \Rightarrow \ln 0.8 = -k \Rightarrow k = -\ln 0.8$ (= 0.223)

(iii) At $x = 1$ $\frac{dy}{dx} = 0.8\ln 0.8$ **OR** $\frac{dy}{dx} = -0.178$ (or – 0.179)

45. (a) (i) $f'(x) = -\frac{3}{2}x + 1$

$f'(2) = -2$, gradient of the normal $\left(\frac{1}{2}\right)$

$y - 3 = \frac{1}{2}(x - 2)$ $\left(\text{or } y = \frac{1}{2}x + 2\right)$

(ii) $-\frac{3}{4}x^2 + x + 4 = \frac{1}{2}x + 2 \Leftrightarrow 3x^2 - 2x - 8 = 0 \Leftrightarrow x = -\frac{4}{3}$ $(= -1.33)$

(b) (i) $\int_{-1}^{2}\left(-\frac{3}{4}x^2 + x + 4\right)dx, \left[-\frac{1}{4}x^3 + \frac{1}{2}x^2 + 4x\right]_{-1}^{2}$

(ii) Area = $\frac{45}{4}$ $(= 11.25)$ (accept 11.3)

46. (a) At A, $x = 0 \Rightarrow y = \sin(e^0) = \sin(1) \Rightarrow$ coordinates of A = (0,0.841)

(b) $\sin(e^x) = 0 \Rightarrow e^x = \pi \Rightarrow x = \ln \pi$ (or $k = \pi$)

(c) (i) Maximum value of sin function = 1

(ii) $\frac{dy}{dx} = e^x \cos(e^x)$

$\frac{dy}{dx} = 0 \Rightarrow e^x \cos(e^x) = 0 \Rightarrow e^x = 0$ (impossible) or $\cos(e^x) = 0$

$\Rightarrow e^x = \frac{\pi}{2} \Rightarrow x = \ln \frac{\pi}{2}$

(d) (Area = $\int_0^{\ln \pi} \sin(e^x)\,dx$ = 0.90585 = 0.906 (3 s.f.)

47. (a) (i) $f'(x) = -2e^{-2x}$

(ii) $f'(x)$ is always negative

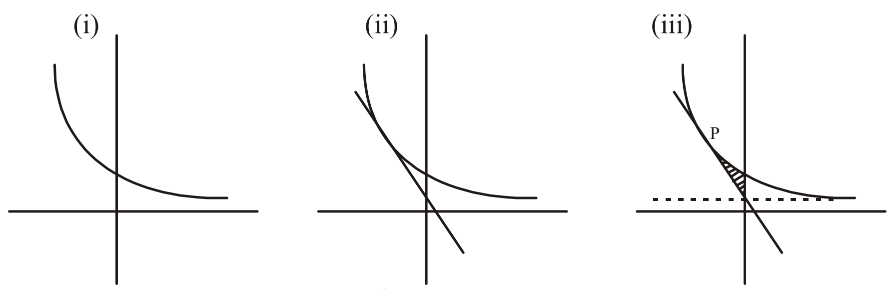
(b) (i) $y = 1 + e^{-2\times-\frac{1}{2}} = 1 + e$

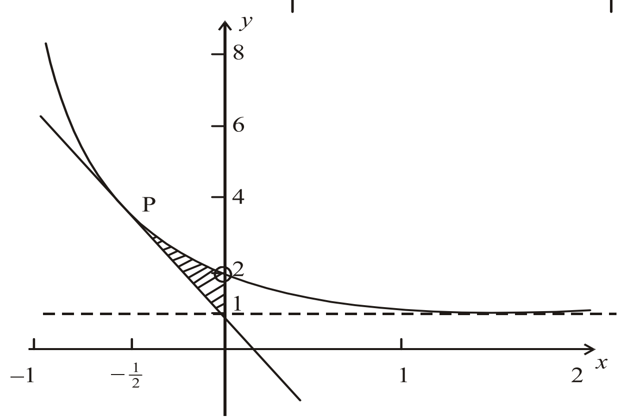
(ii) $f'\left(-\frac{1}{2}\right) = -2e^{-2\times-\frac{1}{2}} = -2e$

(c) $y - (1 + e) = -2e\left(x + \frac{1}{2}\right)$

$y = -2ex + 1$ ($y = -5.44x + 1$)

(d)





(iv) Area = $\int_{-\frac{1}{2}}^{0} [(1 + e^{-2x}) - (-2ex)]dx = \int_{-\frac{1}{2}}^{0} [(e^{-2x} + 2ex)dx = \left[-\frac{1}{2}e^{-2x} + ex^2\right]_{-\frac{1}{2}}^{0}$

= 0.1795 …= 0.180 (3 s.f.)

OR directly by GDC Area = 0.180

48. (a) $a = 1 - \pi \quad b = 1 + \pi$

(b) (i) $\int_{-2.14}^{1} h(x)\,\mathrm{d}x - \int_{1}^{2} h(x)\,\mathrm{d}x$ **OR** $\int_{-2.14}^{1} h(x)\,\mathrm{d}x + \left| \int_{1}^{2} h(x)\,\mathrm{d}x \right|$

(ii) $5.141\ldots - (-0.1585\ldots) = 5.30$

(c) (i) $y = 0.973$ (ii) $-0.240 < k < 0.973$

49. (a) (i) $f'(x) = -x + 2$ (ii) $f'(0) = 2$

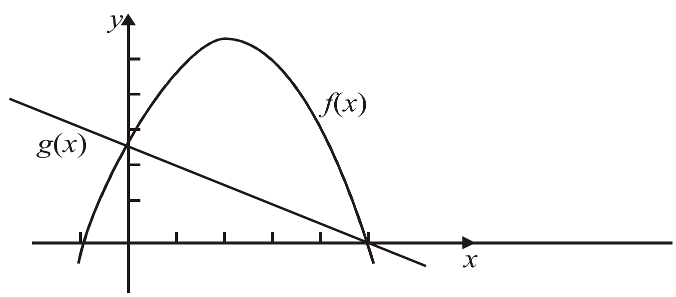
(b) y-intercept : At $x = 0$, $y = 2.5$

Gradient of tangent $= f'(0) = 2 \Rightarrow$ gradient of normal $= \frac{1}{2}$ $(= -0.5)$

the normal is $y - 2.5 = -0.5\,(x - 0) \Leftrightarrow (y = -0.5x + 2.5$

(c) (i) **EITHER** solving $-0.5x^2 + 2x + 2.5 = -0.5x + 2.5 \Leftrightarrow x = 0$ or $x = 5$

OR



Curves intersect at $x = 0$, $x = 5$

(ii) Curve and normal intersect when $x = 0$ or $x = 5$

Other point is when $x = 5 \Rightarrow y = -0.5(5) + 2.5 = 0$ (so other point (5, 0)

(d) (i) Area $= \int_{0}^{5} (f(x) - g(x))\mathrm{d}x = \int_{0}^{5} (-0.5x^2 + 2x + 2.5)\mathrm{d}x - \frac{1}{2} \times 5 \times 2.5\Big)$

(the second integral is the area of triangle)

(ii) $A1 = \frac{50}{3}, A_2 = \frac{25}{4}$

Area $= \frac{50}{3} - \frac{25}{4} = \frac{125}{12}$ (or 10.4 (3s.f.)

50. (a) intersection points at $x = -1$ and $x = 1$

$\int_{-1}^{1} e^x (1 - x^2)\mathrm{d}x$

(b) $f(0) = 1$. Thus P(0,1)

(c) $f'(0) = 1$, gradient of the normal $= -1$

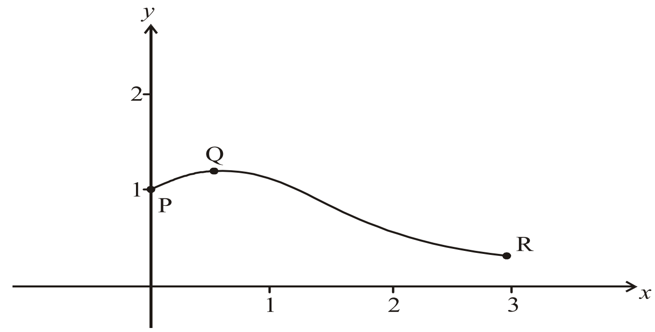
$y - 1 = -1(x - 0) \Leftrightarrow y - 1 = -x \Leftrightarrow y = -x + 1 \Leftrightarrow x + y = 1$

(d) (i) intersection points at $x = 0$ and $x = 1$

$\int_{0}^{1} \left(\mathrm{e}^x\left(1 - x^2\right) - (1 - x)\right)\mathrm{d}x$ **OR** $\int_{0}^{1} f(x)\,dx - \int_{0}^{1} (1 - x)\,\mathrm{d}x$

(ii) area $R = 0.5$

51. (a)



(b) (i) $f'(x) = 2e^{-x} + (2x+1)(-e^{-x}) = (1-2x)e^{-x}$

(ii) At **Q**, $f'(x) = 0$

$x = 0.5, y = 2e^{-0.5}$ **Q** is $(0.5, 2e^{-0.5})$

(c) $1 \le k < 2e^{-0.5}$

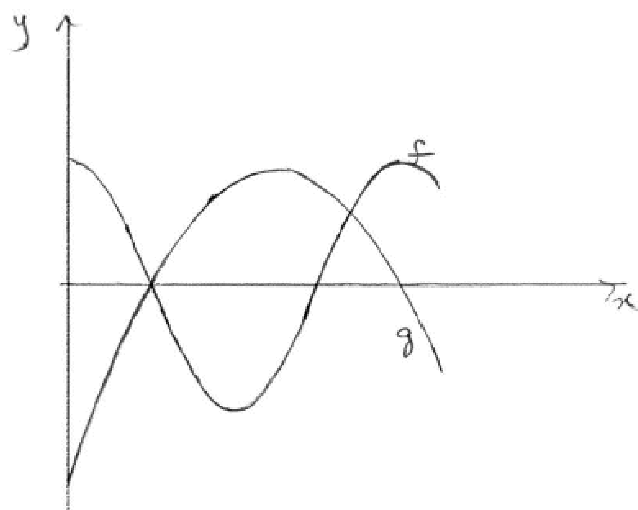
(d) At R, $y = 7e^{-3}$ (= 0.34850 ...)

Gradient of (PR) is $\frac{7e^{-3}-1}{3}\ (=-0.2172)$

Equation of (PR) is $y = \left(\frac{7e^{-3}-1}{3}\right)x+1$ OR $y = -0.2172x+1$

Shaded area is $\int_0^3 \left((2x+1)e^{-x} - \left(\frac{7e^{-3}-1}{3}x+1\right)\right)dx = 0.529$

52. (a)



(b) (i) (2,0) (accept $x = 2$)

(ii) period = 8

(iii) amplitude = 5

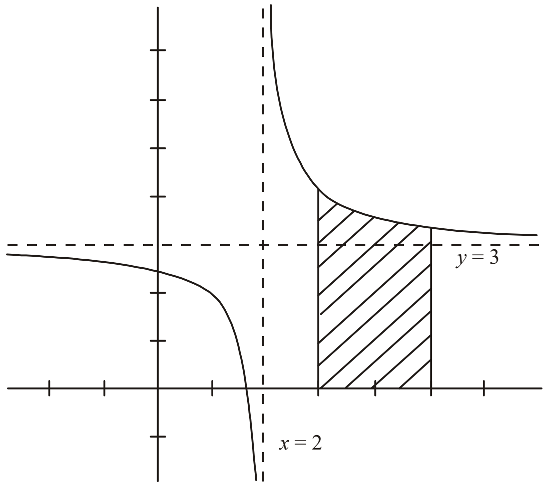
(c) (i) (2, 0), (8, 0) (accept $x = 2$, $x = 8$)

(ii) $x = 5$ (must be an equation)

(d) intersect when $x = 2$ and $x = 6.79$

area = $\int_2^{6.79} \left((-0.5x^2 + 5x - 8 - \left(5\cos\frac{\pi}{4}x\right)\right) = 27.6$

53. (a) (i)



(ii) (Vertical asymptote) $x = 2$, (Horizontal asymptote) $y = 3$

(b) (i) $3x + \ln(x-2) + C$

(ii) $\left[3x+\ln(x-2)\right]_3^5 = (15 + \ln 3) - (9 + \ln 1) = 6 + \ln 3$

(c) See graph

54. (a) limits $x = 0$, $x = 5$, area $= \int_0^5 f(x)\mathrm{d}x = 52.1$

(b) area is $\int_0^a x(a-x)\mathrm{d}x = \left[\frac{ax^2}{2} - \frac{x^3}{3}\right]_0^a = \frac{a^3}{2} - \frac{a^3}{3}$

$\frac{a^3}{2} - \frac{a^3}{3} = 52.1 \Leftrightarrow a^3 = 6 \times 52.1 \Leftrightarrow a = 6.79$

55. (a) $f(-x) = \frac{a(-x)}{(-x)^2+1} = \frac{-ax}{x^2+1} = -f(x)$

(b) $f''(x) = 0 \Leftrightarrow 2ax(x^2-3) = 0 \Leftrightarrow x = 0$ or $x^2 = 3$

$(0, 0)$, $\left(\sqrt{3}, \frac{a\sqrt{3}}{4}\right)$, $\left(-\sqrt{3}, -\frac{a\sqrt{3}}{4}\right)$

(c) (i) area $= \left[\frac{a}{2}\ln(x^2+1)\right]_3^7 = \frac{a}{2}(\ln 50 - \ln 10) = \frac{a}{2}\ln 5$

(ii) **METHOD 1**

the shift does not change the area: $\int_4^8 f(x-1)dx = \int_3^7 f(x)\mathrm{d}x, \frac{a}{2}\ln 5$

the factor of 2 doubles the area: $\int_4^8 2f(x-1)\mathrm{d}x = 2\int_4^8 f(x-1)\mathrm{d}x \quad \left(= 2\int_3^7 f(x)\mathrm{d}x\right)$

$\int_4^8 2f(x-1)\mathrm{d}x = a\ln 5$

METHOD 2

changing variable: let $w = x - 1$, so $\frac{\mathrm{d}w}{\mathrm{d}x} = 1$

Integral $= 2\int f(w)\mathrm{d}w = \frac{2a}{2}\ln(w^2+1) + c$

Limits: when $x=4 \Rightarrow w = 3$, when $x=8 \Rightarrow w = 7$,

$\int_4^8 2f(x-1)\mathrm{d}x = \left[a\ln(w^2+1)\right]_3^7 = a\ln 50 - a\ln 10 = a\ln 5$

56. (a) (i) gradient $= \dfrac{y_1 - y_2}{x_1 - x_2} = \dfrac{f(a) - 0}{a - \frac{2}{3}} =. \dfrac{a^3 - 0}{a - \frac{2}{3}} = \dfrac{a^3}{a - \frac{2}{3}}$

(ii) $f'(x) = 3x^2,\ f'(a) = 3a^2$

(iii) $3a^2 = \dfrac{a^3}{a - \frac{2}{3}} \Leftrightarrow 3a^2\left(a - \frac{2}{3}\right) = a^3 \Leftrightarrow 3a^3 - 2a^2 = a^3 \Leftrightarrow 2a^3 - 2a^2 = 0$

$\Leftrightarrow 2a^2(a - 1) = 0 \Leftrightarrow a = 1$

(b) Area $= \int_{-2}^{k} (x^3 - 3x + 2)\mathrm{d}x = \left[\frac{1}{4}x^4 - \frac{3}{2}x^2 + 2x\right]_{-2}^{k}$

$\left(\frac{1}{4}k^4 - \frac{3}{2}k^2 + 2k\right) - (4 - 6 - 4) = \frac{1}{4}k^4 - \frac{3}{2}k^2 + 2k + 6$

$\frac{1}{4}k^4 - \frac{3}{2}k^2 + 2k + 6 = 2k + 4$

$\frac{1}{4}k^4 - \frac{3}{2}k^2 + 2 = 0$

$k^4 - 6k^2 + 8 = 0$

57. (a) (i) $\sin x = 0 \Leftrightarrow x = 0, x = \pi$

(ii) $\sin x = -1 \Leftrightarrow x = \dfrac{3\pi}{2}$

(b) $\dfrac{3\pi}{2}$

(c) $k = \int_{0}^{\frac{3\pi}{2}} (6 + 6\sin x)\mathrm{d}x = [\ 6x - 6\cos x\ ]_0^{3\pi/2} = 6\left(\frac{3\pi}{2}\right) - 6\cos\left(\frac{3\pi}{2}\right) - (-6\cos 0)$

$k = 9\pi + 6$

(d) translation of $\begin{pmatrix} \frac{\pi}{2} \\ 0 \end{pmatrix}$

(e) the area under g is the same as the shaded region in f

$p = \dfrac{\pi}{2}, p = 0$

58. (a) $13 = Ae^0 + 3 \Leftrightarrow 13 = A + 3 \Leftrightarrow A = 10$

(b) $f(15) = 3.49 \Leftrightarrow 3.49 = 10e^{15k} + 3 \Leftrightarrow k = -0.201$ $\left(\text{accept } \dfrac{\ln 0.049}{15}\right)$

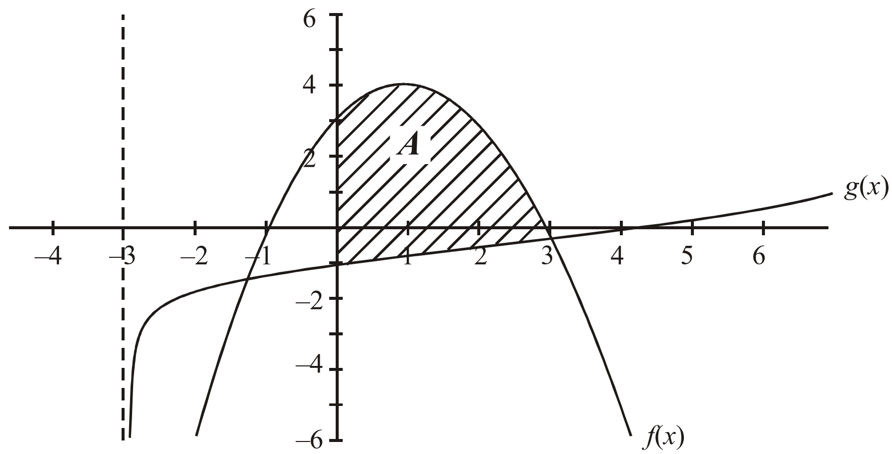
(c) (i) $f(x) = 10e^{-0.201x} + 3$
$f'(x) = 10e^{-0.201x} \times -0.201 = -2.01e^{-0.201x}$

(ii) $f'(x) < 0$, derivative always negative

(iii) $y = 3$

(d) finding limits 3.8953…, 8.6940…
Area $= \int_{3.90}^{8.69} g(x) - f(x)\mathrm{d}x, \int_{3.90}^{8.69} [(-x^2 + 12x - 24) - (10e^{-0.201x} + 3)]\mathrm{d}x = 19.5$

59. (a)



(b) (i) $x = -3$ is the vertical asymptote.

(ii) x-intercept: $x = 4.39$ ($= e^2 - 3$)

y-intercept: $y = -0.901$ ($= \ln 3 - 2$)

(c) $f(x) = g(x)$

$x = -1.34$ or $x = 3.05$

(d) (i) See graph

(ii) Area of $A = \int_0^{3.05} \left(4 - (1 - x)^2\right) - (\ln(x + 3) - 2)\, dx$

(iii) Area of $A = 10.6$

(e) $y = f(x) - g(x)$

$y = 5 + 2x - x^2 - \ln(x + 3)$

$\frac{dy}{dx} = 2 - 2x - \frac{1}{x+3}$

Maximum occurs when $\frac{dy}{dx} = 0$

$2 - 2x = \frac{1}{x+3}$

$5 - 4x - 2x^2 = 0$

$x = 0.871$

$y = 4.63$

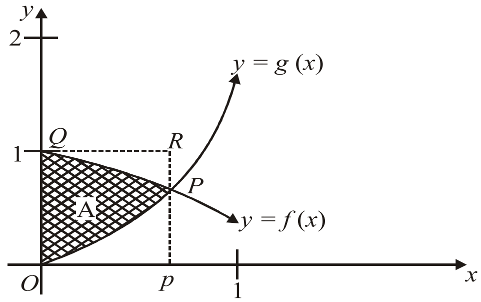
OR

Vertical distance is the difference $f(x) - g(x)$.

Maximum of $f(x) - g(x)$ occurs at $x = 0.871$.

The maximum value is 4.63.

60. (a)



(b) area ΔOPQ < area of region A < area of rectangle $OSRQ$

$\frac{1}{2}(1)(p)$ < area of region A < $(p)(1)$

$\frac{p}{2}$ < area of region A < p

(c) Solving the equation $e^{-p^2} - e^{p^2} + 1 = 0$ using a calculator gives $p = 0.6937$ (4 decimal places)

OR the value of p may be found as follows:

$e^{-p^2} = e^{p^2} - 1 \Rightarrow e^{2p^2} - e^{p^2} - 1 = 0$

$\Rightarrow e^{p^2} = \frac{1 \pm \sqrt{1+4}}{2} \Rightarrow e^{p^2} = \frac{1 \pm \sqrt{5}}{2}$ since $e^{p^2} > 0$

$\Rightarrow$ This gives $p = \sqrt{\ln\left(\frac{1+\sqrt{5}}{2}\right)} \approx 0.6937$ (4 decimal places)

(d) Area of region $A = \int_0^p \left(e^{-x^2} - \left[e^{x^2} - 1\right]\right) dx = 0.467$ (using a GDC)