Lesson 13: Two graphs for each relationship

Goals

- Coordinate (orally and in writing) tables, graphs, and equations that represent the same proportional relationship.
- Interpret two different graphs that represent the same proportional relationship, but have reversed which quantity is represented on each axis.
- Write an equation to represent a proportional relationship given only one pair of values or one point on the graph.

Learning Targets

- I can interpret a graph of a proportional relationship using the situation.
- I can write an equation representing a proportional relationship from a graph.

Lesson Narrative

In this lesson students focus on the relationship between the graph and the equation of a proportional relationship. They start with an activity designed to help them see all the different ways in which the graph and the equation are connected, for example the relation between a point (a, b) on the graph and the constant of proportionality $k = \frac{b}{a}$ in the equation and the fact that the point $(1, k)$ on the graph tells you the constant of proportionality. This prepares them for the next two activities where they see two ways to graph a proportional relationship, depending on which quantity goes on which axis. This connects with previous work with tables and equations, and gives students an opportunity to remember the fact that the constants of proportionality in the two ways are reciprocals.

Addressing

- Recognise and represent proportional relationships between quantities.

Building Towards

- Use properties of operations to generate equivalent expressions.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Collect and Display
- True or False

Required Materials

Rulers

Student Learning Goals

Let's use tables, equations, and graphs to answer questions about proportional relationships.

13.1 True or False: Fractions and Decimals

Warm Up: 5 minutes

This warm-up encourages students to connect and reason algebraically about various computational relationships and patterns from previous exercises. While students may evaluate each side of the equation to determine if it is true or false, encourage students to think about the following ideas in each:

1. The multiplicative relationships between the factors. Multiplying one factor by 2 and dividing the other by 2 on the left side of the equation results in the two factors on the right hand side.
2. In this case, the factors on the left hand side of the equation are adjusted in the same manner as the first equation, however since the operation is division, this strategy results in one side being 4 times the value of the other.
3. This equation applies the same reasoning as the first equation except the factors are adjusted by multiplying and dividing by 4.

Instructional Routines

- True or False

Launch

Display one problem at a time. Tell students to give a signal when they have an answer and a strategy. After each problem, give students 1 minute of quiet think time and follow with a whole-class discussion.

Anticipated Misconceptions

Students may think the same strategies that work for multiplication can be applied to division. For these students ask them for a context to demonstrate what happens when we double a dividend and halve the divisor.

Student Task Statement

Decide whether each equation is true or false. Be prepared to explain your reasoning.

1. $\frac{3}{2} \times 16 = 3 \times 8$

2. $\frac{3}{4} \div \frac{1}{2} = \frac{6}{4} \div \frac{1}{4}$

3. $(2.8) \times (13) = (0.7) \times (52)$

Student Response

1. True. $\frac{3}{2} \times 16 = 2 \times \frac{3}{2} \times \frac{1}{2} \times 16 = 3 \times 8 = 24$

2. False. $\frac{3}{4} \div \frac{1}{2} = \frac{6}{4}$ and $\frac{6}{4} \div \frac{1}{4} = 6$

3. True. $(2.8) \times 13 = (0.7 \times 4) \times 13 = 0.7 \times (4 \times 13) = 0.7 \times (52)$

Activity Synthesis

Ask students to share their strategies for each problem. Record and display their explanations for all to see. Ask students if or how the factors in the problem impacted the strategy choice. To involve more students in the conversation, consider asking:

- "Do you agree or disagree? Why?"
- "Who can restate ___'s reasoning in a different way?"
- "Does anyone want to add on to _____'s strategy?"
- "Will that strategy always work? How do you know?"

After each true equation, ask students if they could rely on the same reasoning to determine if other similar problems are equivalent. After each false equation, ask students how the problem could be changed to make the equation true.

13.2 Tables, Graphs, and Equations

20 minutes (there is a digital version of this activity)

Note: if it is possible in your local environment, we recommend using the digital version of this activity.

This activity is intended to help students identify correspondences between parts of a table, graph, and an equation of a line through the origin in the first quadrant. Students are guided to notice:

- Any pair of positive values (a, b) determine a proportional relationship.
- Given a point (a, b) other than the origin on the graph of a line through the origin, the constant of proportionality is always $\frac{b}{a}$.
- In an equation $y = \frac{b}{a}x$ that represents the relationship, the constant of proportionality appears as the coefficient of x.

- The constant of proportionality is the y-coordinate when x is 1, that is, $(1, \frac{b}{a})$ is a point on the graph.

In the print version, students plot one point and draw a ray that starts at the origin and passes through their point. They also create a table and an equation to represent the relationship. They respond to a series of questions about their representations. Then, they compare their representations of their relationship with the representations of other relationships created by members of their group.

In the digital version (recommended), students interact with a dynamic sketch while recording observations and responding to prompts. In the sketch, students are able to manipulate the graph of a proportional relationship, while changes to an associated table and equation automatically update.

Instructional Routines

- Collect and Display

Launch

For the print version: Arrange students in groups of 3. Assign each student in each group a letter: A, B, or C. Provide access to rulers.

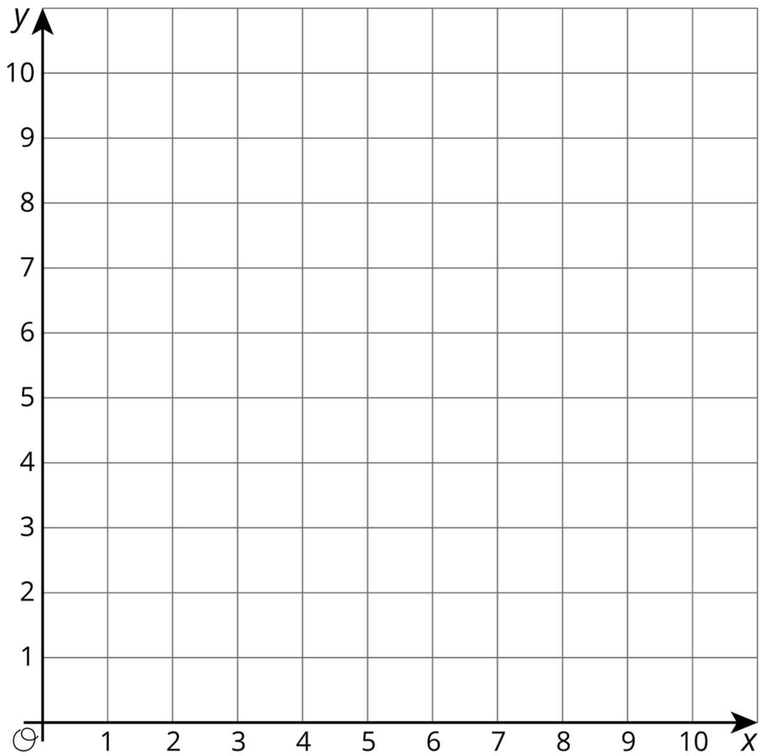
For the digital version: Arrange students in groups of 2–3 and have them complete the task.

Action and Expression: Internalise Executive Functions. Chunk this task into more manageable parts to support students who benefit from support with organisation and problem solving. For example, present one question at a time and monitor students to ensure they are making progress throughout the activity.
Supports accessibility for: Organisation; Attention Speaking, Writing: Collect and Display. While pairs or groups are working, circulate and listen to student talk about the connections they see between the tables, characteristics of the graphs, and the equations. Write down common or important phrases you hear students say about the connections onto a visual display. Throughout the remainder of the lesson, continue to update collected student language and remind students to borrow language from the display as needed. This will help students read and use mathematical language during paired and whole-class discussions.
Design Principle(s): Optimise output (for explanation); Maximise meta-awareness

Student Task Statement

Your teacher will assign you *one* of these three points:

$A = (10,4)$, $B = (4,5)$, $C = (8,5)$.



x	y	$\frac{y}{x}$
0		NA
1		
2		
3		
4		
5		
6		
7		
8		
9		
10		

1. On the graph, plot and label *only* your assigned point.

2. Use a ruler to line up your point with the origin, (0,0). Draw a line that starts at the origin, goes through your point, and continues to the edge of the graph.

3. Complete the table with the coordinates of points on your graph. Use a fraction to represent any value that is not a whole number.

4. Write an equation that represents the relationship between x and y defined by your point.

5. Compare your graph and table with the rest of your group. What is the same and what is different about:

   a. your tables?

   b. your equations?

   c. your graphs?

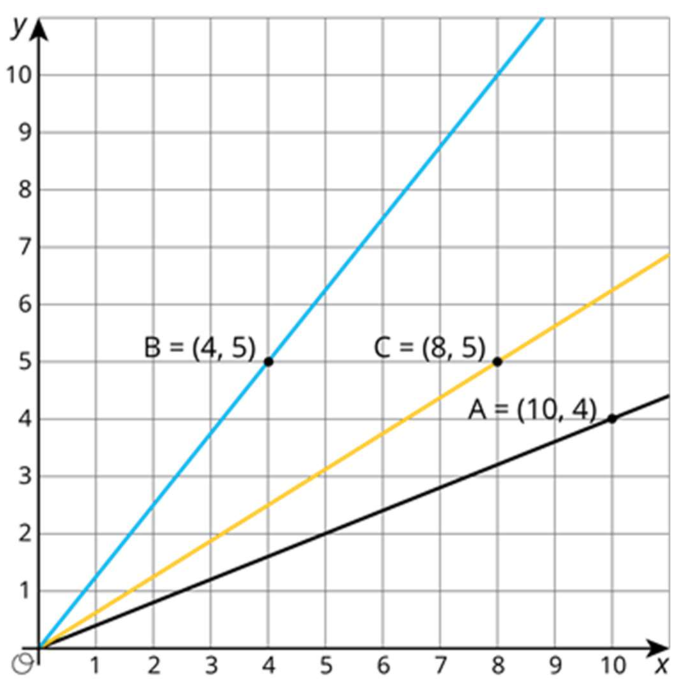
6. What is the y-coordinate of your graph when the x-coordinate is 1? Plot and label this point on your graph. Where do you see this value in the table? Where do you see this value in your equation?

7. Describe any connections you see between the table, characteristics of the graph, and the equation.

Student Response

1. See the graph

2. Students should create one of the following graphs.



3. Tables will differ depending on which point a student has been assigned. Any fraction equivalent to the one shown in the answer is acceptable.

Point A:

x	y	$\frac{y}{x}$
0	0	NA
1	$\frac{2}{5}$	$\frac{2}{5}$
2	$\frac{4}{5}$	$\frac{2}{5}$
3	$\frac{6}{5}$	$\frac{2}{5}$
4	$\frac{8}{5}$	$\frac{2}{5}$
5	2	$\frac{2}{5}$
6	$\frac{12}{5}$	$\frac{2}{5}$
7	$\frac{14}{5}$	$\frac{2}{5}$
8	$\frac{16}{5}$	$\frac{2}{5}$
9	$\frac{18}{5}$	$\frac{2}{5}$
10	4	$\frac{2}{5}$

Point B:

x	y	$\frac{y}{x}$
0	0	NA
1	$\frac{5}{4}$	$\frac{5}{4}$
2	$\frac{10}{4}$	$\frac{5}{4}$
3	$\frac{15}{4}$	$\frac{5}{4}$
4	5	$\frac{5}{4}$
5	$\frac{25}{4}$	$\frac{5}{4}$
6	$\frac{30}{4}$	$\frac{5}{4}$
7	$\frac{35}{4}$	$\frac{5}{4}$
8	10	$\frac{5}{4}$

9	$\frac{45}{4}$	$\frac{5}{4}$
10	$\frac{50}{4}$	$\frac{5}{4}$

Point C:

x	y	$\frac{y}{x}$
0	0	NA
1	$\frac{5}{8}$	$\frac{5}{8}$
2	$\frac{10}{8}$	$\frac{5}{8}$
3	$\frac{15}{8}$	$\frac{5}{8}$
4	$\frac{20}{8}$	$\frac{5}{8}$
5	$\frac{25}{8}$	$\frac{5}{8}$
6	$\frac{30}{8}$	$\frac{5}{8}$
7	$\frac{35}{8}$	$\frac{5}{8}$
8	5	$\frac{5}{8}$
9	$\frac{45}{8}$	$\frac{5}{8}$
10	$\frac{50}{8}$	$\frac{5}{8}$

4. A: $y = \frac{2}{5}x$ or equivalent. B: $y = \frac{5}{4}x$ or equivalent. C: $y = \frac{5}{8}x$ or equivalent.

5. Answers vary. Students may notice:

    - When the y-coordinates are written as fractions, there are consistencies among them, like perhaps they all have the same denominator.
    - In each table, all of the $\frac{y}{x}$ values are equal.
    - All of the graphs are lines through the origin, but they have different steepnesses.
    - The equations all include a y and an x. They all include a different number, but the number corresponds to a value in its table.

6. A: $\frac{2}{5}$ or equivalent. B: $\frac{5}{4}$ or equivalent. C: $\frac{5}{8}$ or equivalent.

7. Answers vary. Connections should be made between the point $(1, k)$ that appears on the graph and in the table and within the equation $y = kx$.

Are You Ready for More?

The graph of an equation of the form $y = kx$, where k is a positive number, is a line through $(0,0)$ and the point $(1, k)$.

1. Name at least one line through $(0,0)$ that cannot be represented by an equation like this.

2. If you could draw the graphs of *all* of the equations of this form in the same coordinate grid, what would it look like?

Student Response

1. The x and y-axes are both examples. Any line through $(0,0)$ and $(1, k)$ where k is negative is also an example.

2. It would look like you completely shaded in the first and third quadrants of the coordinate grid.

Activity Synthesis

When all students have completed the activity, ask them to share their responses with a partner or small group. After a minute, go around the room asking each group to share one thing from the last question, and display these for all to see. When each group has shared, ask if there were any important observations that were missed.

Ensure that all the important connections are highlighted:

- A graph of a line through the origin and passing through the first quadrant represents a proportional relationship.
- The value of $\frac{b}{a}$ computed from any point (a, b) on that line (other than the origin) is the constant of proportionality.
- An equation of the relationship is given by $y = kx$ where k is $\frac{b}{a}$ for any point (a, b) on the graph other than the origin.

13.3 Hot Dog Eating Contest

10 minutes (there is a digital version of this activity)

The purpose of this activity is to help students understand derived units and rates. It is intended to help students see that a proportional relationship between two quantities is associated with two rates. The first rate indicates how many hot dogs someone eats in one minute (number of hot dogs per minute), and the second indicates how many minutes it takes to eat one hot dog (number of minutes per hot dog). The need for care with units will become very clear should any students express rates in seconds per hot dog. If any students make this choice, it can be an opportunity to make connections between different ways of representing the same situation.

Monitor for students who take each approach.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect

Launch

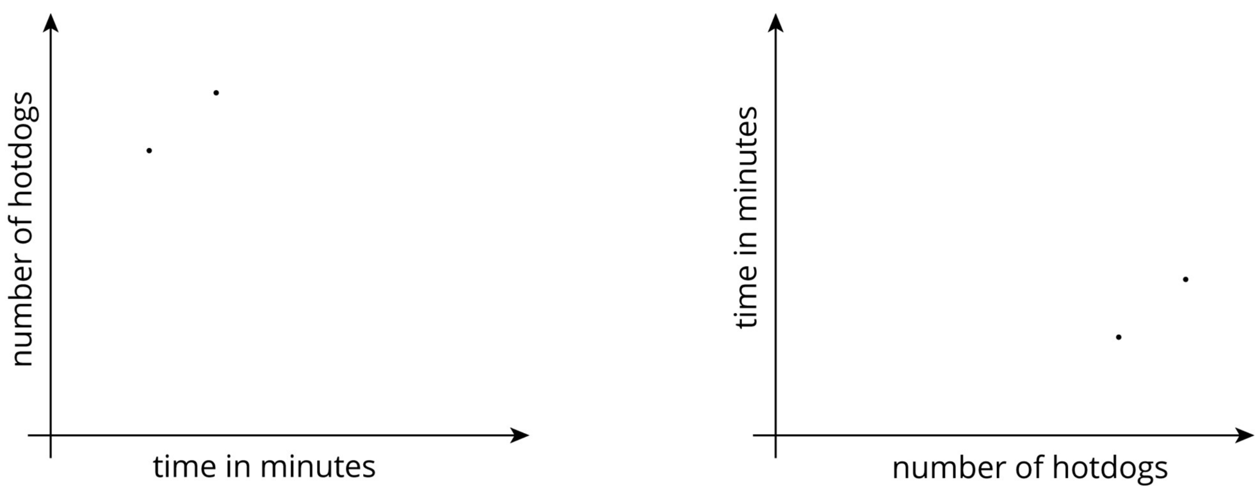
Keep students in the same groups.

Representation: Internalise Comprehension. Demonstrate and encourage students to use colour coding and annotations to highlight connections between representations in a problem. For example, use a different colour for each person to highlight the connection between the graph, equation, and constant of proportionality.
Supports accessibility for: Visual-spatial processing

Student Task Statement

Andre and Jada were in a hot dog eating contest. Andre ate 10 hot dogs in 3 minutes. Jada ate 12 hot dogs in 5 minutes.

Here are two different graphs that both represent this situation.



1. On the first graph, which point shows Andre's consumption and which shows Jada's consumption? Label them.
2. Draw two lines: one through the origin and Andre's point, and one through the origin and Jada's point.
3. Write an equation for each line. Use t to represent time in minutes and h to represent number of hot dogs.
    a. Andre:
    b. Jada:

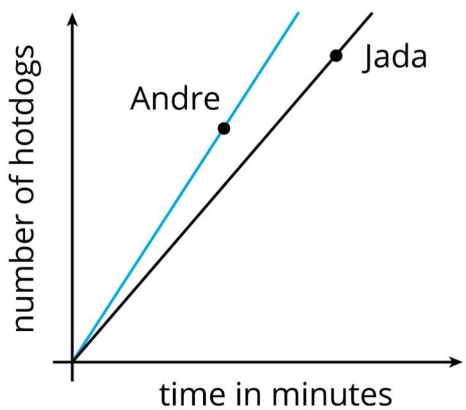
4. For each equation, what does the constant of proportionality tell you?

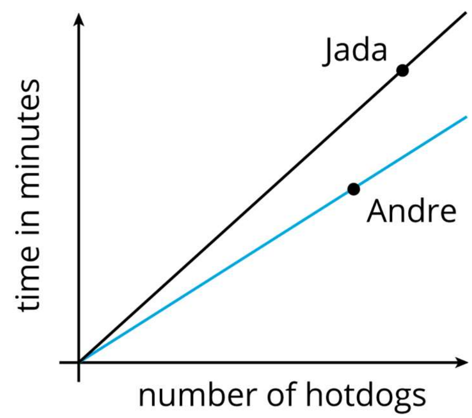
5. Repeat the previous steps for the second graph.

    a. Andre:

    b. Jada:

Student Response





1. Points are labelled.

2. See graph.

3. Andre: $h = \frac{10}{3}t$; Jada: $h = \frac{12}{5}t$ or $h = 2.4t$.

4. Andre eats $\frac{10}{3}$ (or $3\frac{1}{3}$ or approximately 3.33) hot dogs per minute. Jada eats $\frac{12}{5}$ (or $2\frac{2}{5}$ or 2.4) hot dogs per minute.

5. Points are labelled. See graph.

    – Andre: $t = \frac{3}{10}h$ or $t = 0.3h$. Jada: $t = \frac{5}{12}h$.

    – Andre takes $\frac{3}{10}$ or 0.3 minutes per hot dog. Jada takes $\frac{5}{12}$ or approximately 0.42 minutes per hot dog. (Possibly: Andre takes 18 seconds per hot dog and Jada takes 25 seconds per hot dog.)

Activity Synthesis

Select students to share their reasoning. An important point to bring out in the discussion is that we can describe the rate of hot dog eating in two different ways as hot dogs per minute or minutes per hot dog.

Consider asking the following sequence of questions:

- "At what rate did Andre eat hot dogs?" (Some students might say $\frac{10}{3}$ or three and a third, while some say $\frac{3}{10}$ or 0.3.)
- "Well, you're both right, but we need more information to know what you're talking about. Can you be more precise?" (This will prompt students to modify their response and say "$\frac{10}{3}$ or three and a third hot dogs per minute" and "$\frac{3}{10}$ or 0.3 minutes per hot dog.")

Reassure students that either response is correct, as long as units are included. The important thing is that we communicate the meaning of the number clearly. Highlight the fact that $\frac{3}{10}$ and $\frac{10}{3}$ are reciprocals of each other.

Lesson Synthesis

Display the graphs and the corresponding equations from the "Hot Dog Eating Contest" activity. Ask student:

- "Do the graphs and equations tell the same story?"
- "How can you see the same information in both?"

When we have two quantities x and y in a proportional relationship, we have two choices for: writing an equation; making a table; drawing a graph, to represent the relationship. Often the choice is arbitrary and if two people have made different choices, i.e. one views x as proportional to y and the other views y as proportional to x, the representations are related and still provide the same information.

Recall the new units used for the constants of proportionality in the activity: hot dogs per minute, minutes per hot dog. Note that if there is a proportional relationship between two quantities with units A and B, then the associated rates are expressed in As per B and Bs per A.

13.4 Spicy Popcorn

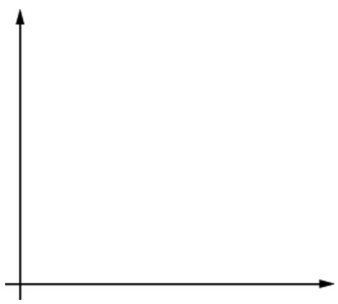
Cool Down: 5 minutes

Student Task Statement

Elena went to a store where you can scoop your own popcorn and buy as much as you want. She bought 10 ounces of spicy popcorn for £2.50.

1. How much does popcorn cost per ounce?
2. How much popcorn can you buy per pound?
3. Write two different equations that represent this situation. Use p for ounces of popcorn and c for cost in pounds.

4. Choose one of your equations, and sketch its graph. Be sure to label the axes.



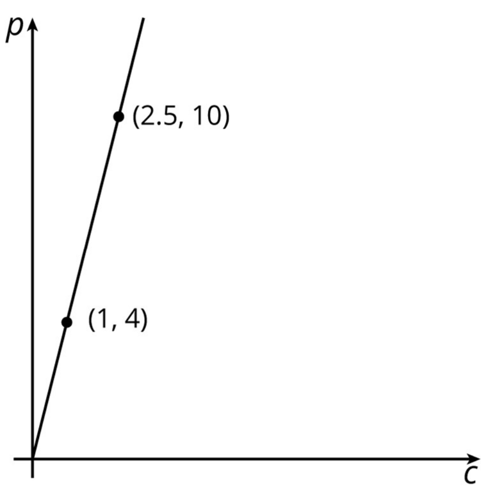
Student Response

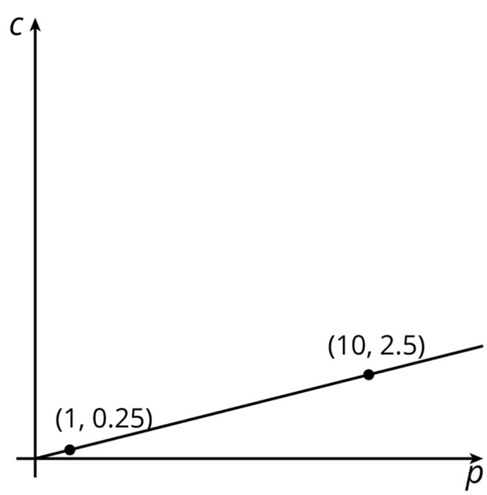
1. £0.25, because $2.50 \div 10 = 0.25$.

2. 4 ounces, because $10 \div 2.5 = 4$.

3. $p = 4c$ or $c = 0.25p$ or equivalent.

4. Students are only asked to create one of these graphs. It is not necessary that they plot and label any points, but it could be a helpful step in creating a reasonably accurate graph.





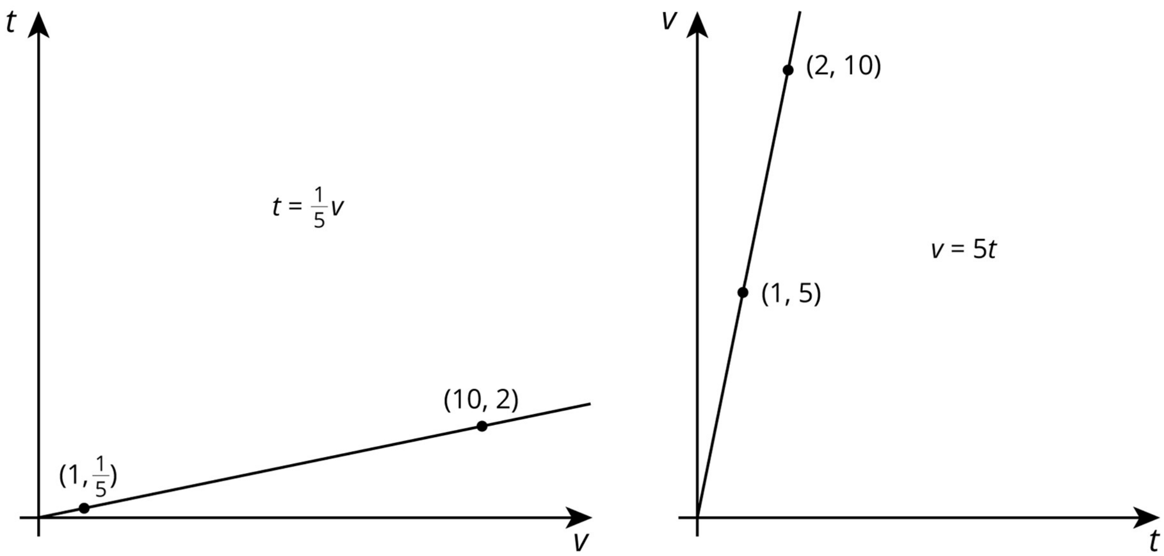
Student Lesson Summary

Imagine that a tap is leaking at a constant rate and that every 2 minutes, 10 millilitres of water leaks from the tap. There is a proportional relationship between the volume of water and elapsed time.

- We could say that the elapsed time is proportional to the volume of water. The corresponding constant of proportionality tells us that the tap is leaking at a rate of $\frac{1}{5}$ of a minute per millilitre.

- We could say that the volume of water is proportional to the elapsed time. The corresponding constant of proportionality tells us that the tap is leaking at a rate of 5 millilitres per minute.

Let's use v to represent volume in millilitres and t to represent time in minutes. Here are graphs and equations that represent both ways of thinking about this relationship:



Even though the relationship between time and volume is the same, we are making a different choice in each case about which variable to view as the independent variable. The graph on the left has v as the independent variable, and the graph on the right has t as the independent variable.

Lesson 13 Practice Problems

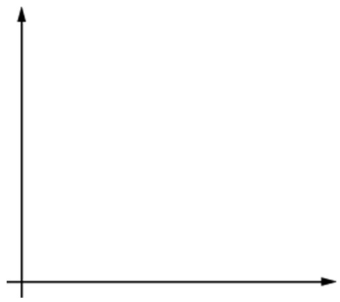
1. **Problem 1 Statement**

   At the supermarket you can fill your own honey bear container. A customer buys 12 oz of honey for £5.40.

   a. How much does honey cost per ounce?

   b. How much honey can you buy per pound?

   c. Write two different equations that represent this situation. Use h for ounces of honey and c for cost in pounds.



– Choose one of your equations, and sketch its graph. Be sure to label the axes.

Solution

a. £0.45 per ounce

b. About 2.2 ounces

c. $c = 0.45h$; $h = 2.2c$

d. Students should have one of two linear graphs going through the origin. Graph 1: $c = 0.45h$, horizontal axis label: h, honey (ounces); vertical axis label c, cost (£); Graph 2: $h = 2.2c$, horizontal axis label: c, cost (£) ; vertical axis label: h, honey (ounces)

2. **Problem 2 Statement**

The point $(3, \frac{6}{5})$ lies on the graph representing a proportional relationship. Which of the following points also lie on the same graph? Select **all** that apply.

a. (1,0.4)

b. $\left(1.5, \frac{6}{10}\right)$

c. $\left(\frac{6}{5}, 3\right)$

d. $\left(4, \frac{11}{5}\right)$

e. (15,6)

Solution ["A", "B", "E"]

3. **Problem 3 Statement**

A trail mix recipe asks for 4 cups of raisins for every 6 cups of peanuts. There is proportional relationship between the amount of raisins, r (cups), and the amount of peanuts, p (cups), in this recipe.

a. Write the equation for the relationship that has constant of proportionality greater than 1. Graph the relationship.

b. Write the equation for the relationship that has constant of proportionality less than 1. Graph the relationship.

Solution

a. $p = \frac{6}{4}r$. Students should have a graph of $p = \frac{6}{4}r$, label horizontal axis r (or "raisins (cups)") and vertical axis p (or "peanuts (cups)"). Since this is a proportional relationship, the graph should be linear and go through the origin.

b. $r = \frac{4}{6}p$. Students should have a graph of $r = \frac{4}{6}p$, label horizontal axis p and vertical axis r. Since this is a proportional relationship, the graph should be linear and go through the origin. The slope of this graph should be less steep than the previous graph.

4. **Problem 4 Statement**

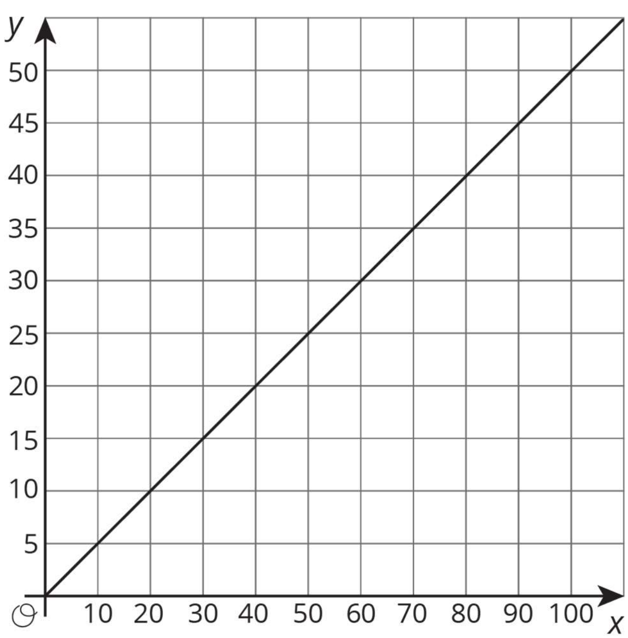
Here is a graph that represents a proportional relationship.

a. Come up with a situation that could be represented by this graph.

b. Label the axes with the quantities in your situation.

c. Give the graph a title.

d. Choose a point on the graph. What do the coordinates represent in your situation?



Solution

Answers vary. Sample response:

a. For every 2 gallons of grey paint created, 1 gallon of black paint is used.

b. Horizontal axis: grey paint (gallons). Vertical axis: black paint (gallons).

c. Title: Amount of Black Paint Needed to Create Grey Paint

d. The point (60,30) means, in order to make 60 gallons of grey paint, 30 gallons of black paint is needed.