Lesson 21: Cylinders, cones, and spheres

Goals

- Calculate the value of the radius of a sphere with a given volume using the structure of the equation, and explain (orally) the solution method.
- Determine what information is needed to solve a problem involving volumes of cones, cylinders, and spheres. Ask questions to elicit that information.

Learning Targets

- I can find the radius of a sphere if I know its volume.
- I can solve mathematical and real-world problems about the volume of cylinders, cones, and spheres.

Lesson Narrative

In this lesson, students use the formula for the volume of a sphere to solve various problems. They have opportunities to analyse common errors that people make when using this formula. They also use the structure of an equation to find the radius of a sphere when they know its volume. Finally, they have opportunities to practise using all of the new volume formulae they have learned in this unit to solve mixed problems with spheres, cylinders, and cones, reasoning about the effect of different dimensions on the volume of different shapes.

Addressing

- Know the formulae for the volumes of cones, cylinders, and spheres and use them to solve real-world and mathematical problems.

Instructional Routines

- Stronger and Clearer Each Time
- Information Gap Cards
- Discussion Supports

Required Materials

Pre-printed slips, cut from copies of the blackline master

Info Gap: Unknown Dimensions Problem Card 1 A cone and a sphere have the same dimensions. What is the volume of the sphere?	Info Gap: Unknown Dimensions Data Card 1 • The volume of the cone is $V = 144\pi$ cm^3. • The radius of the cone is the same as the radius of the sphere. • $4^3 = 64$, $5^3 = 125$, $6^3 = 216$, $7^3 = 343$
Info Gap: Unknown Dimensions Problem Card 2 A cone and a sphere have the same height. What is the volume of the sphere?	Info Gap: Unknown Dimensions Data Card 2 • The volume of the cone is $V = 18\pi$ cm^3. • The radius of the sphere is half the height of the cone. • The height of the cone is twice the value of the radius of the cone. • $4^3 = 64$, $5^3 = 125$, $6^3 = 216$, $7^3 = 343$

Tools for creating a visual display

Any way for students to create work that can be easily displayed to the class. Examples: chart paper and markers, whiteboard space and markers, shared online drawing tool, access to a document camera.

Required Preparation

Print and cut up slips from the Info Gap: Unknown Dimensions blackline master. Prepare 1 set of slips for every 2 students. Provide students with access to make a visual display during the Lesson Synthesis.

Student Learning Goals

Let’s find the volume of shapes.

21.1 Sphere Arguments

Warm Up: 5 minutes

The purpose of this warm-up is to catch errors students are making in calculating the volume of a sphere. Monitor for students who use the formula for a cylinder or cone, who use r^2 instead of r^3, or who forget to include π as a factor in the calculation.

Launch

Arrange students in groups of 2. Tell students that if they have a hard time visualising this sphere, they can sketch it. Give students 1–2 minutes of quiet work time followed by time to discuss their responses with their partner.

Student Task Statement

Four students each calculated the volume of a sphere with a radius of 9 centimetres and they got four different answers.

- Han thinks it is 108 cubic centimetres.
- Jada got 108π cubic centimetres.
- Tyler calculated 972 cubic centimetres.
- Mai says it is 972π cubic centimetres.

Do you agree with any of them? Explain your reasoning.

Student Response

Mai's calculation is correct. Explanations vary. Sample explanation: The volume of a sphere is found with the formula $V = \frac{4}{3}\pi r^3$. Using 9 for the radius, the volume is $\frac{4}{3}\pi(9^3) = \frac{4}{3}\pi(729) = 972\pi$.

Activity Synthesis

For each answer, ask students to indicate whether or not they agree. Display the number of students who agree with each answer all to see. Invite someone who agreed with 972π to explain their reasoning. Ask students if they think they know what the other students did incorrectly to get their answers. (To get 108, Han and Jada likely used r^2 instead of r^3, and Tyler probably didn't realise that multiplying $\frac{4}{3}r^3$ by π is necessary.)

21.2 Sphere's Radius

Optional: 5 minutes

The purpose of this activity is for students to think about how to find the radius of a sphere when its volume is known. Students can examine the structure of the equation for volume and reason about a number that makes the equation true. They can also notice that π is a factor on each side of the equation and divide each side by π. Both strategies simplify the solution process and minimise the need for rounding. Watch for students who substitute a value for π to each side of the equation, who use the structure of the equation to reason about the solution, or who solve another way, so strategies can be shared and compared in the whole-class discussion.

Instructional Routines

- Stronger and Clearer Each Time

Launch

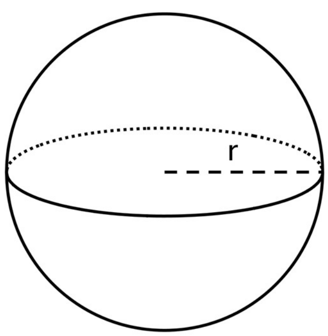
Allow students 3–4 minutes work time followed by a whole-class discussion.

Writing, Speaking: Stronger and Clearer Each Time. Use this routine to give students a structured opportunity to revise and refine their response to the question, "What is the value of r for this sphere? Explain how you know." Ask each student to meet with 2–3 other partners in a row for feedback. Provide students with prompts for feedback that will help them to strengthen their ideas and clarify their language (e.g., "What did you do first?", "Why did you…?", and "How did you deal with r^3?", etc.). Students can borrow ideas and language from each partner to strengthen their final version.
Design Principle(s): Optimise output (for explanation)

Anticipated Misconceptions

Students who substituted a value for π and solved the resulting equation might have rounded along the way, making the value for the radius slightly less than 6 while the actual value is exactly 6. This is a good opportunity to talk about the effects of rounding and how to minimise the error that rounding introduces.

Student Task Statement



The volume of this sphere with radius r is $V = 288\pi$. This statement is true:

$288\pi = \frac{4}{3}r^3\pi$. What is the value of r for this sphere? Explain how you know.

Student Response

6 units. Explanations vary. Sample responses: Examine the equation $288\pi = \frac{4}{3}r^3\pi$. π appears on both sides of the equation. This means that the remaining factors on each side must be equal, so $288 = \frac{4}{3}r^3$. Multiplying each side by $\frac{3}{4}$ gives $216 = r^3$. The number 6 yields 216 when cubed.

Activity Synthesis

The purpose of the discussion is to examine how students reasoned through each step in solving for the unknown radius. Ask previously identified students to share their responses.

Consider asking students the following questions to help clarify the different approaches students took:

- "π appears on both sides of the volume equation. Did you deal with this as a first step or later in the solution process?"
- "How did you deal with the fraction in the equation?"
- "If the final step in your solution was solving for r when $r^3 = 216$, how did you solve? If you found that r^3 was a different number, how did you solve?"

Engagement: Develop Effort and Persistence. Break the class into small discussion groups and then invite a representative from each group to report back to the whole class. All group members should be prepared to share if invited.
Supports accessibility for: Attention; Social-emotional skills

21.3 Info Gap: Unknown Dimensions

20 minutes

In this info gap activity, students determine and request the information needed to answer questions related to volume equations of cylinders, cones, and spheres.

The info gap structure requires students to make sense of problems by determining what information is necessary, and then to ask for information they need to solve it. This may take several rounds of discussion if their first requests do not yield the information they need. It also allows them to refine the language they use and ask increasingly more precise questions until they get the information they need.

Instructional Routines

- Information Gap Cards

Launch

Tell students that they will continue to refresh their skills of working with proportional relationships. Explain the Info Gap structure and consider demonstrating the protocol if students are unfamiliar with it.

Arrange students in groups of 2. In each group, distribute a problem card to one student and a data card to the other student. After you review their work on the first problem, give them the cards for the second problem and instruct them to switch roles.

Engagement: Develop Effort and Persistence. Display or provide students with a physical copy of the written directions. Check for understanding by inviting students to rephrase

directions in their own words. Keep the display of directions visible throughout the activity.
Supports accessibility for: Memory; Organisation Conversing: This activity gives students a purpose for discussing information necessary solve problems involving the volume of shapes. Display questions or question starters for students who need a starting point such as: "Can you tell me . . . (specific piece of information)", and "Why do you need to know . . . (that piece of information)?"
Design Principle(s): Cultivate Conversation

Anticipated Misconceptions

Students may have a difficulty making sense of the relationships between the dimensions of the two shapes. Encourage these students to sketch the shapes and label them carefully.

Student Task Statement

Your teacher will give you either a *problem card* or a *data card*. Do not show or read your card to your partner.

If your teacher gives you the *problem card*:

1. Silently read your card and think about what information you need to be able to answer the question.
2. Ask your partner for the specific information that you need.
3. Explain how you are using the information to solve the problem.

   Continue to ask questions until you have enough information to solve the problem.
4. Share the *problem card* and solve the problem independently.
5. Read the *data card* and discuss your reasoning.

If your teacher gives you the *data card*:

1. Silently read your card.
2. Ask your partner *"What specific information do you need?"* and wait for them to *ask* for information.

   If your partner asks for information that is not on the card, do not do the calculations for them. Tell them you don't have that information.
3. Before sharing the information, ask "*Why do you need that information?*" Listen to your partner's reasoning and ask clarifying questions.
4. Read the *problem card* and solve the problem independently.
5. Share the *data card* and discuss your reasoning.

Pause here so your teacher can review your work. Ask your teacher for a new set of cards and repeat the activity, swapping roles with your partner.

Student Response

Problem card 1: The volume of the sphere is 288π cm^3. Students can calculate this value by finding the radius of the cone and then using the volume formula for a sphere, or they can use the fact that the volume of a sphere is twice that of the volume of a cone with the same dimensions.

Problem Card 2: Possible solution paths:

- Use the volume of the cone and $h = 2r$ to find $r = 3$, so the volume of the sphere is 36π cm^3.
- Since the radius and height of the cone and sphere are equal, the volume of the sphere must be twice the volume of the cone, or 36π cm^3.

Activity Synthesis

Select several groups to share their answers and reasoning for each of the problems. If any students made sketches with labelled dimensions, display these for all to see. In particular, contrast students who used volume formulae versus those who remembered that the volume of a cone is half the volume of a sphere with the same dimensions (radius and height).

Consider asking these discussion questions:

- For students who had a Problem Card:
  - "How did you decide what information to ask for? How did the information on your card help?"
  - "How easy or difficult was it to explain why you needed the information you were asking for?"
  - "Give an example of a question that you asked, the clue you received, and how you made use of it."
  - "How many questions did it take for you to be able to solve the problem? What were those questions?"
  - "Was anyone able to solve the problem with a different set of questions?"
- For students who had a Data Card:
  - "When you asked your partner why they needed a specific piece of information, what kind of explanations did you consider acceptable?"
  - "Were you able to tell from their questions what volume question they were trying to answer? If so, how? If not, why might that be?"

21.4 The Right Fit

10 minutes

In this activity, students once again consider different shapes with given dimensions, this time comparing their capacity to contain a certain amount of water. The goal is for students to not only apply the correct volume formulae, but to slow down and think about how the dimensions of the shapes compare and how those measurements affect the volume of the shapes.

Instructional Routines

- Discussion Supports

Launch

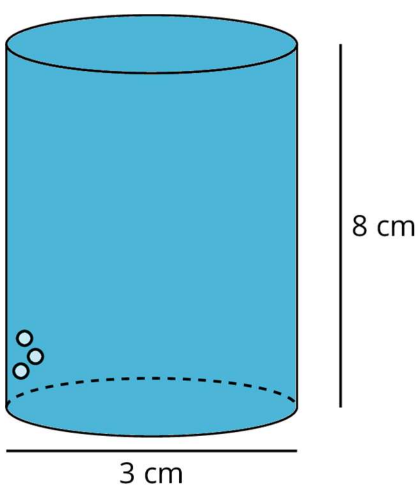
Arrange students in groups of 2. Allow 5 minutes of quiet work time, then a partner discussion followed by a whole-class discussion.

Representation: Internalise Comprehension. Provide appropriate reading accommodations and supports to ensure students access to written directions, word problems and other text-based content.
Supports accessibility for: Language; Conceptual processing Speaking: Discussion Supports. Provide sentence frames to support students as they work with their partner and explain their reasoning. For example, "I think ____ , because _____ .", or "I (agree/disagree) because ______ .", or "The volume is (greater than/less than) _____ , which means _____ ."
Design Principle(s): Support sense-making; Optimise output for (explanation)

Anticipated Misconceptions

Students might think that doubling a dimension doubles the volume. Help students recall and reason that changes in the radius, because it is squared or cubed when calculating volume, have a greater affect.

Student Task Statement



A cylinder with diameter 3 centimetres and height 8 centimetres is filled with water. Decide which shapes described here, if any, could hold all of the water from the cylinder. Explain your reasoning.

1. Cone with a height of 8 centimetres and a radius of 3 centimetres.

2. Cylinder with a diameter of 6 centimetres and height of 2 centimetres.

3. Cuboid with a length of 3 centimetres, width of 4 centimetres, and height of 8 centimetres.

4. Sphere with a radius of 2 centimetres.

Student Response

The cone, cylinder, and cuboid can hold the water. The sphere cannot. Explanations vary. Sample response:

The volume of the given cylinder is 18π, or around 56.5 cubic centimetres: ($V = \pi\left(\frac{3}{2}\right)^2(8) = \pi\left(\frac{9}{4}\right)(8) = 18\pi$).

1. Cone: $V = \frac{1}{3}\pi(3^2)(8) = \frac{1}{3}\pi(9)(8) = 24\pi$. Since the volume is greater than 18π, the cone can hold the water.

2. Cylinder: $V = \pi(3^2)(2) = \pi(9)(2) = 18\pi$. Since the volume is equal to 18π, the cylinder can hold the water.

3. Cuboid: $V = (3)(4)(8) = 96$. Since the volume is greater than 56.5, the cuboid can hold the water.

4. Sphere: $V = \frac{4}{3}\pi(2^3) = \frac{4}{3}\pi(8) = \frac{32}{3}\pi$. Since the volume is less than 18π, the sphere cannot hold the water.

Are You Ready for More?

A thirsty crow wants to raise the level of water in a cylindrical container so that it can reach the water with its beak.

- The container has diameter of 2 inches and a height of 9 inches.
- The water level is currently at 6 inches.
- The crow can reach the water if it is 1 inch from the top of the container.

In order to raise the water level, the crow puts spherical pebbles in the container. If the pebbles are approximately $\frac{1}{2}$ inch in diameter, what is the fewest number of pebbles the crow needs to drop into the container in order to reach the water?

Student Response

The current volume of water is $V_1 = \pi \times 1^2 \times 6$ cubic inches and the volume required is $V_2 = \pi \times 1^2 \times 8$ cubic inches. This means that the difference in volumes is $\pi \times 1^2 \times 2 = 6.28$ cubic inches. Each pebble has volume $V_p = \frac{4}{3} \times \pi \times (\frac{1}{4})^3 = 0.065$ cubic inches. In order to raise the water by a volume of 6.28 cubic inches, $6.28/0.065 = 96.62$ pebbles need to be added. Since fractional pebbles are not possible, the crow needs to add 97 pebbles.

Activity Synthesis

The purpose of this discussion is to compare volumes of different shapes by calculation and also by considering the effect that different dimensions have on volume.

Ask students if they made any predictions about the volumes before directly computing them and, if yes, how they were able to predict. Students might have reasoned, for example, that the second cylinder had double the radius of the first, which would make the volume 4 times as great, but the height was only $\frac{1}{4}$ as great so the volume would be the same. Or they might have seen that the cone would have a greater volume since the radius was double and the height the same, making the volume (if it were another cylinder) 4 times as great, so the factor of $\frac{1}{3}$ for the cone didn't bring the volume down below the volume of the cylinder.

Lesson Synthesis

In this unit, students have learned how to find the volume of cylinders, cones, and spheres, how to find an unknown dimension when the volume and another dimension are known, and how to reason about the effects of different dimensions on volume. Assign groups of 2–3 students one of the questions shown here and provide them with the tools to make a visual display explaining their response. Encourage students to make their displays as though they are explaining the answer to the question to someone who is not in the class and to make up values for dimensions to use to illustrate their ideas. Suggest sketches of shapes where appropriate.

- "Describe some relationships between the volumes of cylinders, cones, and spheres."
- "How do we find a missing dimension when we know the volume and another dimension of a cylinder, cone, or sphere (or just the volume in the case of the sphere)?"
- "What happens to the volume of a cylinder or cone when its height is doubled? Tripled?"
- "What happens to the volume of a sphere when its height is doubled? Tripled?"
- "What happens to the volume of a cylinder, cone, or sphere when its radius is doubled? Tripled?"

- "What happens to the volume of a cylinder or cone when its height is doubled and its radius is halved?"
- "What happens to the volume of a cylinder or cone when its radius is doubled and its height is halved?"

21.5 New Four Spheres

Cool Down: 5 minutes

Students synthesise the material they have learned about using the volume of a sphere formula by sorting different representations of the equation.

Student Task Statement

Some information is given about each sphere. Order them from least volume to greatest volume. You may sketch a sphere to help you visualise if you prefer.

Sphere A: Has a radius of 4

Sphere B: Has a diameter of 6

Sphere C: Has a volume of 64π

Sphere D: Has a radius double that of sphere B.

Student Response

B, C, A, D

Sphere A: Has a radius of 4, so its volume is $\frac{256}{3}\pi$.

Sphere B: Has a diameter of 6, so its radius is 3, and its volume is 36π.

Sphere C: Has a volume of 64π.

Sphere D: Has a radius twice as large as sphere B, so its radius is 6, and its volume is 288π.

Student Lesson Summary

The formula

$$V = \frac{4}{3}\pi r^3$$

gives the volume of a sphere with radius r. We can use the formula to find the volume of a sphere with a known radius. For example, if the radius of a sphere is 6 units, then the volume would be

$$\frac{4}{3}\pi(6)^3 = 288\pi$$

or approximately 904 cubic units. We can also use the formula to find the radius of a sphere if we only know its volume. For example, if we know the volume of a sphere is 36π cubic units but we don't know the radius, then this equation is true:

$$36\pi = \frac{4}{3}\pi r^3$$

That means that $r^3 = 27$, so the radius r has to be 3 units in order for both sides of the equation to have the same value.

Many common objects, from water bottles to buildings to balloons, are similar in shape to cuboids, cylinders, cones, and spheres—or even combinations of these shapes! Using the volume formulae for these shapes allows us to compare the volume of different types of objects, sometimes with surprising results.

For example, a cube-shaped box with side length 3 centimetres holds less than a sphere with radius 2 centimetres because the volume of the cube is 27 cubic centimetres ($3^3 = 27$), and the volume of the sphere is around 33.51 cubic centimetres ($\frac{4}{3}\pi \times 2^3 \approx 33.51$).

Lesson 21 Practice Problems

1. **Problem 1 Statement**

   A scoop of ice cream has a 3-inch diameter. How tall should the ice cream cone of the same diameter be in order to contain all of the ice cream inside the cone?

   Solution

   6 inches (The volume of the ice cream is $\frac{4}{3}\pi(1.5)^3$, which must be the same as the volume of the cone given by $\frac{1}{3}\pi(1.5)^2 h$. Set these two expressions equal to each other and solve for h.)

2. **Problem 2 Statement**

   Calculate the volume of the following shapes with the given information. For the first three questions, give each answer both in terms of π and by using 3.14 to approximate π. Make sure to include units.

   a. Sphere with a diameter of 6 inches

   b. Cylinder with a height of 6 inches and a diameter of 6 inches

   c. Cone with a height of 6 inches and a radius of 3 inches

   d. How are these three volumes related?

   Solution

   a. 36π, about 113.04 cubic inches

b. 54π, about 169.56 cubic inches

c. 18π, about 56.52 cubic inches

d. Answers vary. Sample response: The volume of the cone plus the volume of the sphere equals the volume of the cylinder. This is like the video in a previous lesson where the sphere fits snugly inside of the cylinder, and after pouring a cone of water, the cylinder fills completely to the top. The cone takes up $\frac{1}{3}$ of the cylinder, and the sphere takes up the other $\frac{2}{3}$.

3. **Problem 3 Statement**

A coin-operated bouncy ball dispenser has a large glass sphere that holds many spherical balls. The large glass sphere has a radius of 9 inches. Each bouncy ball has radius of 1 inch and sits inside the dispenser.

If there are 243 bouncy balls in the large glass sphere, what proportion of the large glass sphere's volume is taken up by bouncy balls? Explain how you know.

Solution

About 33% (or $\frac{1}{3}$). The large glass sphere has radius 9 inches, so its volume in cubic inches is $\frac{4}{3}\pi(9)^3$. This is about 3 052 cubic inches. Each bouncy ball has radius 1 inch. The volume of 243 bouncy balls in cubic inches is $243 \times \frac{4}{3}\pi(1)^3$. This is about 1 017 cubic inches. To find the proportion taken up by bouncy balls, divide $\frac{1017}{3052} \approx 0.33$. About 33% of the space is taken up by bouncy balls. By using the exact volume, the result is exactly $\frac{1}{3}$.

4. **Problem 4 Statement**

A farmer has a water tank for cows in the shape of a cylinder with radius of 7 ft and a height of 3 ft. The tank comes equipped with a sensor to alert the farmer to fill it up when the water falls to 20% capacity. What is the volume of the tank when the sensor turns on?

Solution

About 92 cubic feet (The volume of the cylinder is given by $V = \pi \times 7^2 \times 3$, which is approximately 462 cubic feet. The sensor turns on when 20% of this volume remains, and 20% of 462 is a little more than 92 cubic feet.)