Lesson 13: Solving systems of equations

Goals

- Coordinate (orally) the solution of an equation with variables on each side to the solution of a system of two linear equations.
- Create a graph of a system of equations, and identify (orally and in writing) the number of solutions of the system of equations.

Learning Targets

- I can graph a system of equations.
- I can solve systems of equations using algebra.

Lesson Narrative

In this lesson, students continue to explore systems where the equations are both of the form $y = mx + c$. They connect algebraic and graphical representations of systems, first by matching graphs to systems, then by drawing their own graphs from given systems. Additionally, students see how to see the number of solutions from both the graphical and algebraic representations. In the graphical representation the number of solutions is equal to the number of points where the graphs intersect. In the algebraic representation, two equations with the same rate of change can have 0 or infinitely many solutions depending on whether the initial values (y-intercepts) are the same or not. If the rates of change are different then there is a single solution, which can be interpreted physically as the point at which two quantities changing at different rates become equal.

Addressing

- Analyse and solve pairs of simultaneous linear equations.
- Understand that solutions to a system of two linear equations in two variables correspond to points of intersection of their graphs, because points of intersection satisfy both equations simultaneously.

Instructional Routines

- Stronger and Clearer Each Time
- Discussion Supports
- Think Pair Share

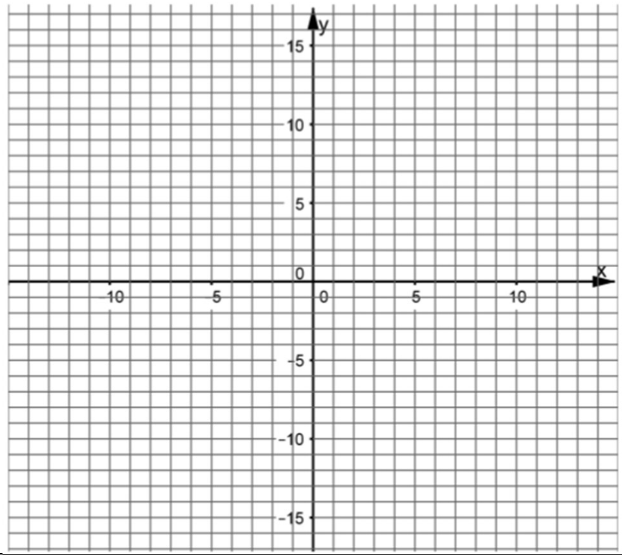
Required Materials

Copies of blackline master

Different Types of Systems

$$A\begin{cases} y = -4(x-2) \\ y = -2\left(x - \dfrac{5}{2}\right) \end{cases}$$

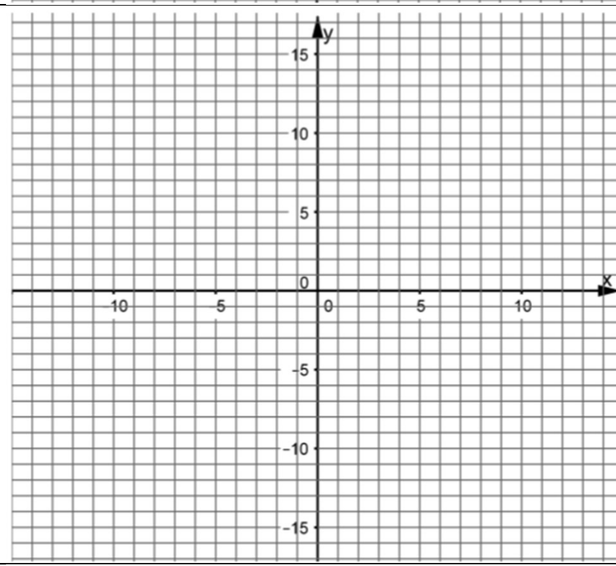
One solution



Different Types of Systems

$$B\begin{cases} y = 5(x-3) \\ y = 2x - 6 \end{cases}$$

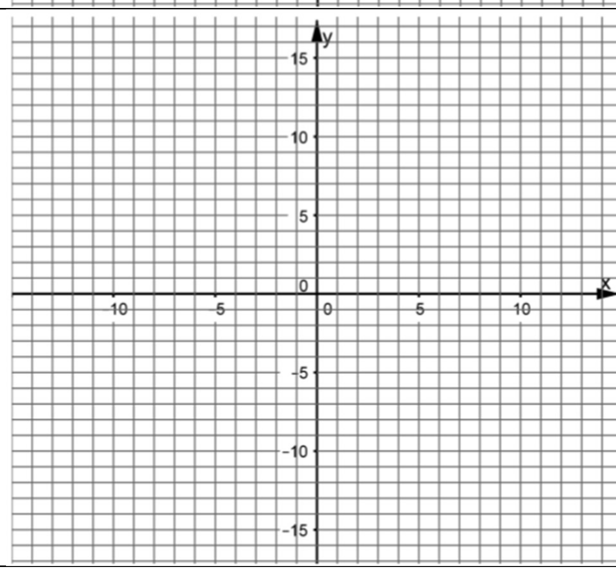
One solution



Different Types of Systems

$$C\begin{cases} y = 2x + 3 \\ y = 2x - 5 \end{cases}$$

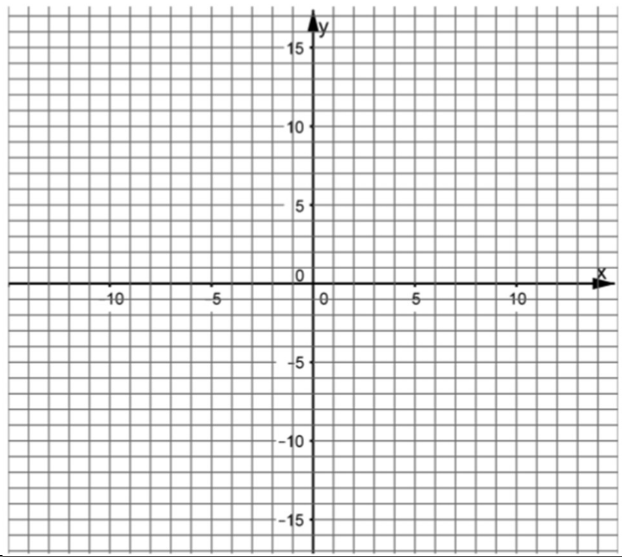
No solutions



Different Types of Systems

$$D\begin{cases} y = -6x \\ y = -5(x-2) - x \end{cases}$$

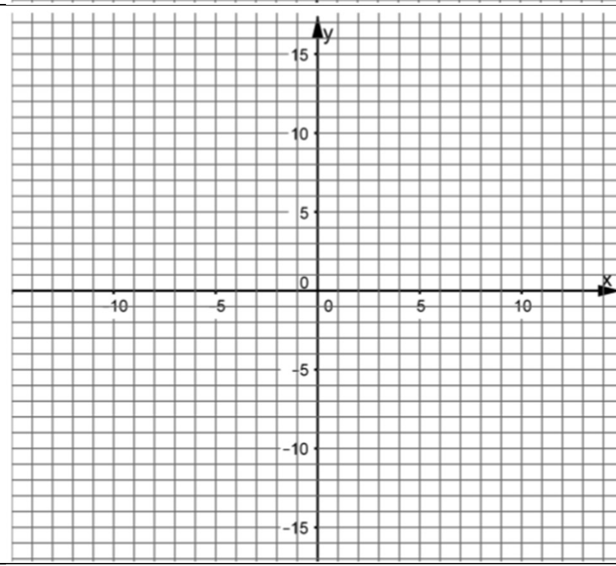
No solutions



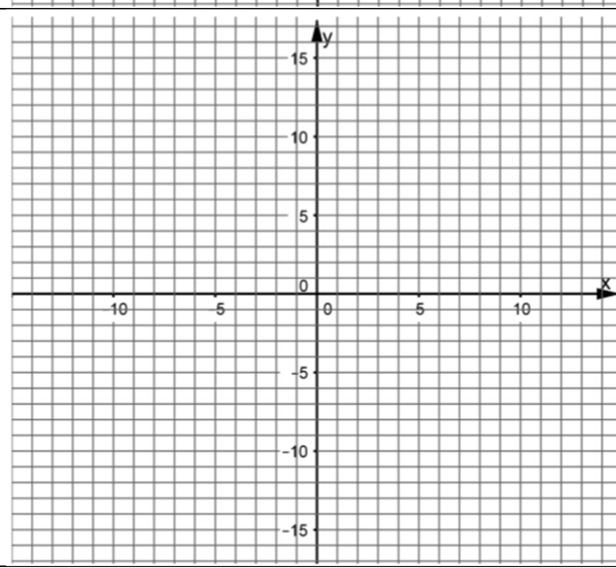
Different Types of Systems

$$E\begin{cases} y = 3(2x+1) - 4x \\ y = 2x + 3 \end{cases}$$

Infinite solutions



Different Types of Systems



Scissors
Straightedges
A rigid edge that can be used for drawing line segments. Sometimes a ruler is okay to use as a straightedge, but sometimes it is preferable to use an unruled straightedge, like a blank index card.

Required Preparation

Print the Different Types of Systems blackline master. Prepare one set for every 2–3 students. Provide access to straightedges for drawing accurate graphs and scissors for groups that wish to cut apart the graphs on the blackline master.

Student Learning Goals

Let's solve systems of equations.

13.1 True or False: Two Lines

Warm Up: 5 minutes

The purpose of this warm-up is to get students to reason about solutions to equations by looking at their structure and reading their graphs. While some students may solve each equation to find if it is true or false without relating it to the graphs, encourage all students to show why their answer is correct based on the graphs of the equations during the whole-class discussion.

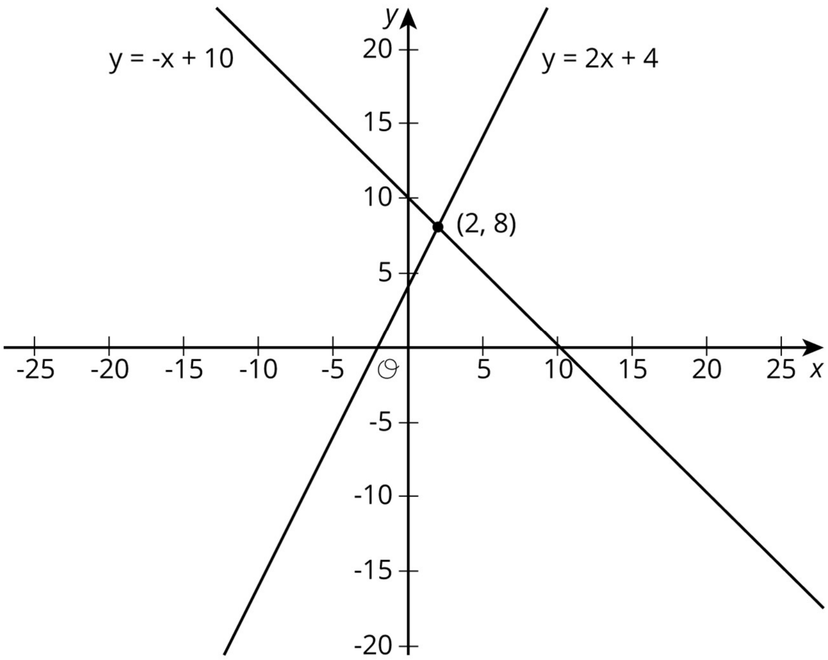
Instructional Routines

- Think Pair Share

Launch

Arrange students in groups of 2. Display the image for all to see. Give students 2 minutes of quiet work time to begin the task individually and then 1 minute to discuss their responses with a partner followed by a whole-class discussion.

Student Task Statement



Use the lines to decide whether each statement is true or false. Be prepared to explain your reasoning using the lines.

1. A solution to $8 = -x + 10$ is 2.
2. A solution to $2 = 2x + 4$ is 8.
3. A solution to $-x + 10 = 2x + 4$ is 8.
4. A solution to $-x + 10 = 2x + 4$ is 2.
5. There are no values of x and y that make $y = -x + 10$ and $y = 2x + 4$ true at the same time.

Student Response

1. True because the line passes through point (2,8) where $x = 2$
2. False
3. False
4. True
5. False

Activity Synthesis

Display the task image for all to see. Ask students to share their solutions and to reference the lines in their explanations. Emphasise the transitive property when students explain that since $y = 8$ at the point of intersection of $y = 2x + 4$ and $y = -x + 10$, then both $2x + 4 = 8$ and $-x + 10 = 8$ are true, which leads to $-x + 10 = 2x + 4$. If students do not mention this idea, bring it to their attention.

Ask students to solve $-x + 10 = 2x + 4$ for x if they've not already done so and confirm that $x = 2$ is the x-coordinate of the solution to the system of equations.

13.2 Matching Graphs to Systems

15 minutes (there is a digital version of this activity)

This activity represents the first time students solve a system of equations using algebraic methods. They first match systems of equations to their graphs and then calculate the solutions to each system. The purpose of matching is so students have a way to check that their algebraic solutions are correct, but not to shortcut the algebraic process since the graphs themselves do not include enough detail to accurately state the coordinates of the solution.

Instructional Routines

- Discussion Supports

Launch

Keep students in groups of 2. Give 2–3 minutes of quiet work time for the first problem and then ask students to pause their work. Select 1–2 students per figure to explain how they matched it to one of the systems of equations. For example, a student may identify the system matching Figure A as the only system with an equation that has negative gradient. Give students 5–7 minutes of work time with their partner to complete the activity followed by a whole-class discussion. If students finish early and have not already done so on their own, ask them how they could check their solutions and encourage them to do so.

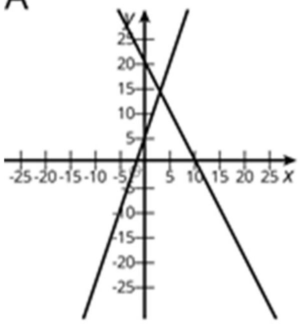
If using the digital activity, implement the lesson as indicated above. The only difference between the print and digital version is the digital lesson has an applet that will simulate the graphs so the students have another way of checking their solutions.

Representation: Internalise Comprehension. Demonstrate and encourage students to use colour coding and annotations to highlight connections between representations in a problem. Invite students to illustrate connections between gradients and y-intercepts of each line to the corresponding parts of each equation using the same colour.
Supports accessibility for: Visual-spatial processing Conversing: Discussion Supports. Use this routine to support small-group discussion as students describe the reasons for their matches. Arrange students in groups of 2. Invite Partner A to begin with this sentence frame: "Figure ____ matches with the system of equations ____, because ____." Invite the listener, Partner B, to press for additional details referring to specific features of the graphs (e.g. positive gradient, negative y-intercept, coordinates of the intersection point, etc). Students should switch roles for each figure. This will help students justify how features of the graph can be used to identify matching equations.
Design Principle(s): Support sense-making; Cultivate conversation

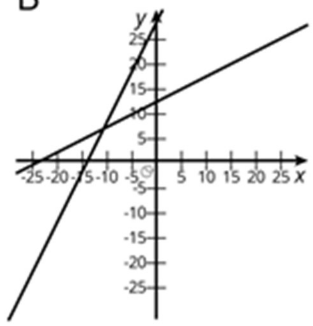
Student Task Statement

Here are three **systems of equations** graphed on a coordinate plane:

A



B



C

1. Match each figure to one of the systems of equations shown here.

   a. $\begin{cases} y = 3x + 5 \\ y = \text{-}2x + 20 \end{cases}$

   b. $\begin{cases} y = 2x - 10 \\ y = 4x - 1 \end{cases}$

   c. $\begin{cases} y = 0.5x + 12 \\ y = 2x + 27 \end{cases}$

2. Find the solution to each system and check that your solution is reasonable based on the graph.

Student Response

a. Graph A since it is the only graph with a negative gradient.

b. Graph C since both graphs have a negative y-intercept.

c. Graph B since both graphs have a positive y-intercept.

1. Graph A: (3,14). Graph B: (-10,7). Graph C: (-4.5,-19)

Activity Synthesis

The goal of this discussion is to deliberately connect the current topic of systems of equations to the previous topic of solving equations with variables on both sides. For each of the following questions, give students 30 seconds of quiet think time and then invite students per question to explain their answer. The final question looks ahead to the following activity.

- "Do you need to see the graphs of the equations in a system in order to solve the system?" (No, but the graphs made me feel more confident that my answer was correct.)

- “How do you know your solution doesn’t contain any errors?" (I know my solution does not have errors because I substituted my values for x and y into the equations and they made both equations true.)
- “How does solving systems of equations compare to solving equations with variables on both sides like we did in earlier lessons?" (They are very similar, only with a system of equations you are finding an x and a y to make both equations true and not just an x to make one equation true.)
- “When you solved equations with variables on both sides, some had one solution, some had no solutions, and some had infinite solutions. Do you think systems of equations can have no solutions or infinite solutions?" (Yes. We have seen some graphs of parallel lines where there were no solutions and some graphs of lines that are on top of one another where there are infinite solutions.)

13.3 Different Types of Systems

15 minutes (there is a digital version of this activity)

While students have encountered equations with different numbers of solutions in earlier activities, this is the first activity where students connect systems of equations with their previous thinking about equations that have no solution, one solution, or infinitely many solutions. The purpose of this activity is for students to connect the features of the graph of the equations of a system to the number of solutions of a system. While students are not asked to solve the systems of equations, they may choose to rewrite the equations in equivalent forms as they work to graph the lines.

Depending on instructional time available, you may wish to alter the activity and ask students to solve one or more of the systems of equations algebraically.

Instructional Routines

- Stronger and Clearer Each Time

Launch

Remind students of the activity they did sorting equations with a single variable where each equation had either one solution, no solution, or infinitely many solutions. Tell them that, just like one variable equations, systems of equations can also have either one solution, no solution, or infinitely many solutions. Point out that in the previous activity, each of the three systems of equations had one solution, which they found algebraically by solving the system, and so the graphs of the equations of the system showed one point where the lines intersected. Ask students what they think the graphs of equations from systems with no or infinitely many solutions might look like. Allow 30 seconds of quiet think time before inviting a few students to suggest possibilities for each type of system while recording and displaying their ideas for all to see. Remind students of the activities in previous lessons where they have seen these situations and their graphs (a bike race between Elena and Jada had infinite solutions and stacking different sized cups had none).

Arrange students in groups of 2–3. Provide each group with access to straightedges and scissors as well as one copy of the blackline master. Encourage partners to split the work by cutting apart the problems, each taking one to three graphs, and then trading pages within their group to check the work. Give 4–6 minutes for groups to complete the graphs and remind students to use straightedges for precision while graphing.

Before beginning the final problem, have each group trade their work with another group and place a question mark next to the graphs they are not sure are correct. Give groups 3–4 minutes to revise as needed and write their descriptions for the second problem followed by a whole-class discussion.

If using the digital activity, use the discussion structure above. The digital applet will make the graphing and solving of systems go quickly so students can spend more time analysing the solutions. Using technology to graph allows students to focus on the main purpose of the lesson and also recognise the value in technology when solving systems in addition to appreciating when the graphing method is efficient. In this activity, one of the main purposes is to notice what is common among systems with the same number of solutions. Therefore, it may be useful to ask students to justify why the lines graphed with no obvious intersections are actually parallel.

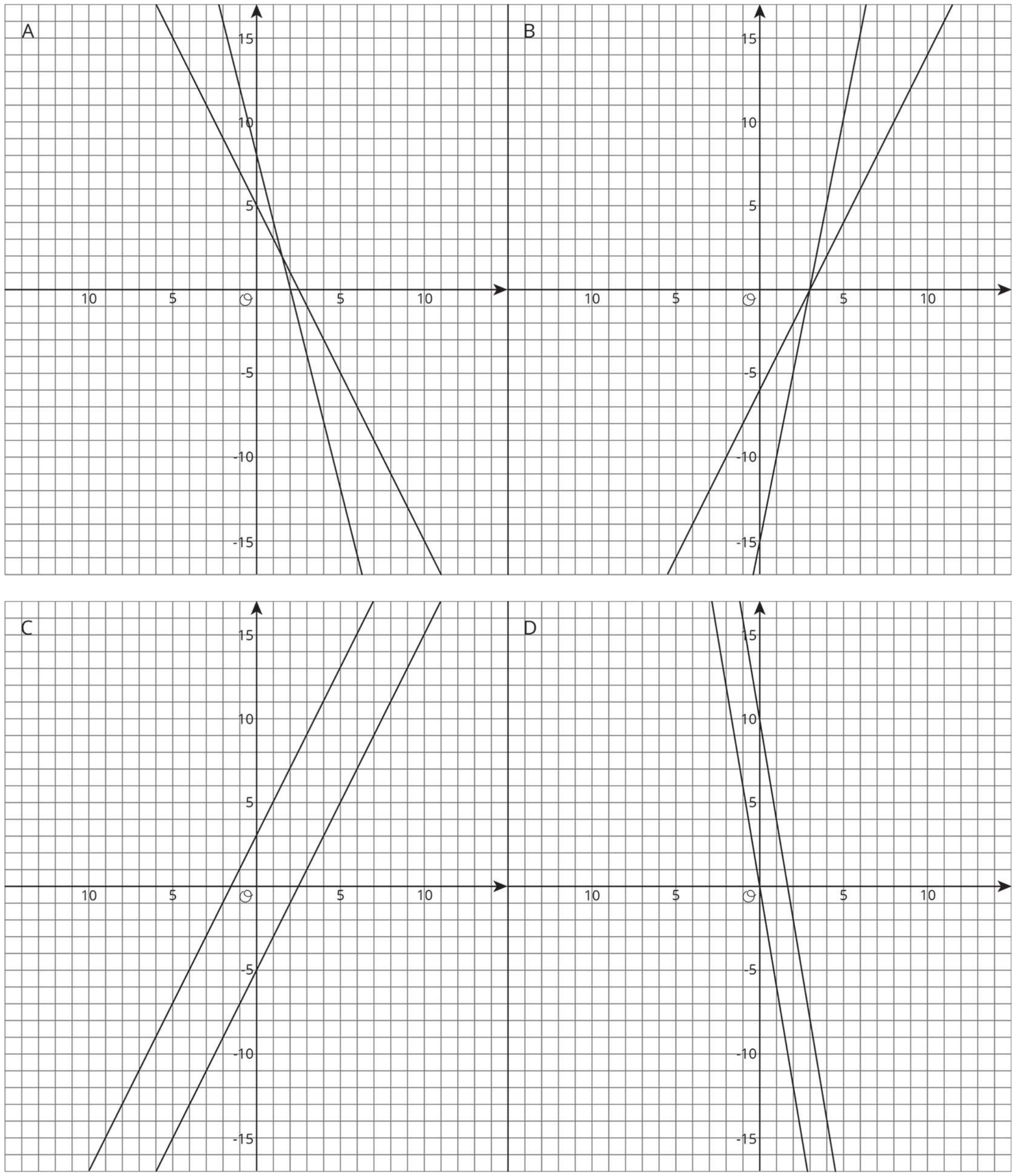
Student Task Statement

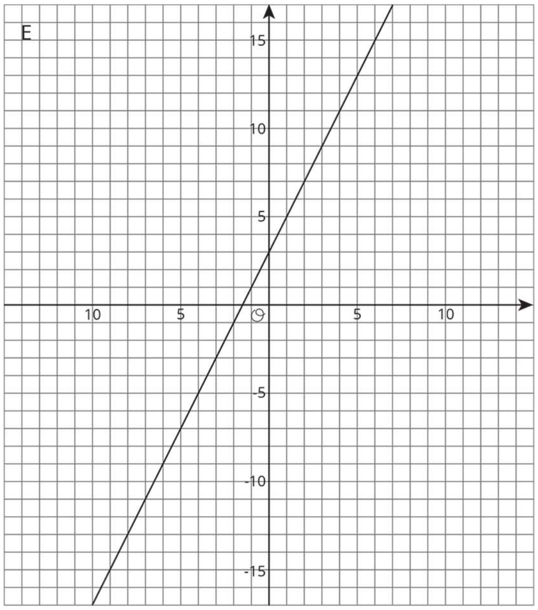
Your teacher will give you a page with some systems of equations.

1. Graph each system of equations carefully on the provided coordinate plane.
2. Describe what the graph of a system of equations looks like when it has . . .
    a. 1 solution
    b. 0 solutions
    c. infinitely many solutions

Student Response

1.


A
B
C
D
15
10
5
-5
-10
-15
10
5
O
5
10



a. Two lines that cross at a single point, which is the only solution they have in common. Graphs A and B have one solution.

b. Two distinct lines that are parallel. They have no solutions in common. Graphs C and D have no solution.

c. A single line since both equations have the same set of solutions. Graph E has infinitely many solutions.

Are You Ready for More?

The graphs of the equations $Ax + By = 15$ and $Ax - By = 9$ intersect at $(2,1)$. Find A and B. Show or explain your reasoning.

Student Response

$A = 6$, $B = 3$.

Explanations vary. Sample response: If the lines intersect at $(2,1)$ then that point is on both lines. So we can substitute $x = 2$, $y = 1$ into both equations and write $2A + B = 15$, $2A - B = 9$. Then there are different ways we can solve.

- Since $2A - B$ has the same value as 9, we can add $2A - B$ to one side of the first equation and 9 to the other.
- We can rewrite the second equation as $B = 2A - 9$ and substitute this expression for B into the first equation.

Activity Synthesis

The goal of this discussion is for students to draw conclusions about the relationship between the number of solutions a system of equations has and the appearance of the graphs of the equations in the system. Select 2–3 students to share and explain their answers to the second problem.

If no students mention it, bring in gradient language and how inspecting the gradients of the equations before graphing or solving can give clues to the possible number of solutions the system has. In particular, students should notice that systems with lines that have different gradients have a single solution, lines that have the same gradient and different y-intercept have no solution, and lines that have the same gradient and y-intercept will have infinitely many solutions.

Assign a number of solutions (one, none, or infinite) to each group and ask them to write a system of equations that would have that number of solutions. Have a few groups share their systems and describe how the graphs of the systems would look. In particular, ask each group to describe how the gradient and y-intercept of their written lines would be seen in the graph and how the number of solutions would appear on the graph. Following the description, display the graph of the system using a digital resource, if possible, or a general sketch on a set of displayed axes.

Writing: Stronger and Clearer Each Time. Use this routine to give students an opportunity to revise and improve their response to the final question, "Describe what the graph of a system of equations looks like when it has one solution, zero solutions, and infinitely many solutions." Give students time to meet with 2–3 partners, to share and get feedback on their response. Encourage the listener to press for supporting details and evidence by asking, "How do the gradients compare?", "How do the y-intercepts compare?" or "What do you notice about the gradients and the y-intercepts?" Students can borrow ideas and language from each partner to strengthen the final product. This will help students produce a written generalisation for how to identify the number of solutions for a system of equations by using the features of a graph.
Design Principle(s): Optimise output (for generalisation)

Lesson Synthesis

To highlight the connection between the number of solutions to a system of equations and features of its graph and equations, ask:

- "How can you know the number of solutions for a system of equations from its graph?" (If the two lines intersect at a point, there is one solution. If the two lines are parallel and do not intersect, there are no solutions. If the two lines are drawn through the same points, there are infinitely many solutions.)

- "How can you know the number of solutions for a system of equations from their equations?" (If the two equations have different gradients, there is one solution. If the two equations have the same gradient and different y-intercepts, there are no solutions. If the two equations have the same gradient and the same y-intercept, there are infinitely many solutions.)

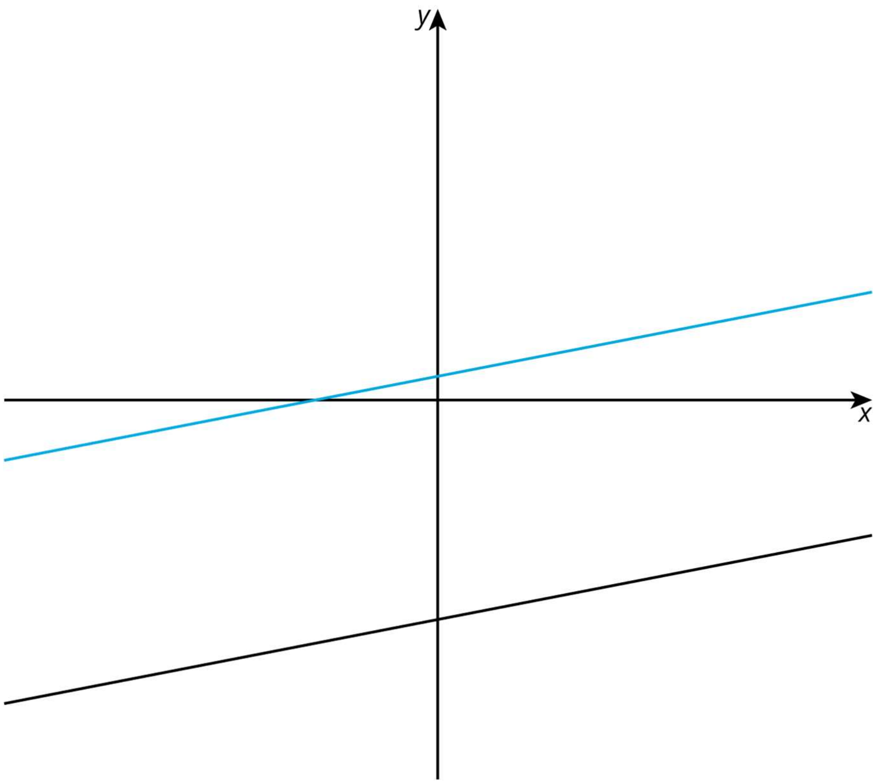
If students do not make the connection themselves, remind them of their earlier conclusions about the number of solutions and equation in one variable has.

13.4 Two Lines

Cool Down: 5 minutes

This cool-down asks students to write equations that could match a given graph and then to identify the number of solutions in the system shown.

Student Task Statement



1. Given the lines shown here, what are two possible equations for this system of equations?

2. How many solutions does this system of equations have? Explain your reasoning.

Student Response

1. Correct answers should have the same positive gradient for each linear equation. One equation should have a negative y-intercept and the other should have a positive y-intercept. Sample response: $\begin{cases} y = \frac{1}{4}x - 10 \\ y = \frac{1}{4}x + 1 \end{cases}$

2. 0. Since the lines are parallel and do not intersect, there are no solutions to the system of equations.

Student Lesson Summary

Sometimes it is easier to solve a system of equations without having to graph the equations and look for an intersection point. In general, whenever we are solving a system of equations written as

$$\begin{cases} y = \text{[some stuff]} \\ y = \text{[some other stuff]} \end{cases}$$

we know that we are looking for a pair of values (x, y) that makes both equations true. In particular, we know that the value for y will be the same in both equations. That means that

[some stuff] $=$ [some other stuff]

For example, look at this system of equations:

$$\begin{cases} y = 2x + 6 \\ y = \text{-}3x - 4 \end{cases}$$

Since the y value of the solution is the same in both equations, then we know $2x + 6 = \text{-}3x - 4$

We can solve this equation for x:

$2x + 6$	$= \text{-}3x - 4$	
$5x + 6$	$= \text{-}4$	add $3x$ to each side
$5x$	$= \text{-}10$	subtract 6 from each side
x	$= \text{-}2$	divide each side by 5

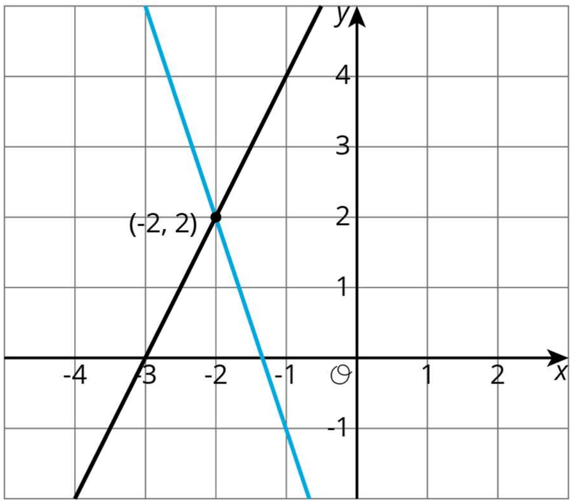
But this is only half of what we are looking for: we know the value for x, but we need the corresponding value for y. Since both equations have the same y value, we can use either equation to find the y-value:

$y = 2(\text{-}2) + 6$

Or

$y = \text{-}3(\text{-}2) - 4$

In both cases, we find that $y = 2$. So the solution to the system is $(\text{-}2,2)$. We can verify this by graphing both equations in the coordinate plane.



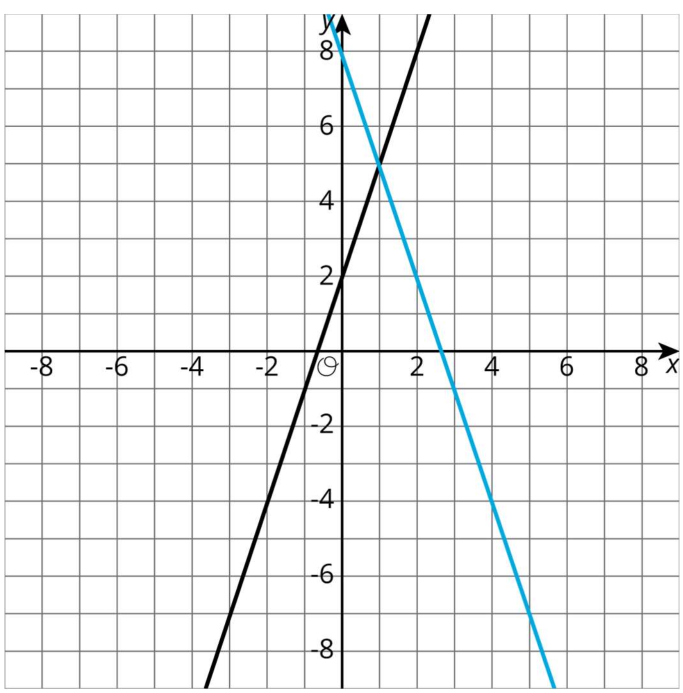
In general, a system of linear equations can have:

- No solutions. In this case, the lines that correspond to each equation never intersect.
- Exactly one solution. The lines that correspond to each equation intersect in exactly one point.
- An infinite number of solutions. The graphs of the two equations are the same line!

Lesson 13 Practice Problems

1. **Problem 1 Statement**

    a. Write equations for the lines shown.



b. Describe how to find the solution to the corresponding system by looking at the graph.

c. Describe how to find the solution to the corresponding system by using the equations.

Solution

a. $y = 3x + 2$ and $y = 8 - 3x$

b. The point where the two lines meet, (1,5)

c. Set the two expressions for y equal to each other and solve: $3x + 2 = 8 - 3x$, $6x = 6$, $x = 1$, $y = 3(1) + 2 = 5$

2. **Problem 2 Statement**

The solution to a system of equations is (5,-19). Choose two equations that might make up the system.

a. $y = \text{-}3x - 6$

b. $y = 2x - 23$

c. $y = \text{-}7x + 16$

d. $y = x - 17$

e. $y = \text{-}2x - 9$

Solution ["C", "E"]

3. **Problem 3 Statement**

Solve the system of equations: $\begin{cases} y = 4x - 3 \\ y = \text{-}2x + 9 \end{cases}$

Solution

(2,5)

4. **Problem 4 Statement**

Solve the system of equations: $\begin{cases} y = \frac{5}{4}x - 2 \\ y = \frac{\text{-}1}{4}x + 19 \end{cases}$

Solution

$\left(14, 15\frac{1}{2}\right)$

5. **Problem 5 Statement**

Here is an equation: $\frac{15(x-3)}{5} = 3(2x-3)$

a. Solve the equation by using the distributive property first.

b. Solve the equation without using the distributive property.

c. Check your solution.

Solution

a. $x = 0$. Responses vary. Sample response:

$\frac{15(x-3)}{5}$	$= 3(2x-3)$	
$\frac{15x-4}{5}$	$= 6x-9$	distributive property on each side
$3x-9$	$= 6x-9$	divide each term in the numerator of the left side by 5
$3x$	$= 6x$	add 9 to each side
0	$= 3x$	subtract 3x on each side
0	$= x$	divide each side by 3

b. $x = 0$. Responses vary. Sample response:

$\frac{15(x-3)}{5}$	$= 3(2x-3)$	
$15(x-3)$	$= 15(2x-3)$	multiply each side by 5
$x-3$	$= 2x-3$	divide each side by 15
x	$= 2x$	add 3 to each side
0	$= x$	subtract x on each side

c. This equation is true, so $x = 0$ is the solution.