Lesson 2: Introducing proportional relationships with tables

Goals

- Comprehend that the phrase "proportional relationship" (in spoken and written language) refers to when two quantities are related by multiplying by a "constant of proportionality."
- Describe (orally and in writing) relationships between rows or between columns in a table that represents a proportional relationship.
- Explain (orally) how to calculate missing values in a table that represents a proportional relationship.

Learning Targets

- I can use a table to reason about two quantities that are in a proportional relationship.
- I understand the terms proportional relationship and constant of proportionality.

Lesson Narrative

The purpose of this lesson is to introduce the concept of a proportional relationship by looking at tables of equivalent ratios. Students learn that all entries in one column of the table can be obtained by multiplying entries in the other column by the *same* number. This number is called the constant of proportionality. The activities use contexts that make using the constant of proportionality the more convenient approach, rather than reasoning about equivalent ratios.

In any proportional relationship between two quantities x and y, there are two ways of viewing the relationship; y is proportional to x, or x is proportional to y. For example, the two tables below represent the same relationship between time elapsed and distance travelled for someone running at a constant rate. The first table shows that distance is proportional to time, with constant of proportionality 6, and the second table, representing the same information, shows that time is proportional to distance, with constant of proportionality $\frac{1}{6}$.

time (h)	distance (mi)	constant of proportionality
2	12	6
1	6	6
$\frac{1}{2}$	3	6
distance (mi)	time (h)	constant of proportionality
12	2	$\frac{1}{6}$

6	1	$\frac{1}{6}$
3	$\frac{1}{2}$	$\frac{1}{6}$

These tables illustrate the convention that when we say "y is proportional to x" we usually put x in the left hand column and y in the right hand column, so that multiplication by the constant of proportionality always goes from left to right. This is not a hard and fast rule, but it prepares students for later work on functions, where they will think of x as the independent variable and y as the dependent variable.

Building On

- Use ratio and rate reasoning to solve real-world and mathematical problems, e.g., by reasoning about tables of equivalent ratios, bar models, double number line diagrams, or equations.

Addressing

- Recognise and represent proportional relationships between quantities.
- Decide whether two quantities are in a proportional relationship, e.g., by testing for equivalent ratios in a table or graphing on a coordinate plane and observing whether the graph is a straight line through the origin.
- Identify the constant of proportionality (unit rate) in tables, graphs, equations, diagrams, and verbal descriptions of proportional relationships.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Three Reads
- Discussion Supports
- Notice and Wonder

Required Materials

Measuring cup
Measuring spoons

Required Preparation

A measuring cup and a tablespoon are optional—they may be handy for showing students who are unfamiliar with these kitchen tools.

Student Learning Goals

Let's solve problems involving proportional relationships using tables.

2.1 Notice and Wonder: Paper Towels by the Case

Warm Up: 5 minutes

The purpose of this warm-up is to elicit the idea that you can use a table to see patterns between related quantities, which will be useful when students discuss how to use tables to learn about proportional relationships in a later activity. While students may notice and wonder many things about these images, the relationship between the rows (multiply both entries by the same number to get another row) and the relationship between the columns (multiply the number of cases by 12 to get the number of paper towels) are the important discussion points.

Instructional Routines

- Notice and Wonder

Launch

Arrange students in groups of 2. Tell students that they will look at an image, and their job is to think of at least one thing they notice and at least one thing they wonder. Display the image for all to see. Ask students to give a signal when they have noticed or wondered about something. Give students 1 minute of quiet think time, and then 1 minute to discuss the things they notice with their partner, followed by a whole-class discussion.

Student Task Statement

Here is a table that shows how many rolls of paper towels a store receives when they order different numbers of cases.

number of cases they order	number of rolls of paper towels
1	12
3	36
5	60
10	120

× 2 × 2

What do you notice about the table? What do you wonder?

Student Response

Answers vary.

Some things to notice:

- To go from one row to another, multiply both columns by the same number.

- To find the number of rolls, multiply the number of cases by 12.
- There are 12 rolls in a case.

Some things to wonder:

- How much does a case cost?
- How many paper towels are on a roll?
- Why would you need 120 rolls of paper towels?

Activity Synthesis

Ask students to share the things they noticed and wondered. Record and display their responses for all to see. If possible, record the relevant reasoning on or near the image. After each response, ask the class if they agree or disagree and to explain alternative ways of thinking, referring back to the images each time. If the relationship between the number of cases and the number of paper towels does not come up during the conversation, ask students to discuss this idea.

2.2 Feeding a Crowd

15 minutes

The purpose of this task is to introduce students to the idea of a proportional relationship. From previous work, students should be familiar with the idea of equivalent ratios, and they may very well recognise the table as a set of equivalent ratios. Here, we are starting to expand this concept and the language associated with it to say that there is a proportional relationship between the cups of rice and number of people as well as the number of spring rolls and number of people. More generally, there is a proportional relationship between two quantities when the quantities are characterised by a set of equivalent ratios.

The context in this activity and the numbers used are intended to be accessible to all students so that they can focus on its mathematical structure and the new terms introduced in the lesson without being distracted.

While students are working, monitor for groups using each of these approaches:

- Drawing that depicts 15 cups of rice and 45 people, organised into three people per cup. This representation of the problem can support all learners in moving forward. The teacher should highlight the organisation of three people per cup, and identify correspondences between parts of the drawing and numbers in the table.
- Moving down the table, multiplying both numbers in a row by the same number. “Since I multiply 2 by 5 to get 10, I will multiply 6 by 5 and get 30. Since 9 times 5 is 45, I will also multiply 3 by 5 to get 15.” If it comes up, the most appropriate name for the multiplier would be *scale factor*, but it is not necessary for students to call it this.

- Calculation and use of a unit rate. “I reasoned that 1 cup of rice must serve 3 people. So 10 cups of rice must serve 30, and 15 cups must serve 45.”
- Moving across the table. “It looks like we always multiply number of cups of rice by 3 to get the number of people it serves, so I will multiply 10 cups by 3 to get 30, and divide the 45 by 3 to get 15.”

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect

Launch

Say, “Rice is a big part of the traditions and cultures of many families. Does your family cook rice, and if so, how?" Invite a student to describe the process (measure rice, measure water, simmer for a while). You use more rice for more people and less rice for fewer people. If students have trouble understanding or representing the context, show them the measuring cup so that they have a sense of its size, or draw a literal diagram that looks something like this:

Similarly, ask students if they have ever eaten a spring roll and invite them to describe what they are. While some spring rolls can be very large, the ones referred to in this activity are smaller.

Engagement: Develop Effort and Persistence. Encourage and support opportunities for peer interactions. Invite students to talk about their ideas with a partner before writing them down. Display sentence frames to support students when they explain their strategy. For example, “First, I _____ because _____. Then, I….,” “I noticed _____ so I….,” and “I tried _____ and what happened was….”
Supports accessibility for: Language; Social-emotional skills

Student Task Statement

1. A recipe says that 2 cups of dry rice will serve 6 people. Complete the table as you answer the questions. Be prepared to explain your reasoning.

    a. How many people will 10 cups of rice serve?

    b. How many cups of rice are needed to serve 45 people?

cups of rice	number of people
2	6
3	9
10	
	45

2. A recipe says that 6 spring rolls will serve 3 people. Complete the table.

number of spring rolls	number of people
6	3
30	
40	
	28

Student Response

a. 30

b. 15

cups of rice	number of people
2	6
3	9
10	30
15	45

number of spring rolls	number of people
6	3
30	15
40	20
56	28

Activity Synthesis

Select students to share their approaches using the sequence listed in the Activity Narrative.

After students have shared their reasoning and connections have been drawn between the different approaches, introduce the term **proportional relationship**. For example, "Whenever we have a situation like this where two quantities are always in the same ratio, we say there is a proportional relationship between the quantities. So the relationship between the number of cups of rice and the number of people is a proportional relationship. The number of cups of rice is proportional to the number of people." Write out some of these statements for all to see (in the next activity, students will need to write a statement like these about a different relationship). If the class has a word wall or students keep track of mathematical vocabulary in their notebooks, these new terms can be included there.

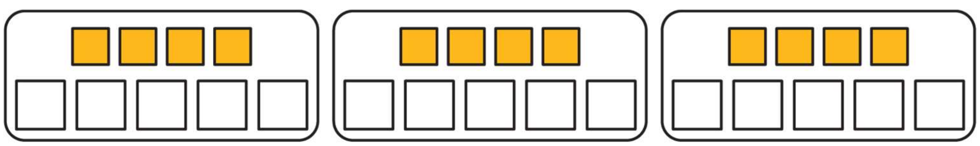
Notice that methods 2 and 3 involve identifying relationships of column entries, namely that the entry in the right-hand column is three times the corresponding entry in the left-hand column. This pattern can be explained in terms of the *unit rate* (people per cup of rice), which tells us how many people we can serve with a given amount of rice. The next activity is intended to build the understanding underlying this explanation.

After discussing the rice context thoroughly, ask students to share their solution approaches to the spring roll context. Ask students to identify and interpret the unit rate in this situation (0.5 people per spring roll may sound strange, but it means that 1 spring roll only halfway satisfies a person.)

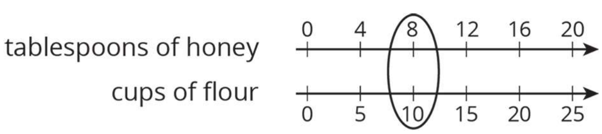
2.3 Making Bread Dough

10 minutes

In this activity, students grapple with finding missing values for ratios of whole numbers presented in a table where identifying a usable scale factor is not as easy as in the previous activity. This task is designed to encourage students to use a unit rate. Its context is intended to be familiar so that students can focus on mathematical structure and the new terms constant of proportionality and proportional relationship. If students are having difficulty understanding the scenario, consider drawing discrete diagrams like this:



This can be followed by a double number line diagram. Correspondences among the diagrams can be identified.



Anticipated approaches include use of scale factors, unit rate, and moving across columns. Because 13 is not a multiple of 4 or 8, students are more likely to use and see the value of using the unit rate or the relationship of the columns. Unit rates might be used: $\frac{5}{4}$ or 1.25 cups of flour per tablespoon of honey, or $\frac{4}{5}$ or 0.8 tablespoon of honey per cup of flour. Both approaches are correct, but the numbers are easier for the first approach, so this is the solution to have students share. While students are working, monitor the approaches used by each student or group.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Three Reads

Launch

Tell students that in this activity, they will think about a different proportional relationship. If necessary, show students the measuring cup and the tablespoon side by side to help make the context more concrete. You could even mime the first sentence in the activity: "measuring" 8 tablespoons of invisible honey and 10 cups of invisible flour.

Reading: Three Reads. In this routine, students are supported in reading a mathematical text, situation, or word problem three times, each with a particular focus. During the first read, students focus on comprehending the situation; during the second read, students identify quantities; during the third read, students brainstorm possible strategies to answer the question. The question to be answered does not become a focus until the third read so that students can make sense of the whole context before rushing to a solution. The purpose of this routine is to support students' reading comprehension as they make sense of mathematical situations and information through conversation with a partner.
Design Principle(s): Support sense-making

How It Happens:

1. In the first read, students read the problem with the goal of comprehending the situation.

   Invite a student to read the problem aloud while everyone else reads with them and then ask, "What is this situation about?"

   Allow one minute to discuss with a partner, and then share with the whole class. A clear response would be: "A bakery uses a recipe to make bread. The recipe includes honey and flour."

2. In the second read, students analyse the mathematical structure of the story by naming quantities.

   Invite students to read the problem aloud with their partner or select a different student to read to the class and then prompt students by asking, "What can be counted or measured in this situation? For now we don't need to focus on how many or how much of anything, but what can we count in this situation?" Give students one minute of quiet think time followed by another minute to share with their partner. Quantities may include: number of tablespoons of honey, number of cups of flour, size of the batches.

   Listen for, and amplify, student language about the relationships among the amount of honey, the amount of flour, and the size of the the batches. Invite students to sketch a diagram to represent these relationships.

3. In the third read, students brainstorm possible strategies to complete the table.

Invite students to read the problem aloud with their partner or select a different student to read to the class. Instruct students to think of ways to approach the questions without actually completing the table. Consider using these questions to prompt students: “How would you approach this question?,” and “What strategy would you try first?”

Give students one minute of quiet think time followed by another minute to discuss with their partner. Provide these sentence frames as partners discuss: “To figure out how much flour is needed for a given amount of honey….”, “One way a constant of proportionality could help is….”

Sample responses include: “I would try to find the factor that gets us from 8 tablespoons to 20 tablespoons,” “I would draw a diagram to figure out how much flour is needed for 1 tablespoon of honey,” and, “I know that 10 to 8 is the same as 5 to 4 is the same as 1.25, and that is the constant of proportionality, so I know I need to multiply by 1.25.” This will help students concentrate on making sense of the situation before rushing to a solution or method.

4. As partners are discussing their strategies, select 1–2 students to share their ideas with the whole class.

   As students are presenting ideas to the whole class, create a display that summarises their ideas about how to complete the table. Listen for quantities that were mentioned during the second read, and include these on the display.

5. Post the summary where all students can use it as a reference.

Student Task Statement

A bakery uses 8 tablespoons of honey for every 10 cups of flour to make bread dough. Some days they bake bigger batches and some days they bake smaller batches, but they always use the same ratio of honey to flour. Complete the table as you answer the questions. Be prepared to explain your reasoning.

1. How many cups of flour do they use with 20 tablespoons of honey?
2. How many cups of flour do they use with 13 tablespoons of honey?
3. How many tablespoons of honey do they use with 20 cups of flour?
4. What is the **proportional relationship** represented by this table?

honey (tbsp)	flour (cup)
8	10
20	
13	

	20

Student Response

1. 25 cups of flour for 20 tablespoons of honey
2. $16\frac{1}{4}$ or 16.25 cups of flour for 13 tablespoons of honey
3. 16 tablespoons of honey for 20 cups of flour

honey (tbsp)	flour (cup)
4	5
8	10
12	15
13	**16.25**
16	20
18	22.5
20	**25**

4. Answers vary. Possible responses:

    - The relationship between the number of tablespoons of honey and the number of cups of flour is proportional.
    - The relationship between the amount of honey and the amount of flour is a proportional relationship.
    - The table represents a proportional relationship between amount of honey and amount of flour.
    - The amount of honey is proportional to the amount of flour.

Activity Synthesis

Select students to share their solution approaches in this order:

1. Double each entry in the first row to get to row 4.
2. Divide each entry in the first row by 8, yielding the pair (1,1.25). (1.25 can be called the unit rate, since 1.25 cups of flour are needed for 1 tablespoon of honey.) Multiply by 20 and 13 to get rows 2 and 3.

3. Notice any entry in the first column can be multiplied by $\frac{5}{4}$ or 1.25 to get the corresponding entry in the second column. Name this value the **constant of proportionality** for the proportional relationship.

Ensure that these are highlighted as part of the discussion and that students describe their methods using mathematical language. For example,

- Note that even though you multiply by a different scale factor to go from row to row in the table, the unit rate is always the same. Rename this "the constant of proportionality." Note that it can always be found by finding how much of the second quantity per one of the first quantity.
- Ask students what the proportional relationship is in this situation.
- Ask students to interpret the constant of proportionality in the context: "What does the 1.25 tell us about?" (That there are 1.25 cups of flour per tablespoon of honey.) Because this can help them connect it to their earlier work in KS3 and prepare them for the next activity.

Engagement: Develop Effort and Persistence. Encourage and support opportunities for peer interactions. Prior to the whole-class discussion, invite students to share their work with a partner. Display sentence frames to support student conversation such as: "First, I _____ because ….," "I noticed _____ so I ….," "Why did you...?," "I agree/disagree because…."
Supports accessibility for: Language; Social-emotional skills

2.4 Inches and Centimetres

Optional: 10 minutes

The purpose of this optional activity is to practice using a proportional relationship in another context. The chosen numbers make the unit rate a useful tool to answer the questions.

Instructional Routines

- Discussion Supports

Student Task Statement

4 inches are approximately equal to 10 centimetres.

1. How many centimetres equal 6 inches?
2. How many centimetres equal 14 inches?
3. What value belongs next to the 1 in the table? What does it mean in this context?

number of inches	number of centimetres

1	
4	10
6	
14	

Student Response

1. 15

2. 35

3. 2.5 means that two and a half centimetres are the same length as a single inch.

number of inches	number of centimetres
1	2.5
4	10
6	15
14	35

Activity Synthesis

Ask students for the value that belongs next to the 1 in the table. Invite several students to explain the significance of this number.

Speaking: Discussion Supports. To provide a support for students in precision of language when describing proportional relationships from a table, provide sentence frames for students to use, such as: "_____ is equal in value to _____, because . . . ."
Design Principle(s): Cultivate conversation

Lesson Synthesis

Briefly revisit the two activities, demonstrating the use of the new terms. It would be helpful to display a filled-in table for each to facilitate the conversation. For example,

- In the first activity, we looked at a **proportional relationship** between the amount of rice and the number of people served. Some people found missing values in the table by multiplying both values in one of the rows by a number, others used a *unit rate*—the number of people served per cup of rice. What was the **constant of proportionality** in that situation?

- In the second activity, we looked at a proportional relationship between the amount of honey and the amount of flour in a recipe. Some people found missing values in the table by multiplying both values in one of the rows by a number, others used a unit rate—the number of cups of flour per tablespoon of honey. What was the constant of proportionality in that situation?

If this has not already happened during the discussion, write a third column in each table that shows the constant of proportionality and label it.

2.5 Green Paint

Cool Down: 5 minutes

Student Task Statement

When you mix two colours of paint in equivalent ratios, the resulting colour is always the same. Complete the table as you answer the questions.

1. How many cups of yellow paint should you mix with 1 cup of blue paint to make the same shade of green? Explain or show your reasoning.
2. Make up a new pair of numbers that would make the same shade of green. Explain how you know they would make the same shade of green.
3. What is the proportional relationship represented by this table?
4. What is the *constant of proportionality*? What does it represent?

cups of blue paint	cups of yellow paint
2	10
1	

Student Response

1. You need 5 cups of yellow paint for 1 cup of blue paint. You can see this by multiplying the first row by a factor of $\frac{1}{2}$. Alternatively, you have to multiply 2 by $\frac{10}{2} = 5$ to get 10: multiplying 1 by 5 gives 5.
2. Answers vary. Sample response: 3 cups of blue paint mixed with 15 cups of yellow paint will also make the same shade of green. This can be obtained by multiplying the second row by a factor of 3 or choosing 3 for blue and then multiplying that by 5.

3. The relationship between the amount of blue paint and the amount of yellow paint is the proportional relationship represented by this table.

4. The constant of proportionality is 5. It represents the cups of yellow paint needed for 1 cup of blue paint.

Student Lesson Summary

If the ratios between two corresponding quantities are always equivalent, the relationship between the quantities is called a **proportional relationship**.

This table shows different amounts of milk and chocolate syrup. The ingredients in each row, when mixed together, would make a different total amount of chocolate milk, but these mixtures would all taste the same.

Notice that each row in the table shows a ratio of tablespoons of chocolate syrup to cups of milk that is equivalent to 4: 1.

About the relationship between these quantities, we could say:

tablespoons of chocolate syrup	cups of milk
4	1
6	$1\frac{1}{2}$
8	2
$\frac{1}{2}$	$\frac{1}{8}$
12	3
1	$\frac{1}{4}$

- The relationship between amount of chocolate syrup and amount of milk is proportional.
- The relationship between the amount of chocolate syrup and the amount of milk is a proportional relationship.
- The table represents a proportional relationship between the amount of chocolate syrup and amount of milk.
- The amount of milk is proportional to the amount of chocolate syrup.

We could multiply any value in the chocolate syrup column by $\frac{1}{4}$ to get the value in the milk column. We might call $\frac{1}{4}$ a *unit rate*, because $\frac{1}{4}$ cup of milk is needed for 1 tablespoon of chocolate syrup. We also say that $\frac{1}{4}$ is the **constant of proportionality** for this relationship.

It tells us how many cups of milk we would need to mix with 1 tablespoon of chocolate syrup.

Glossary

- constant of proportionality
- proportional relationship

Lesson 2 Practice Problems

1. **Problem 1 Statement**

   When Han makes chocolate milk, he mixes 2 cups of milk with 3 tablespoons of chocolate syrup. Here is a table that shows how to make batches of different sizes. Use the information in the table to complete the statements. Some terms are used more than once.

cups of milk	tablespoons of chocolate syrup
2	3
8	12
1	$\frac{3}{2}$
10	15

   a. The table shows a proportional relationship between ____________ and ____________.

   b. The scale factor shown is ____________.

   c. The constant of proportionality for this relationship is____________.

   d. The units for the constant of proportionality are ____________ per ____________.

   Bank of Terms: tablespoons of chocolate syrup, 4, cups of milk, cup of milk, $\frac{3}{2}$

   Solution

   a. cups of milk, tablespoons of chocolate syrup

   b. 4

   c. $\frac{3}{2}$

   d. tablespoons of chocolate syrup, cup of milk

2. **Problem 2 Statement**

A certain shade of pink is created by adding 3 cups of red paint to 7 cups of white paint.

a. How many cups of red paint should be added to 1 cup of white paint?

cups of white paint	cups of red paint
1	
7	3

b. What is the constant of proportionality?

Solution

a. $\frac{3}{7}$ cups of red paint

b. $\frac{3}{7}$

3. **Problem 3 Statement**

A map of a rectangular park has a length of 4 inches and a width of 6 inches. It uses a scale of 1 inch for every 30 miles.

a. What is the actual area of the park? Show how you know.

b. The map needs to be reproduced at a different scale so that it has an area of 6 square inches and can fit in a brochure. At what scale should the map be reproduced so that it fits on the brochure? Show your reasoning.

Solution

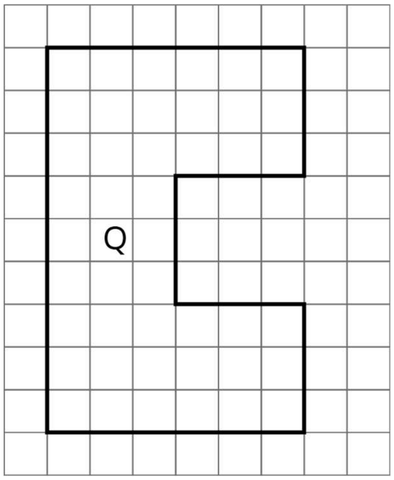
a. 21 600 square miles. Sample reasoning: The area on the map is 24 square inches. 1 square inch represents 900 square miles, since $30 \times 30 = 900$. The actual area is 24×900, which equals 21 600 square miles.

b. 1 inch to 60 miles. Sample explanations:

- If 21 600 square miles need to be represented by 6 square inches, each square inches needs to represent 3 600 square miles: $21\,600 \div 6 = 3\,000$. This means each 1-inch side of the square needs to be 60 miles.
- The area of this new map is $\frac{1}{4}$ of the first map, since 6 is $\frac{1}{4}$ of 24. This means each 1 inch square has to represent 4 times as much area than the first. $900 \times 4 = 3\,600$. If each 1 inch square represents 3 600 square miles, every 1 inch represents 60 miles.

4. **Problem 4 Statement**

Noah drew a scaled copy of Polygon P and labeled it Polygon Q.



If the area of Polygon P is 5 square units, what scale factor did Noah apply to Polygon P to create Polygon Q? Explain or show how you know.

Solution

The area of polygon Q is 45 square units, so the area has scaled by a factor of 9, since $5 \times 9 = 45$. Since the area of a scaled copy varies from the original area by the square of the scale factor, the scale factor is 3.

5. **Problem 5 Statement**

Select **all** the ratios that are equivalent to each other.

a. $4:7$

b. $8:15$

c. $16:28$

d. $2:3$

e. $20:35$

Solution ["A", "C", "E"]