Lesson 7: Representations of linear relationships

Goals

- Create an equation that represents a linear relationship.
- Generalise (orally and in writing) a method for calculating gradient based on coordinates of two points.
- Interpret the gradient and y-intercept of the graph of a line in context.

Learning Targets

- I can use patterns to write a linear equation to represent a situation.
- I can write an equation for the relationship between the total volume in a graduated cylinder and the number of objects added to the graduated cylinder.

Lesson Narrative

In this lesson, students develop an equation for a linear relationship by expressing regularity in repeated calculations. In an activity, students measure the volume of water in a graduated cylinder, repeatedly adding objects to the cylinder. Each additional object increases the volume of water by the same amount. They graph the relationship and interpret the initial water volume as the vertical intercept; they also interpret the gradient as the rate of change, that is the amount by which the volume increases when one object is added.

In the second activity, students explicitly formulate a procedure to compute the gradient of a line from *any* two points that lie on the line, including two different general points. This was covered in another unit, in order to calculate and make sense of gradient as well as to find an equation satisfied by all points on a line. In this unit, students have continued to graph lines, draw gradient triangles, and calculate gradient; this time with an emphasis on understanding the gradient as a rate of change: how much vertical displacement is there per unit of horizontal displacement? This lesson continues to focus on positive gradients; in future lessons, students will start to investigate non-positive gradient values.

Building On

- Recognise volume as an attribute of solid figures and understand concepts of volume measurement.

Addressing

- Understand the connections between proportional relationships, lines, and linear equations.
- Use similar triangles to explain why the gradient m is the same between any two distinct points on a non-vertical line in the coordinate grid; derive the equation $y = mx$ for a line through the origin and the equation $y = mx + b$ for a line intercepting the vertical axis at b.

Building Towards

- Use similar triangles to explain why the gradient m is the same between any two distinct points on a non-vertical line in the coordinate grid; derive the equation $y = mx$ for a line through the origin and the equation $y = mx + b$ for a line intercepting the vertical axis at b.

Instructional Routines

- Stronger and Clearer Each Time
- Collect and Display

Required Materials

Rulers

Required Preparation

Students will work in groups of 2–3.

If doing the video presentation of spheres being added to a cylinder, prepare the video for presentation. Video 'Glasses' available here: https://player.vimeo.com/video/304138068.

If doing the 20 minute version of the water level task with an initial demonstration, you will need one graduated cylinder partially filled with water and 10 to 15 identical solid objects that fit into the cylinder and don't float (marbles, dice, cubes, or hardware items such as nuts or bolts).

If doing the 45 minute version of the water level task, each group will need one graduated cylinder partially filled with water. Each group will also need about 15 identical solid objects that fit into the cylinder and don't float. Determine a good initial water level and the approximate volume of each type of equally sized object.

Student Learning Goals

Let's write equations from real situations.

7.1 Estimation: Which Holds More?

Warm Up: 5 minutes

This warm-up prompts students to estimate the volume of different glasses by reasoning about characteristics of their shape. As students discuss their reasoning with a partner, monitor the discussions and identify students who identified important characteristics of each of the glasses in their response.

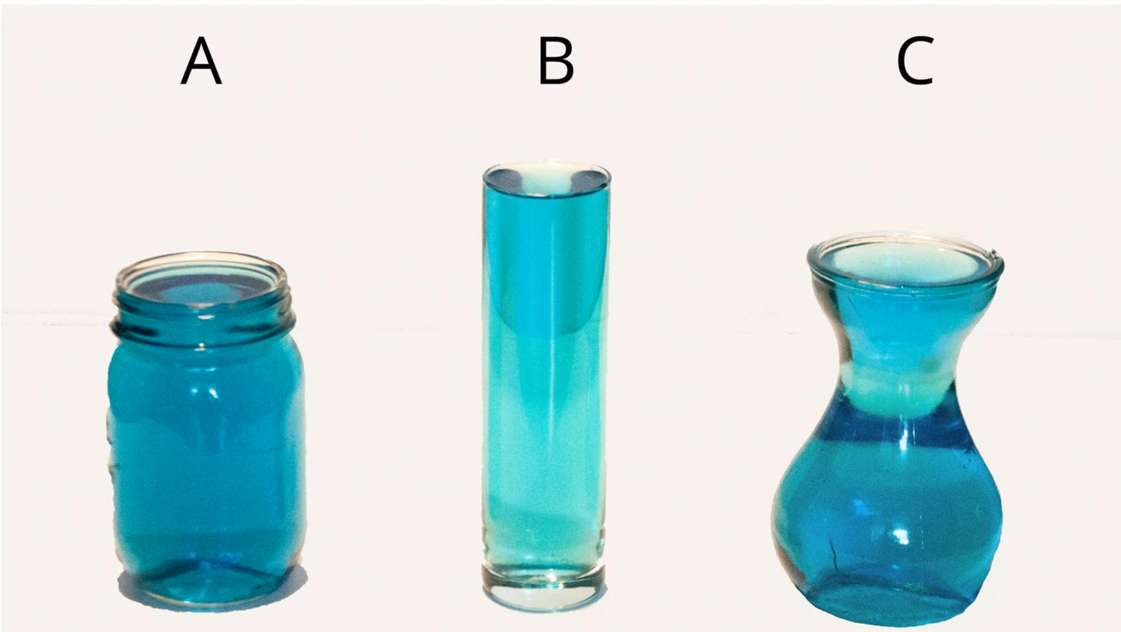
Launch

Arrange students in groups of 2. Tell students they will be estimating which glass would hold the most liquid. Ask students to give a signal when they have an estimate and reasoning. Give students 1 minute of quiet think time followed by 2 minutes to discuss their estimates with a partner. Ask them to discuss the following questions, displayed for all to see:

- "What was important to you in the image when making your decision?"
- "What information would be helpful in finding the answer to this question?"

Student Task Statement

Which glass will hold the most water? The least?



Student Response

Answers vary.

Activity Synthesis

For each glass, ask students to indicate if they think it holds the most or least amount of water. Invite a few students to share their reasoning and the characteristics of the object that were important in making their decision. After each explanation, solicit from the class questions that could help the student clarify his or her reasoning. Record the characteristics, and display them for all to see. If possible, record these characteristics on the images themselves during the discussion.

If there is time, discuss the information they would need to find the answer to the question accurately. Note that no measurements needs to be taken to answer the question; to compare the volume of two containers, it is enough to pour the liquid from one into the other.

It turns out that B holds the least and A and C hold the same. If the right technology is available, consider showing the answer video.

Video 'Glasses' available here: https://player.vimeo.com/video/304138068.

7.2 Rising Water Levels

20 minutes (there is a digital version of this activity)

The goal of this task is to analyse a linear relationship for data gathered in context. Students examine data gathered by successively submerging equal-size objects in a graduated cylinder, partially filled with water, and then measuring the level of the water. Here and in the sample solutions, we use a 100 ml graduated cylinder, 60 ml as the initial amount of water, and marbles whose volume is approximately 3 ml each. Your measurements may be different. You will want to make sure to leave enough space so many marbles can be added before the water reaches the top (but not too much space as the marbles need to all be submerged in the water).

After gathering the data, students plot it and notice that it lies on a line, up to measurement error. They estimate the gradient using gradient triangles and then interpret the meaning of the y-intercept and the gradient in terms of the context. With x marbles and an initial amount of 60 ml in the cylinder, $60 + 3x$ is an expression giving the level of the water:

- The number 60 represents the initial amount of water in the cylinder.
- 3 is the volume per marble in ml (and the rate of change in water level).
- x represents how many marbles have been added to the cylinder.

There are two versions of this task, one for a shorter 20 minute time and one for a longer time (40–45 minutes). The difference is in how the data is gathered. For the shorter time, the teacher performs a demonstration, adding marbles to the cylinder, and then the whole class works with this data. For the second, longer version, students gather their own data.

Note that because of measurement error the data may not lie *exactly* on a line, and different gradient triangles may lead to slightly different values for the gradient. Monitor for students who get slightly different values, and invite them to share during the discussion.

Instructional Routines

- Collect and Display

Launch

Tell students, "Have you ever noticed that when you put ice cubes in your drink, the level of the liquid goes up? Today, we want to investigate what happens when we drop objects into a container with water."

For the 20 minute version, begin with a demonstration, either using the applet with the digital version of the task or a physical demonstration. For a physical demonstration, consider measuring the volume after putting in 1, 2, 5, 8, and 10 marbles. Record the measurements for all to see or choose students to do so. After students have the information for the table, they work in small groups to complete the activity.

For the 45 minute version, distribute materials to each group:

- 1 graduated cylinder.
- 15 identical solid objects that fit into the cylinder and have a higher density than water and don't float (marbles, dice, cubes, hardware items such as nuts or bolts, etc.). Tell students how much water to put initially into their cylinders. Groups spend 10 minutes conducting the experiment. The analysis can be done as a combination of class discussion and student work.

For classrooms without materials to create the lab, there is a digital activity to recreate the experience. The applet might also be useful to show classes before they begin the lab and after they have predictions. Adapted from an applet made in GeoGebra by John Golden.

Arrange students in groups of 2–3.

Action and Expression: Provide Access for Physical Action. Provide access to tools and assistive technologies. If possible, allow students to use the applet for this activity to facilitate observing the impact of adding marbles to the cylinder.
Supports accessibility for: Visual-spatial processing; Conceptual processing; Organisation
Conversing, Representing, Writing: Collect and Display. Use of the longer version of this task, where students gather their own data, will increase opportunities for student discourse. As students collect their data, listen for and collect vocabulary and phrases students use to describe what happens to the water level as they add items. Amplify phrases that relate to volume, rate, gradient, and vertical intercepts. Remind students to borrow language from the display as needed as they complete the follow-up questions. This will help students use mathematical language during their paired discussions and in their written work.
Design Principle(s): Optimise output (for justification); Maximise meta-awareness

Anticipated Misconceptions

Students may think the marks on the cylinder indicate the height of the water. Millilitres are a measure of volume. However, in a cylinder, the height is proportional to the volume, so it does make sense to measure the height. But then the objects will be measured by what height of water they displace, not volume.

Student Task Statement

1. Record data from your teacher's demonstration in the table. (You may not need all the rows.)
2. What is the volume, V, in the cylinder after you add x objects? Explain your reasoning.
3. If you wanted to make the water reach the highest mark on the cylinder, how many objects would you need?

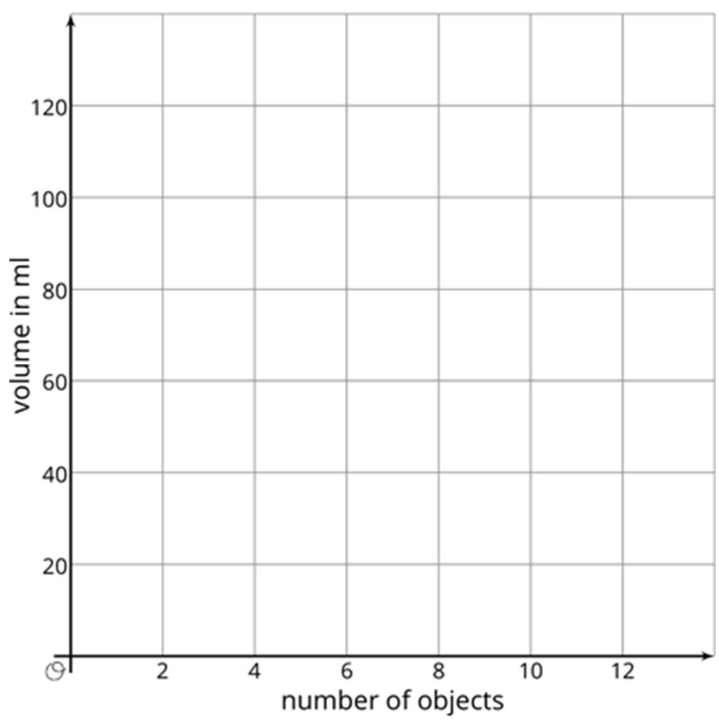
4. Plot and label points that show your measurements from the experiment.

5. The points should fall on a line. Use a ruler to graph this line.

6. Work out the gradient of the line. What does the gradient mean in this situation?

7. What is the vertical intercept? What does vertical intercept mean in this situation?

number of objects	volume in ml



Student Response

1. This narrative uses a 100 ml graduated cylinder, 60 ml as the initial amount of water, and marbles whose volume was approximately 3 ml each. Your measurements may be different.

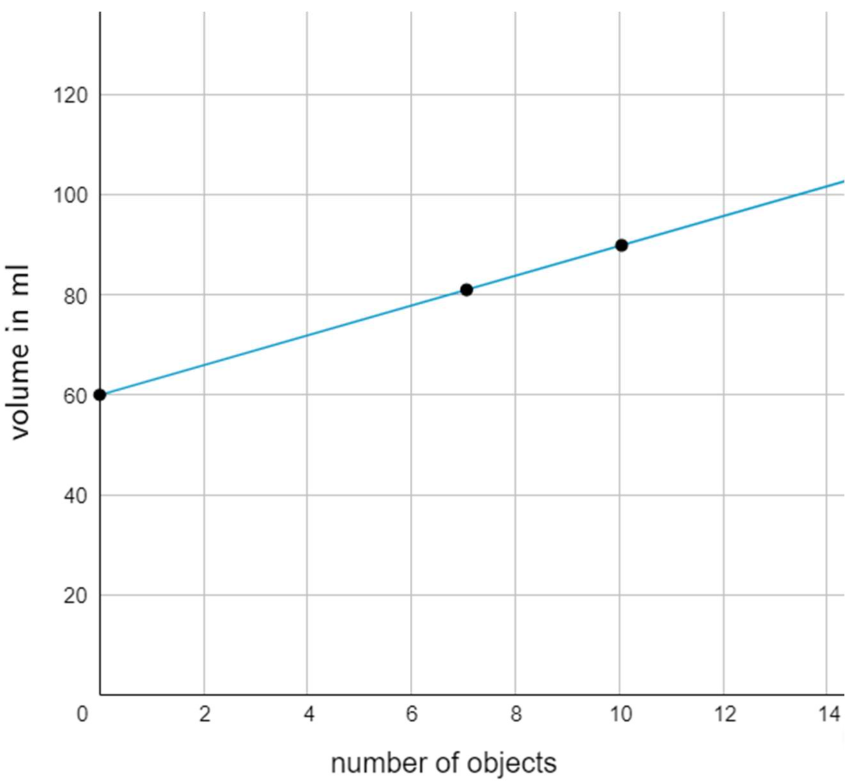
number of objects	volume in ml
3	69

7	81
11	93

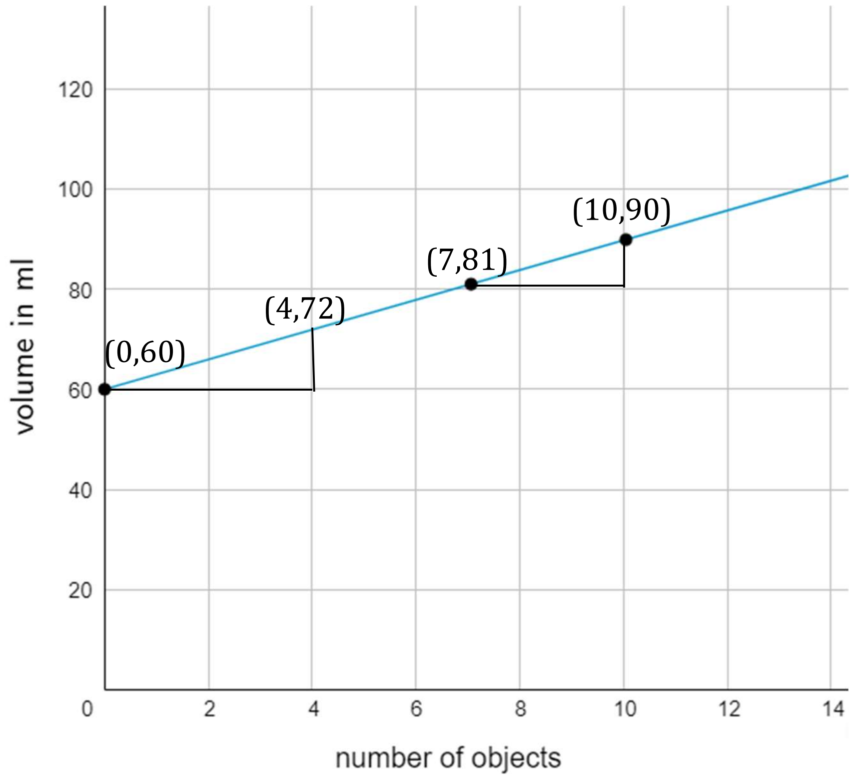
2. $60 + (3) \times x$ ml. Since each marble adds about 3 ml of volume, start with 60 and add 3 ml for each object.

3. With 14 marbles, the water overflows a little bit. With 13 marbles, it is not yet to the top.

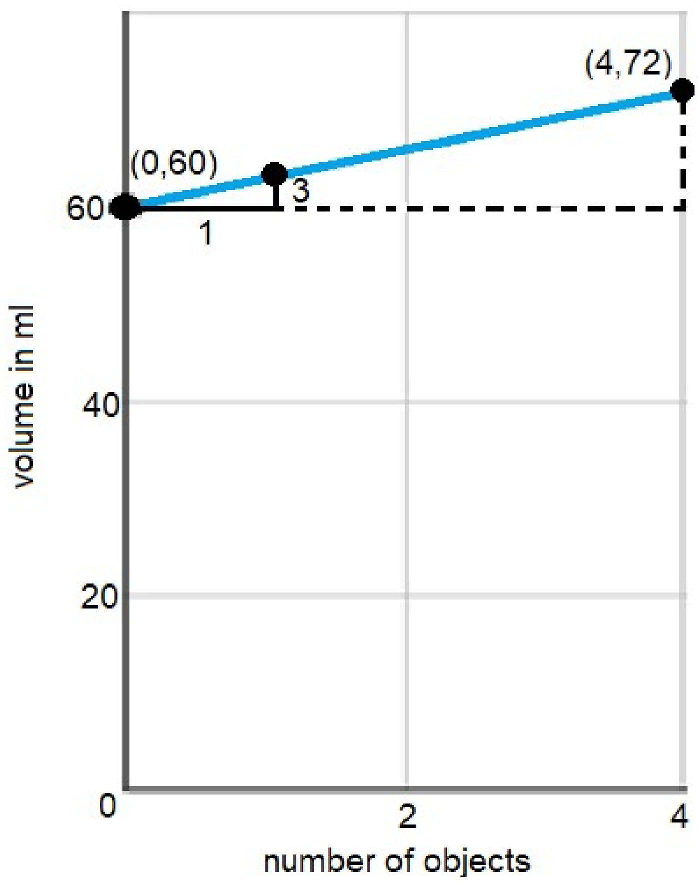
4, 5 Graph:



We can use any two points on the graph to compute the gradient since all gradient triangles are similar and therefore the quotient (vertical side length)/(horizontal side length) is the same for all of them. In particular, for a triangle with horizontal side length of 1, the vertical side length is the rate of change. Each additional object added will increase the volume by the same amount.



120
100
80
60
40
20
0
2
4
6
8
10
12
14
volume in ml
number of objects
(0,60)
(4,72)
(7,81)
(10,90)



(4,72)
(0,60)
3
1
60
40
20
0
2
4
volume in ml
number of objects

The vertical intercept is 60. It means that there were 60 ml of water in the container before any objects were added.

Are You Ready for More?

A situation is represented by the equation $y = 5 + \frac{1}{2}x$.

1. Invent a story for this situation.
2. Graph the equation.
3. What do the $\frac{1}{2}$ and the 5 represent in your situation?
4. Where do you see the $\frac{1}{2}$ and 5 on the graph?

Student Response

Answers vary. Possible response:

1. When you plant a tree in your backyard, it is 5 ft tall. It grows $\frac{1}{2}$ foot every year after that. The total height of the tree is y ft after x years since the tree was planted.
2. A graph showing (x, y) with a vertical intercept of 5, gradient of 0.5.
3. The $\frac{1}{2}$ is the rate of growth of the tree in ft per year, the tree grows $\frac{1}{2}$ ft every year. The 5 is the initial height of the tree; when it was planted, the tree was 5 ft tall.
4. For every one unit increase in x, the line rises by $\frac{1}{2}$ unit in y, so the gradient of the line is $\frac{1}{2}$. The vertical intercept is 5. The line crosses the vertical axis at $(0,5)$, when the tree was planted, i.e., at 0 years, the height was 5 ft.

Activity Synthesis

Display the equation $V = 3x + 60$ for all to see. Note: if you used a different amount of water or objects with a different volume, display the equation corresponding to your class data. Elicit understanding of each part of this equation by asking these questions:

- "How does this equation represent the situation we saw today?" (This is the equation for the line we graphed of the cylinder with the marbles being added.)
- "What do the variables represent?"(V is the total volume in the cylinder in ml. x is the number of objects added to the cylinder.)
- "Where do you see rate of change in this equation? What does it mean in this situation?" (3 is the volume of each object in ml.)

- "What does the number 60 represent?" (60 is the initial amount of water in the cylinder in ml.)

In other words, the equation $V = 3x + 60$ can be interpreted as saying total volume $=$ (3 ml per object) $*$ (number of objects) $+$ initial volume. In terms of the graph, 3 is the gradient and 60 is the vertical intercept.

It might be necessary to discuss the precision of the calculated gradient. It makes sense that every object increases the volume by the same amount (because the marbles, for example, all have the same size). This amount is approximately 3 ml for the marble shown in the pictures. For smaller objects or graduated cylinders with bigger diameters, it may be difficult to accurately measure the change for 3 objects. If so, have students add objects in larger sets. The gradient calculation will tell the amount of change for one object.

Ignoring precision issues, we can use any two points on the graph to compute the gradient since all gradient triangles are similar and therefore (vertical side length)/(horizontal side length) is the same for all of them. In particular, for a triangle with horizontal side length of 1, the vertical side length is the rate of change (3 ml per object for our example)—each additional object added will increase the volume by this same amount.

7.3 Calculate the Gradient

10 minutes

The purpose of this task is to develop a quick method or formula to calculate the gradient of a line based on the coordinates of two points. It is not critical or even recommended to use the traditional formula invoking subscripts. The goal is for students to realise that they can calculate the vertical and horizontal side lengths of the gradient triangle without a grid (eventually without even drawing the gradient triangle) and that the gradient is the quotient of these side lengths.

After students compute the gradients of three specific lines they generalise the procedure. Students are asked to express regularity in repeated reasoning to both describe a procedure in words and then as an algebraic expression. It is more important that students know a technique or way of thinking about it that works for them than it is that they memorise a particular way to express a formula.

Instructional Routines

- Stronger and Clearer Each Time

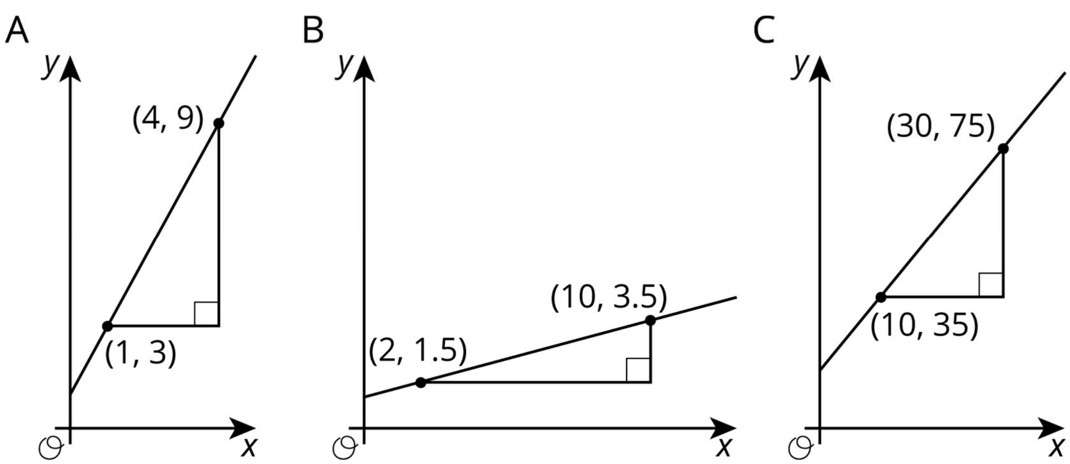
Launch

Quiet work time followed by a class discussion.

Representation: Internalise Comprehension. Encourage and support opportunities for peer interactions. After students complete the table, invite them to compare solutions as well as their procedure for finding gradient with a partner. Once students agree on procedure, they can write it down and continue with the task.

Supports accessibility for: Conceptual processing; Social-emotional skills Writing, Speaking: Stronger and Clearer Each Time. Use this routine to give students a structured opportunity to revise and refine their response to "Describe a procedure for finding the gradient between any two points on a line." Ask each student to meet with 2–3 other partners in a row for feedback. Provide listeners with prompts for feedback that will help students strengthen their ideas and clarify their language (e.g., "Can you explain how...?" and "Will your procedure always work?", etc.). Students can borrow ideas and language from each partner to strengthen the final product. This will help students use more precise language and solidify their process for finding the gradient between any two points on a line.
Design Principle(s): Optimise output (for generalisation)

Student Task Statement

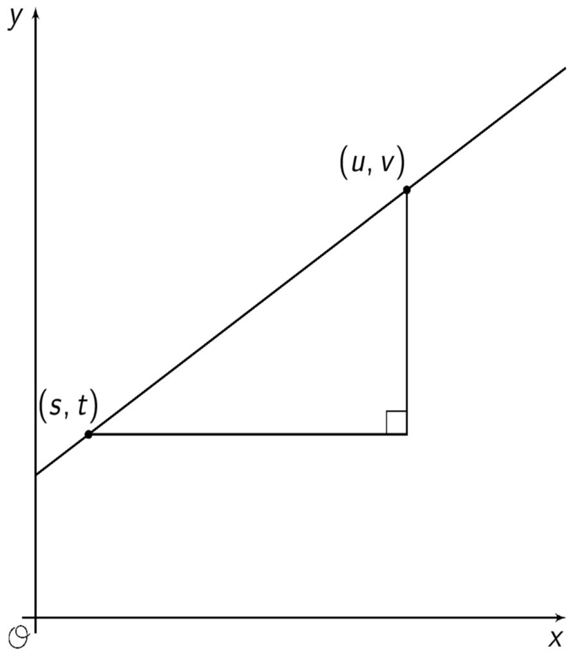


1. For each graph, record:

vertical change	horizontal change	gradient

2. Describe a procedure for finding the gradient between any two points on a line.

3. Write an expression for the gradient of the line in the graph using the letters u, v, s, and t.



Student Response

vertical change	horizontal change	gradient
6	3	2
2	8	$\frac{1}{4}$
40	20	2

1. Divide the vertical change between the two points by the horizontal change between the two points.

2. In general: gradient $= \frac{\text{vertical change}}{\text{horizontal change}} = \frac{v-t}{u-s}$

Activity Synthesis

Ask students to share solutions to the first three problems, and ensure that everyone understands why the correct answers are correct. There is no requirement that students simplify fractions; a student who comes up with $\frac{40}{20}$ for graph C is correct. The convention of simplifying a fraction can be especially helpful if further calculations need to be made (or in this case, to give an immediate sense of the size of the number).

Invite students to share the procedure they came up with. Ideally, they will share several versions of "Subtract the y-coordinates, subtract the x-coordinates, and then divide the difference in y's by the difference in x's." Note that it is important to subtract the x-coordinates for the two points *in the same order* as the y-coordinates, that is $\frac{9-3}{4-1}$ for graph A, not $\frac{9-3}{1-4}$.

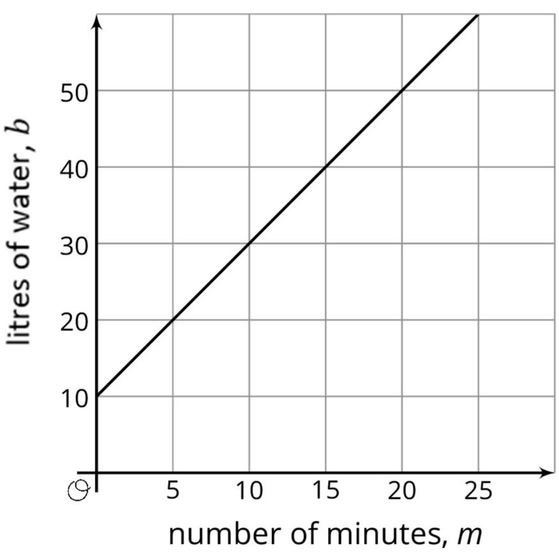
For the last question, invite students to share the expressions they came up with. Acknowledge any response that is equivalent to the correct answer, but be on the lookout for expressions like $v - t \div u - s$, or $(v - t) \div (s - u)$, which are incorrect for different reasons.

Lesson Synthesis

Ask students to create and interpret an equation for a situation with linear growth. For example, imagine you have a bucket of water that already contains 10 litres of water and you turn on the water tap, which adds 2 litres of water every minute. Ask students:

- "Can we use a linear equation to represent this situation?" (Yes.)
- "Why or why not?" (Each minute, 2 more litres of water are added to the bucket; the rate of change is constant.)
- "What is the equation?" ($b = 10 + 2m$ where b is the number of litres of water in the bucket and m is the number of minutes since you turned on the tap.)

Sketch a graph of the line $b = 10 + 2m$.



Ask students the meaning of 10 from the equation and where they see it on the graph. (It's the y-intercept, and it is how many litres of water were in the bucket at the beginning.) Ask the meaning of 2 from the equation and where they see it on the graph. (It's the gradient. A gradient triangle with horizontal side length 1 will have vertical side length 2.)

We can find the gradient of a line using *any* two points on the line. We use the coordinates of the two points to find the vertical and horizontal side lengths of the gradient triangle. The gradient is the quotient of the vertical and horizontal lengths. Label two general points (a, b) and (x, y) on the triangle. The vertical side has length $y - b$, and the horizontal side has length $x - a$ so $\frac{y-b}{x-a} = 2$.

7.4 Graphing a Line

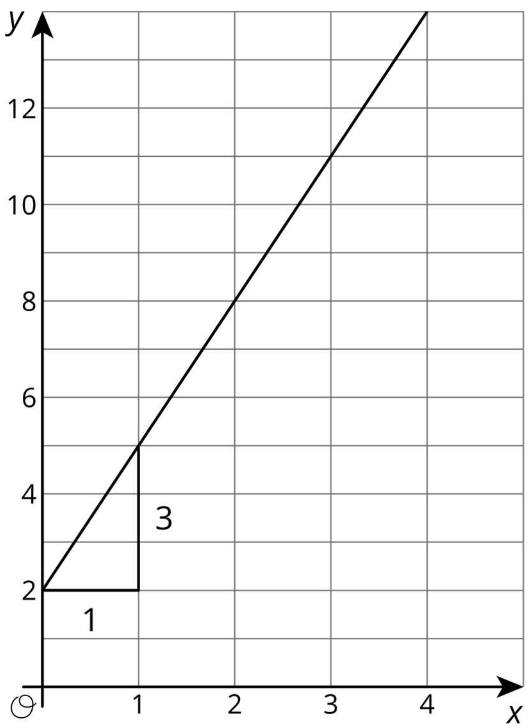
Cool Down: 5 minutes

Student Task Statement

Make a sketch of a linear relationship with gradient of 3 that is not a proportional relationship. Show how you know that the gradient is 3. Write an equation for the line.

Student Response

Answers vary. Sample response:



The gradient triangle has vertical side length 3 and horizontal side length 1, so the gradient of the line is 3.

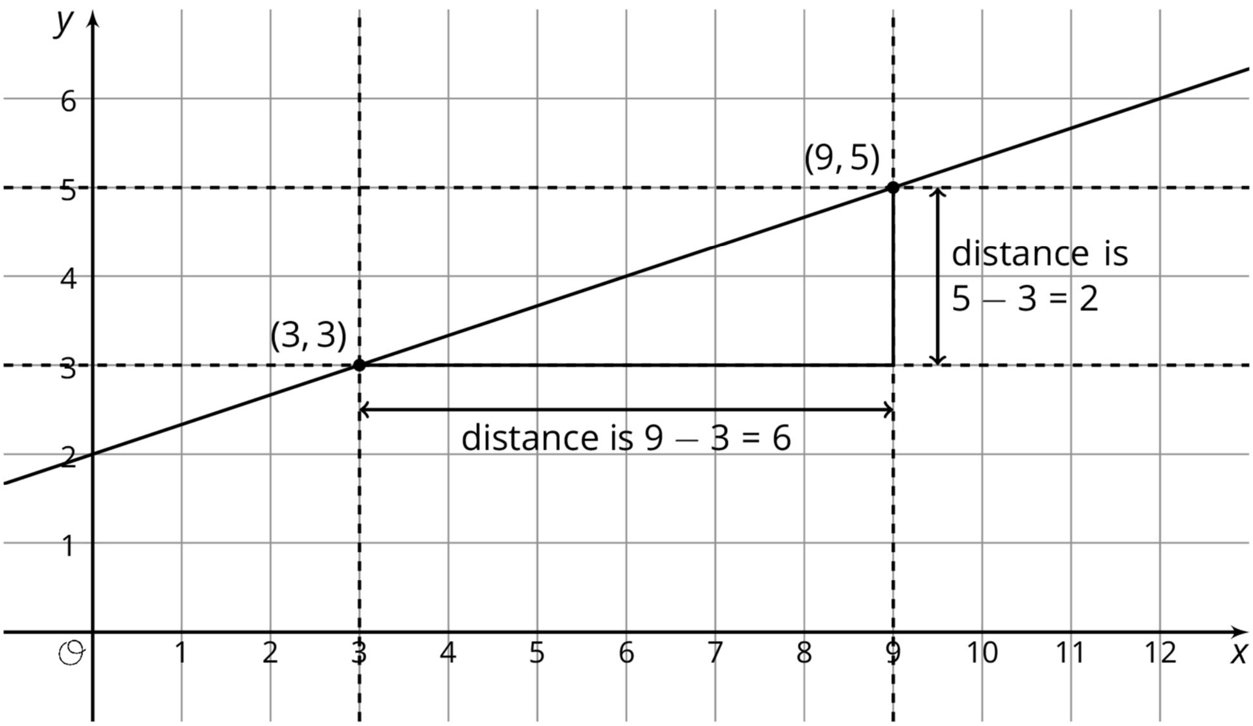
$y = 3x + 2$

Student Lesson Summary

Let's say we have a glass cylinder filled with 50 ml of water and a bunch of marbles that are 3 ml in volume. If we drop marbles into the cylinder one at a time, we can watch the height of the water increase by the same amount, 3 ml, for each one added. This constant rate of change means there is a linear relationship between the number of marbles and the height of the water. Add one marble, the water height goes up 3 ml. Add 2 marbles, the water height goes up 6 ml. Add x marbles, the water height goes up $3x$ ml.

Reasoning this way, we can calculate that the height, y, of the water for x marbles is $y = 3x + 50$. Any linear relationships can be expressed in the form $y = mx + b$ using just the rate of change, m, and the initial amount, b. The 3 represents the rate of change, or gradient of the graph, and the 50 represents the initial amount, or vertical intercept of the graph. We'll learn about some more ways to think about this equation in future lessons.

Now what if we didn't have a description to use to figure out the gradient and the vertical intercept? That's okay so long as we can find some points on the line! For the line graphed here, two of the points on the line are (3,3) and (9,5) and we can use these points to draw in a gradient triangle as shown:



The gradient of this line is the quotient of the length of the vertical side of the gradient triangle and the length of the horizontal side of the gradient triangle. So the gradient, m, is $\frac{\text{vertical change}}{\text{horizontal change}} = \frac{2}{6} = \frac{1}{3}$. We can also see from the graph that the vertical intercept, b, is 2. Putting these together, we can say that the equation for this line is $y = \frac{1}{3}x + 2$.

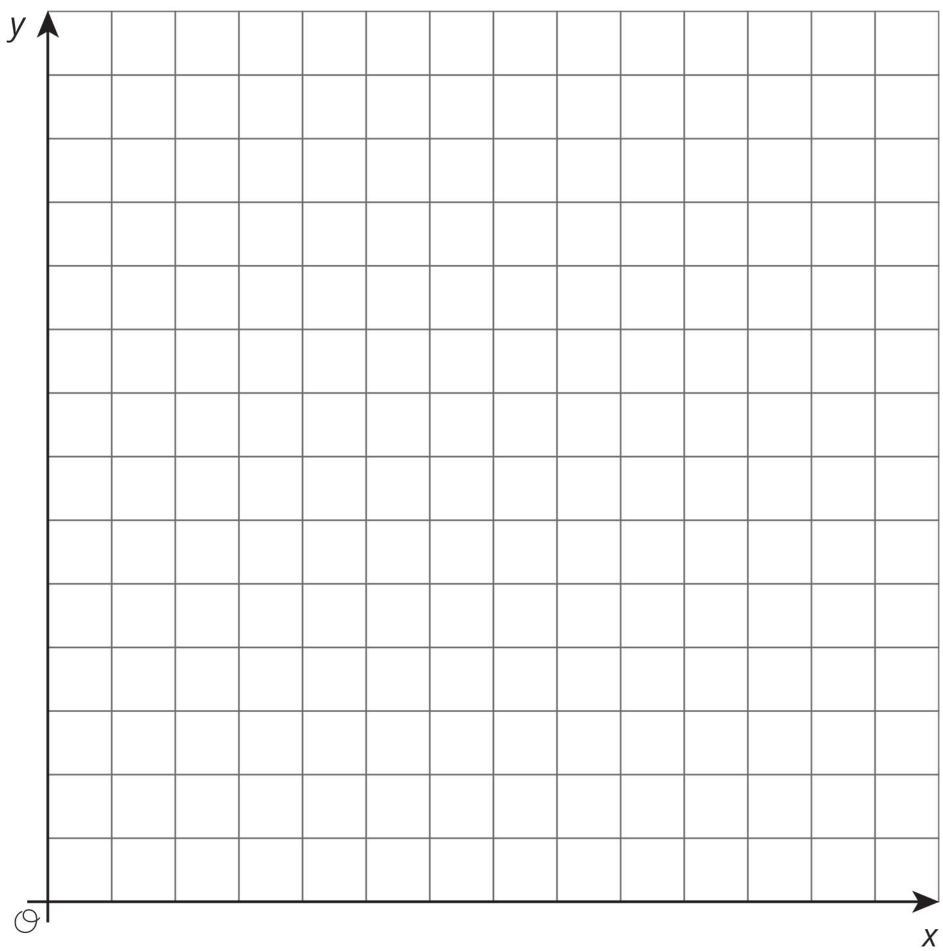
Lesson 7 Practice Problems

1. **Problem 1 Statement**

   Create a graph that shows three linear relationships with different y-intercepts using the following gradients, and write an equation for each line.

   Gradients:

   - $\frac{1}{5}$
   - $\frac{3}{5}$
   - $\frac{6}{5}$



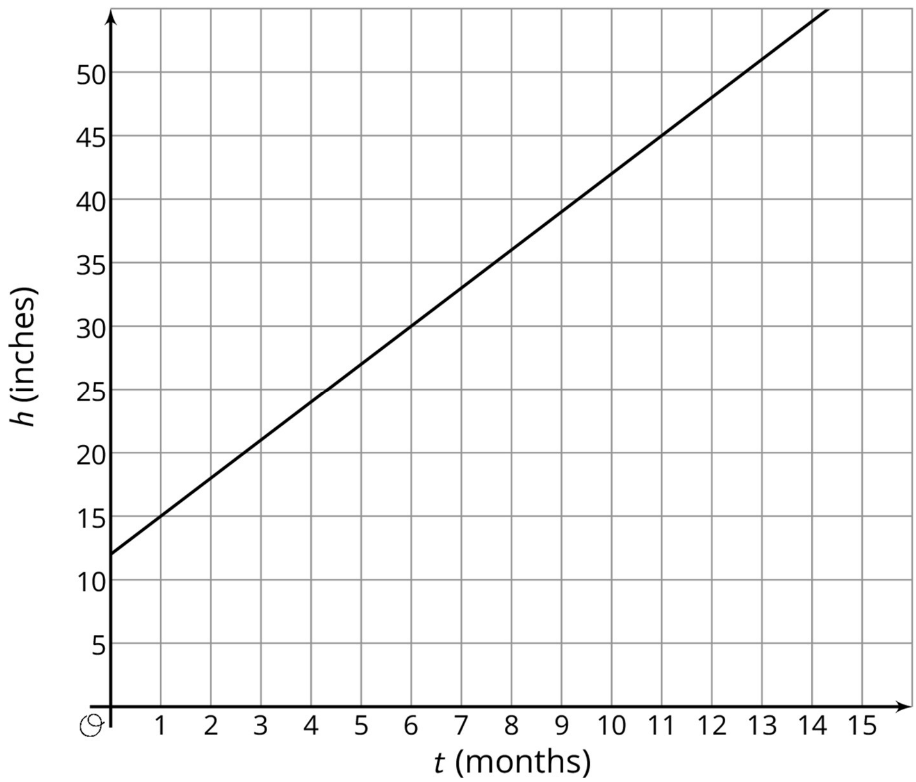
Solution

Answers vary. Sample response: Three graphs, one a line through (0,2) and (5,3), one a line through (0,3) and (5,6), and one a line through (0,4) and (5,10).

$y = \frac{1}{5}x + 2, y = \frac{3}{5}x + 3, y = \frac{6}{5}x + 4$

2. **Problem 2 Statement**

The graph shows the height in inches, h, of a bamboo plant t months after it has been planted.



a. Write an equation that describes the relationship between h and t.

b. After how many months will the bamboo plant be 66 inches tall? Explain or show your reasoning.

Solution

a. $h = 3t + 12$

b. 18 months. Explanations vary. Sample response: Substitute $h = 66$, and solve the equation $66 = 3t + 12$. $3t = 54$, $t = 18$.

3. **Problem 3 Statement**

Here are recipes for two different banana cakes. Information for the first recipe is shown in the table.

sugar (cups)	flour (cups)
$\frac{1}{2}$	$\frac{3}{4}$
$2\frac{1}{2}$	$3\frac{3}{4}$
3	$4\frac{1}{2}$

The relationship between cups of flour y and cups of sugar x in the second recipe is $y = \frac{7}{4}x$

a. If you used 4 cups of sugar, how much flour does each recipe need?

b. What is the constant of proportionality for each situation and what does it mean?

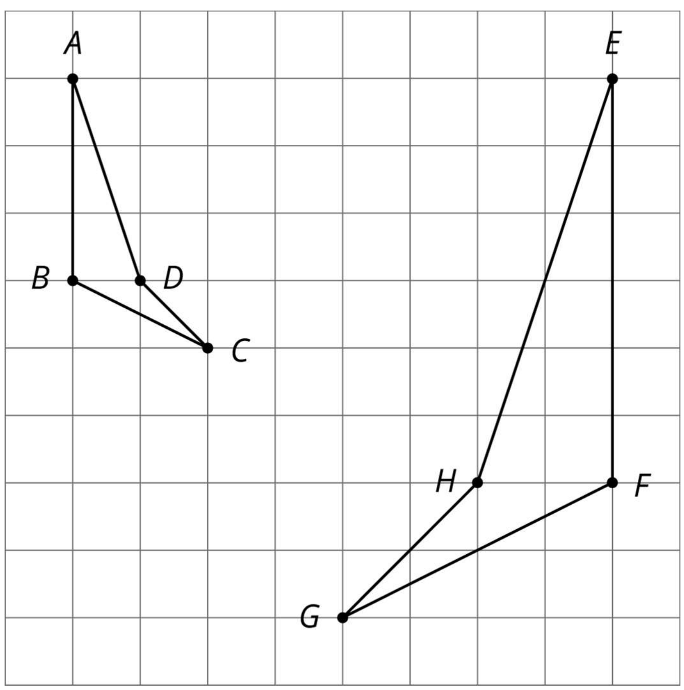
Solution

a. First: 6 cups, second: 7 cups

b. First: $1\frac{1}{2}$ cups of flour per cup of sugar, second: $1\frac{3}{4}$ cups of flour per cup of sugar

4. **Problem 4 Statement**

Show that the two figures are similar by identifying a sequence of translations, rotations, reflections, and enlargements that takes the larger figure to the smaller one.



Solution

Translate H to D (5 units left, 3 units up), reflect across a vertical line through D, and then enlarge using a scale factor of $\frac{1}{2}$ centred at D.