Lesson 12: Systems of equations

Goals

- Comprehend that solving a system of equations means finding values of the variables that makes both equations true at the same time.
- Coordinate (orally and in writing) graphs of parallel lines and a system of equations that has no solutions.
- Create a graph of two lines that represents a system of equations in context.

Learning Targets

- I can explain the solution to a system of equations in a real-world context.
- I can explain what a system of equations is.
- I can make graphs to find an ordered pair that two real-world situations have in common.

Lesson Narrative

This lesson formally introduces the concept of **system of equations** with different contexts. Students recognise that they have found solutions to systems of equations using graphing in the past few lessons by examining the intersection of graphed lines. The next activity introduces students to a system that has no solution and asks them to recognise this by connecting the concepts of parallel lines having no intersection points to the algebraic representations having no common solution.

Addressing

- Analyse and solve pairs of simultaneous linear equations.
- Understand that solutions to a system of two linear equations in two variables correspond to points of intersection of their graphs, because points of intersection satisfy both equations simultaneously.
- Solve systems of two linear equations in two variables algebraically, and estimate solutions by graphing the equations. Solve simple cases by inspection. For example, $3x + 2y = 5$ and $3x + 2y = 6$ have no solution because $3x + 2y$ cannot simultaneously be 5 and 6.

Instructional Routines

- Collect and Display
- Compare and Connect
- Think Pair Share

Required Materials

Straightedges

A rigid edge that can be used for drawing line segments. Sometimes a ruler is okay to use as a straightedge, but sometimes it is preferable to use an unruled straightedge, like a blank index card.

Required Preparation

Provide access to straightedges for students to accurately draw graphs of lines.

Student Learning Goals

Let's learn what a system of equations is.

12.1 Milkshakes

Warm Up: 5 minutes

In the warm-up, students are given a situation and asked to describe the graph without actually graphing the lines. Identify students who correctly use mathematically correct terminology such as y-intercept, gradient, x-intercept, and intersection to describe the graph.

Instructional Routines

- Think Pair Share

Launch

Arrange students in groups of 2. Give 3 minutes quiet work time followed by brief partner discussion for the last question.

Student Task Statement

Diego and Lin are drinking milkshakes. Lin starts with 12 ounces and drinks $\frac{1}{4}$ ounce per second. Diego starts with 20 ounces and drinks $\frac{2}{3}$ ounce per second.

1. How long will it take Lin and Diego to finish their milkshakes?
2. Without graphing, explain what the graphs in this situation would look like. Think about gradient, intercepts, axis labels, units, and intersection points to guide your thinking.
3. Discuss your description with your partner. If you disagree, work to reach an agreement.

Student Response

1. Lin will take 48 seconds and Diego will take 30 seconds.

2. The horizontal axis is labelled "time (seconds)" and the vertical axis is labelled "amount of milkshake left (ounces)." The line for Lin's graph starts at (0,12) and decreases to the right to the point (48,0). The line for Diego's graph starts at (0,20) and decreases more steeply than Lin's line to the point (30,0). The two lines intersect somewhere.

3. No response needed.

Activity Synthesis

The purpose of the discussion is for students to practise describing graphs in words using correct mathematical terminology.

Select previously identified students to share their descriptions of the graphs. After each student shares, ask the class if the description is clear and to identify any vocabulary they heard that made the description precise or any vocabulary that is unclear. If any vocabulary needs to be reinforced for student understanding, this is a good time to discuss these words.

12.2 Passing on the Trail

20 minutes (there is a digital version of this activity)

In this activity, students start with an equation relating distance and time for Han's hike and enough information to write a second equation relating distance and time for Jada's hike. After writing Jada's equation and graphing both lines, students then use the lines to identify the point of intersection and make sense of the point's meaning in the context.

This activity is a culmination of student's work writing, solving, and graphing equations along with the thinking they have done on what it means for an equation to be true. From this foundation, students are ready to understand solving systems of equations from an algebraic standpoint in the following lessons. Fluently solving systems algebraically is not expected at this time.

Instructional Routines

- Collect and Display

Launch

Provide students with access to straightedges. Keep students in groups of 2.

Read the context of the problem with the students to help them understand the situation. Consider asking these questions to help them understand:

- "When t is zero, where is Han? Where is Jada?" (Han is at the lake. Jada is 0.6 miles from the car park.)

- "For times shortly after 0, is d decreasing or increasing for Han? Is d decreasing or increasing for Jada?" (Decreasing for Han and increasing for Jada.)

Give 2–3 minutes quiet work time and ask students to pause after they have completed the first problem to discuss their equation with a partner before starting to graph the equations. Give 5–7 minutes for students to complete the remaining problems with their partners followed by a whole-class discussion.

Representation: Internalise Comprehension. Provide appropriate reading accommodations and supports to ensure students access to written directions, word problems, and other text-based content.
Supports accessibility for: Language; Conceptual processing

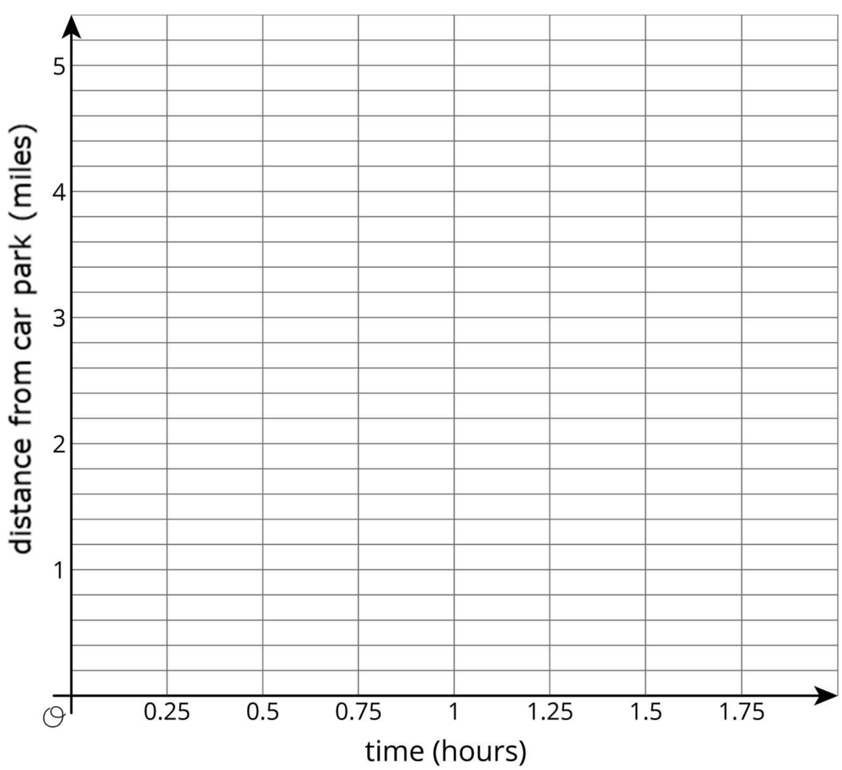
Student Task Statement

There is a hiking trail near the town where Han and Jada live that starts at a car park and ends at a lake. Han and Jada both decide to hike from the car park to the lake and back, but they start their hikes at different times.

At the time that Han reaches the lake and starts to turn back, Jada is 0.6 miles away from the car park and hiking at a constant speed of 3.2 miles per hour towards the lake. Han's distance, d, from the car park can be expressed as $d = -2.4t + 4.8$, where t represents the time in hours since he left the lake.

1. What is an equation for Jada's distance from the car park as she heads toward the lake?

2. Draw both graphs: one representing Han's equation and one representing Jada's equation. It is important to be very precise! Be careful, work in pencil, and use a ruler.



3. Find the point where the two graphs intersect each other. What are the coordinates of this point?

4. What do the coordinates mean in this situation?

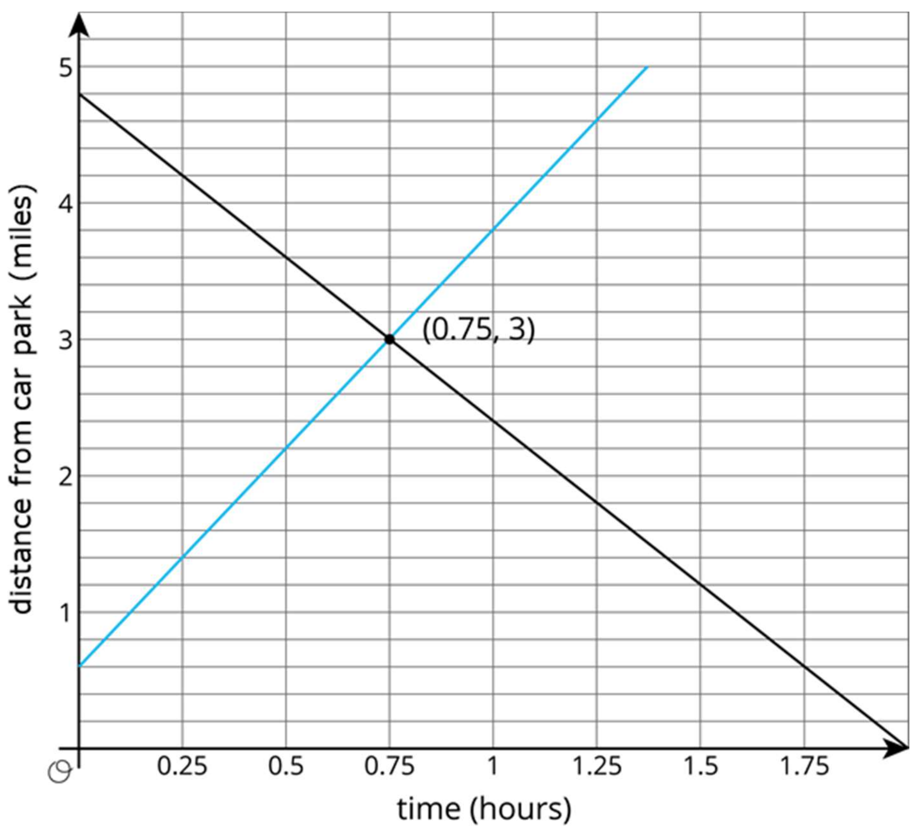
5. What has to be true about the relationship between these coordinates and Jada's equation?

6. What has to be true about the relationship between these coordinates and Han's equation?

Student Response

1. $d = 3.2t + 0.6$ or equivalent.

2.



3. (0.75,3)

4. 0.75 hours or 45 minutes after Han left the lake, he passed Jada on the trail. This happened at a distance of 3 miles from the car park.

5. These values of t and d make Jada's equation true.

6. These values of t and d also make Han's equation true.

Activity Synthesis

The purpose of this discussion is to strengthen the connection between graphs and equations and formally introduce the vocabulary for **systems of equations**.

Begin the discussion by asking groups to share their responses for the last three questions. Ask students how they could check to make sure the coordinates they found with the graph are correct and give a brief quiet think time before selecting students to share their strategies. While students may suggest ideas like checking to make sure the lines are graphed correctly, it is important to point out that graphs are not always perfect. If not brought up by students, mention that a better strategy would be to substitute the coordinate values in for the variables to see if those values make both equations true.

Display a graph of the two equations for all to see alongside the system of equations:

$$\begin{cases} d = -2.4t + 4.8 \\ d = 3.2t + 0.6 \end{cases}$$

Explain to students that this is called a **system of equations**, and "solving a system of equations" means to find the values of the variables that make both equations true at the same time. Point out that in this problem, the solution to the system of equations is the point where Han and Jada are at the same distance from the car park at the same time (point to (0.75,3)). Tell students that it is also possible to solve a system of equations without graphing. To find when Jada and Han are the same distance away from the car park, we can set the expression for Jada's distance equal to the expression for Han's distance to get the equation $-2.4t + 4.8 = 3.2t + 0.6$, which is an equation that can be solved for t. Ask students to solve this equation and confirm that $t = 0.75$, which is the same value they found earlier by carefully graphing the lines of each equation. Emphasise that the intersection point gave a value of both t *and* d, so it is important when solving algebraically to substitute t back into one of the equations to find the value for d. Since Han and Jada are at the same distance from the car park when $t = 0.75$, it doesn't matter which equation is used to find the value of d.

Speaking, Listening: Collect and Display. Listen for and record language students use to discuss the question "How can you check to make sure the coordinates found are correct?" Organise and group similar strategies in the display for students to refer back to throughout the lesson. For instance, group strategies that focus on checking whether the lines are graphed accurately and strategies that refer to substituting the coordinate points for the variables. Emphasise words or phrases referring to these strategies such as "rate of change," "gradient," "initial value," "y-intercept," "substitute," and "both equations are true." Use this opportunity to introduce the term "system of equations." This will help students solidify their understanding on how to check their solutions to a system of equations.
Design Principle(s): Support sense-making; Maximise meta-awareness

12.3 Stacks of Cups

10 minutes (there is a digital version of this activity)

Students explore a system of equations with no solutions in the familiar context of cup stacking. The context reinforces a discussion about what it means for a system of equations to have no solutions, both in terms of a graph and in terms of the equations. Over the next few lessons, the concept of one solution, no solutions, and infinitely many solutions will be abstracted to problems without context. In those situations, it may be useful to refer back to the context in this activity and others as a way to guide students towards abstraction.

Instructional Routines

- Compare and Connect

Launch

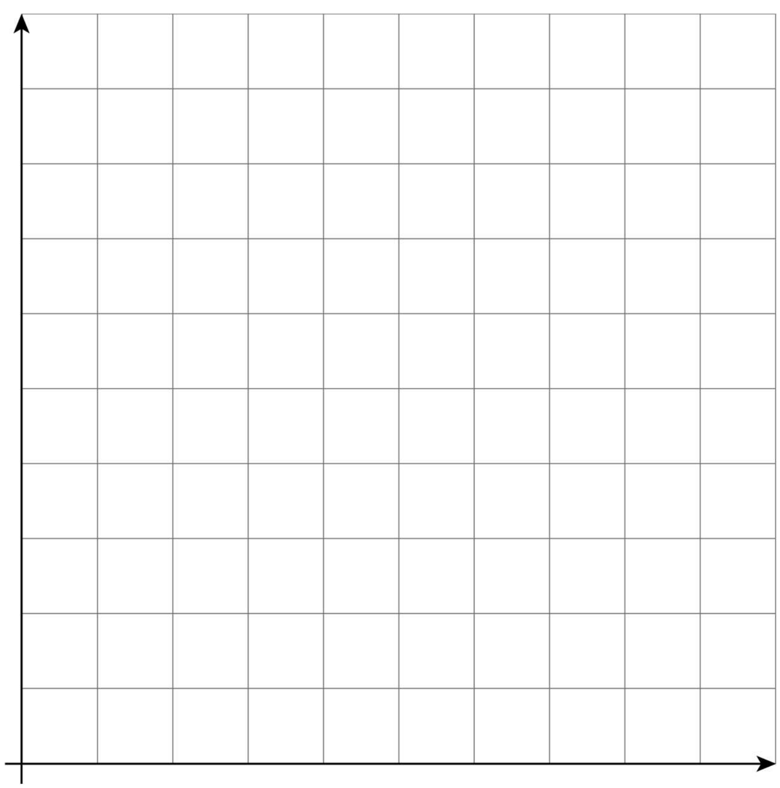
5–7 minutes of quiet work time followed by a whole-class discussion.

Representation: Internalise Comprehension. Check in with students after the first 2–3 minutes of work time. Check to make sure students have attended to all parts of the graph and have selected an appropriate scale.
Supports accessibility for: Conceptual processing; Organisation

Student Task Statement

A stack of n small cups has a height, h, in centimetres of $h = 1.5n + 6$. A stack of n large cups has a height, h, in centimetres of $h = 1.5n + 9$.

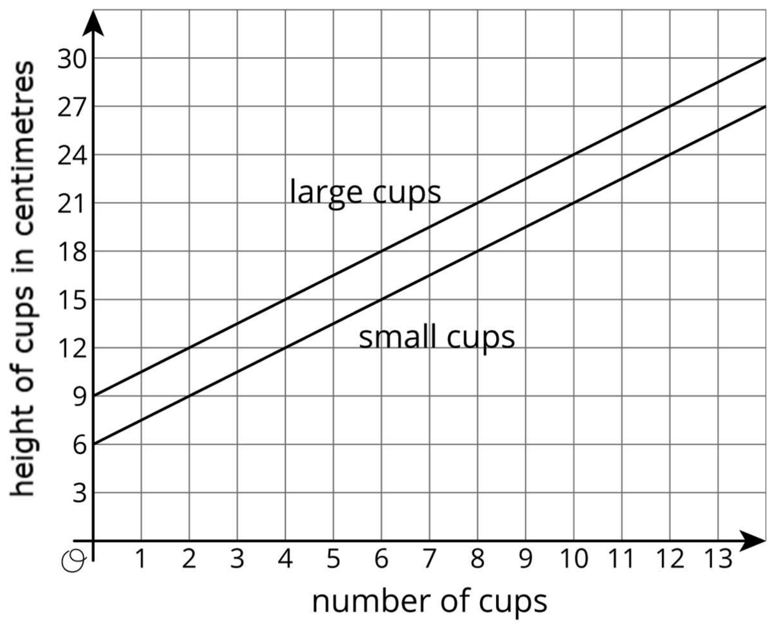
1. Graph the equations for each cup on the same set of axes. Make sure to label the axes and decide on an appropriate scale.



2. For what number of cups will the two stacks have the same height?

Student Response

1.



2. There is no number of cups for which the two stacks have the same height.

Activity Synthesis

The key point for discussion is to connect what students observed about the graph to the concept of "no solutions" from earlier lessons. Graphically, students see that the lines are parallel and always separated by a distance of 3 cm. This means the stack of n large cups will always be 3 cm taller than a stack of n small cups. Connecting this to the equations, this means that there is no value of n that is a solution to both $1.5n + 6$ and $1.5n + 9$ at the same time. By the end of the discussion, students should understand that the following are equivalent:

- The lines don't intersect.
- The lines are parallel.
- There is no value of n for which the stacks have the same height.
- There is no value of n that makes $1.5n + 6 = 1.5n + 9$ true.

Invite students to explain how they used the graph or equations to answer the second question. Ask other students if they answered the question with a different line of reasoning. If not brought up by students, demonstrate that setting the expression for the height of the large cup $1.5n + 9$ equal to the expression for the height of the small cup

$1.5n + 6$ and subtracting $1.5n$ from both sides gives $6 = 9$, which is false no matter what value of n is used.

Representing, Conversing: Compare and Connect. Before the whole-class discussion, use this routine to give students an opportunity to explain how they used the graph or equations to answer the question, "For what number of cups will the two stacks have the same height?" Invite students to demonstrate their strategy using a visual or numerical representation. Display one example of each representation to discuss. Invite students to compare their strategies with a partner. Ask students to discuss how their strategies are the same or different, and then share with the whole class. This will help students connect different strategies that lead to the same conclusion of "no solution" for a system of equations.
Design Principle(s): Optimise output; Maximise meta-awareness

Lesson Synthesis

To highlight some of the main concepts from the lesson, ask:

- "Suppose Jada and Han had met up with another person at the exact same time they met each other along their hikes."
  - "What might the graph look like that represents that person's distance from the car park over time?" (There are an infinite number of lines, but they all pass must through the same intersection point as the lines for Jada and Han.)
  - "What information is known and what information might you need to write an equation representing their distance from the car park?" (I would need to know either something about the speed of the third person or their distance from the car park at another point in time.)
- "What is a system of equations?" (Two or more equations for which you want to find values for all of the variables so that all of the equations are true.)
- "What does the solution to a system of equations represent?" (The values for all of the variables that make all of the equations true.)

12.4 Milkshakes, Revisited

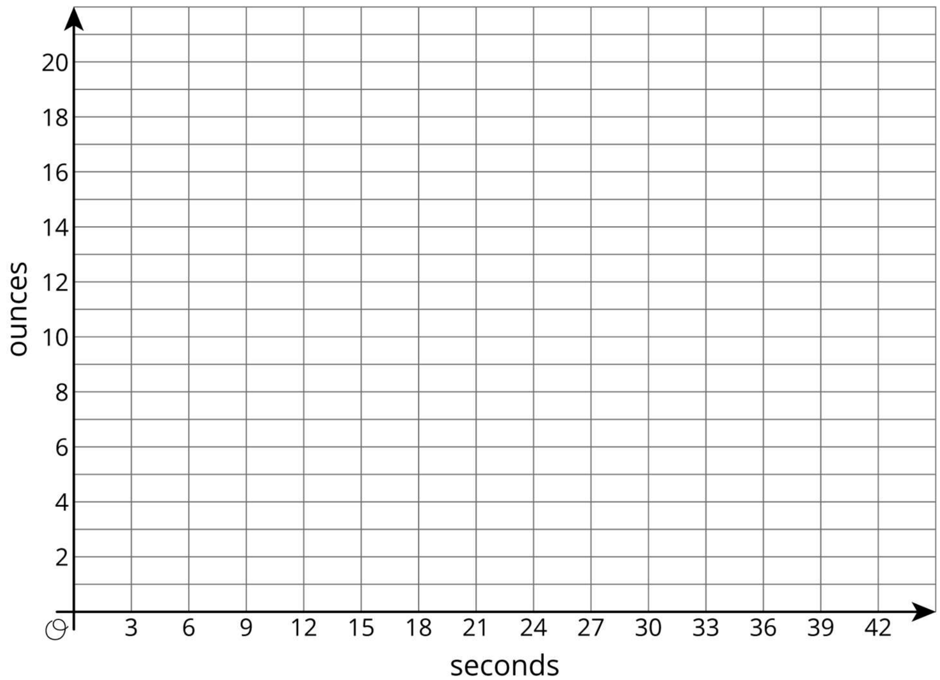
Cool Down: 5 minutes

This cool-down assesses student ability to graph lines based on descriptions and interpret any intersections in the context of the problem.

Student Task Statement

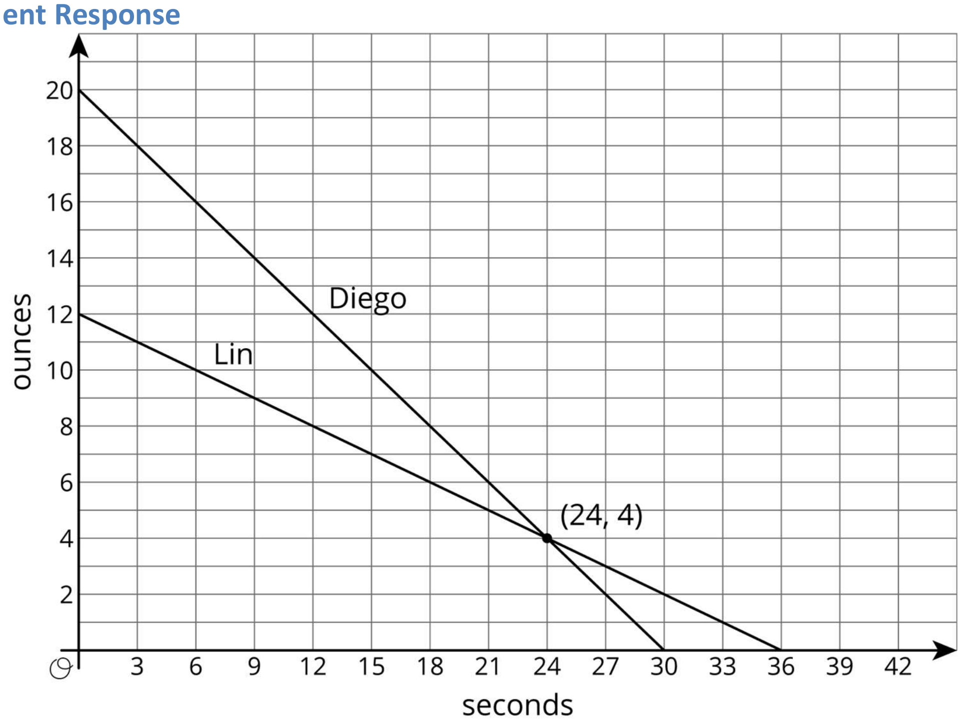
Determined to finish her milkshake before Diego, Lin now drinks her 12 ounce milkshake at a rate of $\frac{1}{3}$ an ounce per second. Diego starts with his usual 20 ounce milkshake and drinks at the same rate as before, $\frac{2}{3}$ an ounce per second.

1. Graph this situation on the axes provided.



2. What does the graph tell you about the situation and how many solutions there are?

Student Response



1.

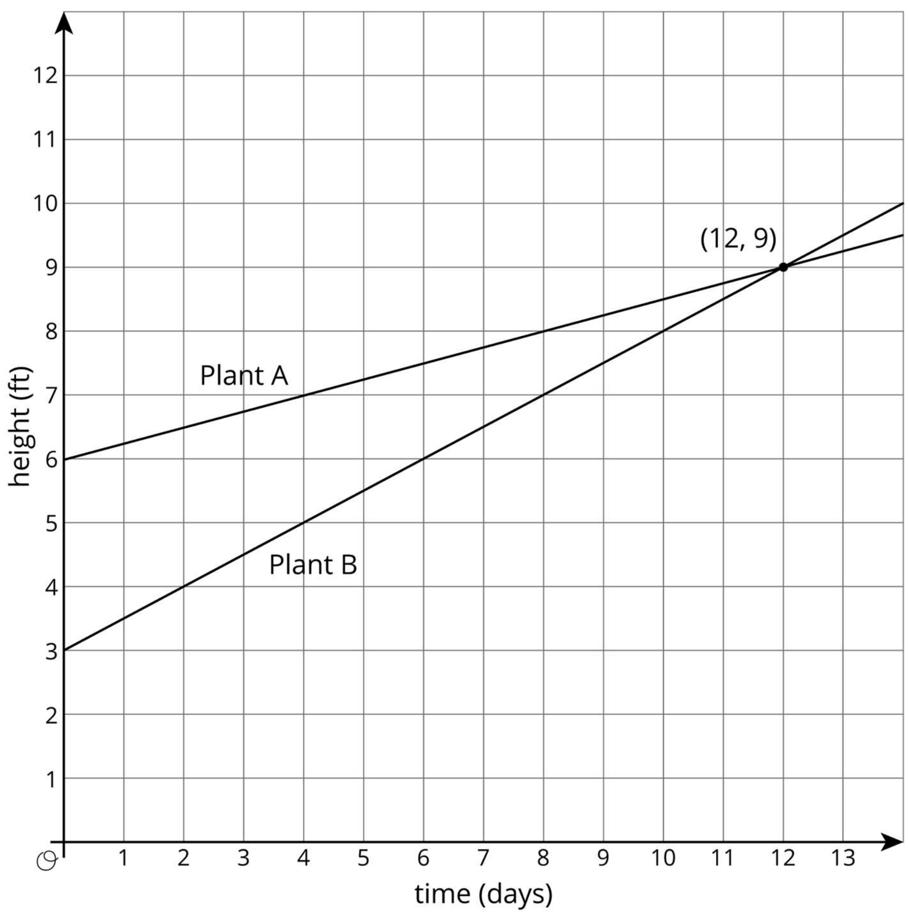
2. There is one solution at (24,4) meaning that after 24 seconds both of them had 4 ounces of milkshake left. Diego still finishes his milkshake first.

Student Lesson Summary

A **system of equations** is a set of 2 (or more) equations where the variables represent the same unknown values. For example, suppose that two different kinds of bamboo are planted at the same time. Plant A starts at 6 ft tall and grows at a constant rate of $\frac{1}{4}$ foot each day. Plant B starts at 3 ft tall and grows at a constant rate of $\frac{1}{2}$ foot each day. We can write equations $y = \frac{1}{4}x + 6$ for Plant A and $y = \frac{1}{2}x + 3$ for Plant B, where x represents the number of days after being planted, and y represents height. We can write this system of equations.

$$\begin{cases} y = \dfrac{1}{4}x + 6 \\ y = \dfrac{1}{2}x + 3 \end{cases}$$

Solving a system of equations means to find the values of x and y that make both equations true at the same time. One way we have seen to find the solution to a system of equations is to graph both lines and find the intersection point. The intersection point represents the pair of x and y values that make both equations true. Here is a graph for the bamboo example:



The solution to this system of equations is $(12,9)$, which means that both bamboo plants will be 9 feet tall after 12 days.

We have seen systems of equations that have no solutions, one solution, and infinitely many solutions.

- When the lines do not intersect, there is no solution. (Lines that do not intersect are *parallel*.)
- When the lines intersect once, there is one solution.
- When the lines are right on top of each other, there are infinitely many solutions.

In future lessons, we will see that some systems cannot be easily solved by graphing, but can be easily solved using algebra.

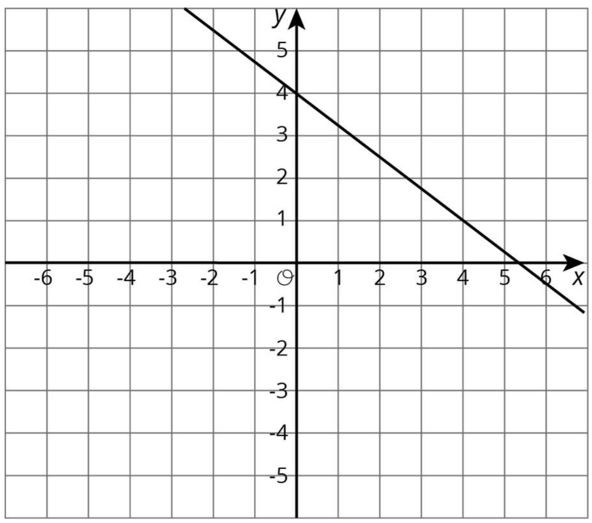
Glossary

- system of equations

Lesson 12 Practice Problems

1. **Problem 1 Statement**

Here is the graph for one equation in a system of equations:



a. Write a second equation for the system so it has infinitely many solutions.

b. Write a second equation whose graph goes through $(0,1)$ so the system has no solutions.

c. Write a second equation whose graph goes through $(0,2)$ so the system has one solution at $(4,1)$.

Solution

a. $y = \frac{-3}{4}x + 4$

b. $y = \frac{-3}{4}x + 1$

c. $y = \frac{-1}{4}x + 2$

2. **Problem 2 Statement**

Create a second equation so the system has no solutions.

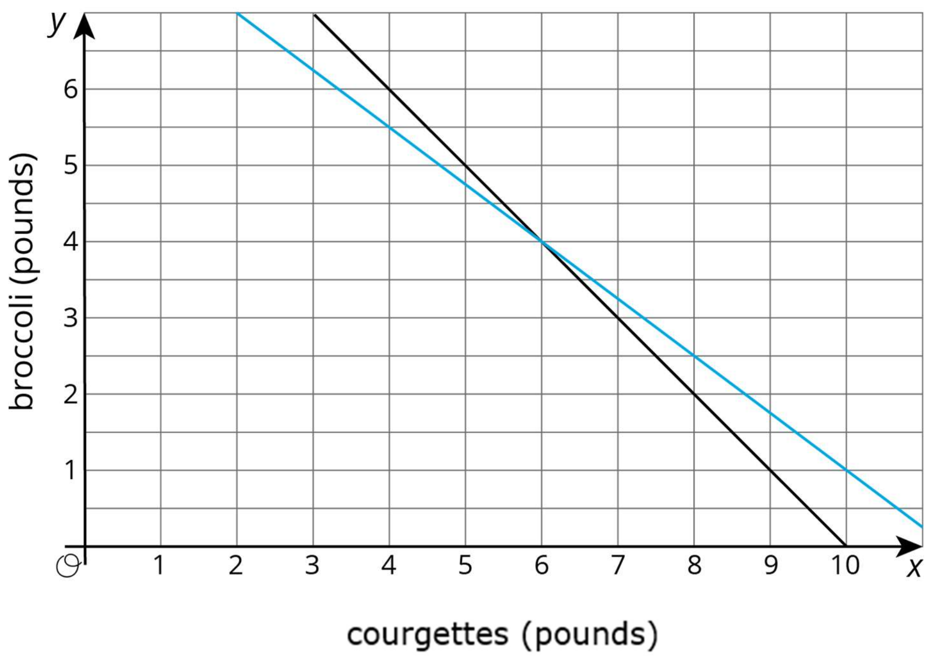
$$\begin{cases} y = \frac{3}{4}x - 4 \\ \end{cases}$$

Solution

Answers vary. Any line of the form $y = \frac{3}{4}x + c$ will make it so the system has no solutions.

3. **Problem 3 Statement**

Andre is in charge of cooking broccoli and courgettes for a large group. He has to spend all £17 he has and can carry 10 pounds of veggies. Courgettes cost £1.50 per pound and broccoli costs £2 per pound. One graph shows combinations of courgettes and broccoli that weigh 10 pounds and the other shows combinations of courgettes and broccoli that cost £17.



a. Name one combination of veggies that weighs 10 pounds but does not cost £17.

b. Name one combination of veggies that costs £17 but does not weigh 10 pounds.

c. How many pounds each of courgettes and broccoli can Andre get so that he spends all £17 and gets 10 pounds of veggies?

Solution

a. Answers vary. Sample response: 4 pounds of courgettes and 6 pounds of broccoli weigh 10 pounds, but do not cost £17 because (4,6) is not on the line of combinations that cost £17.

b. Answers vary. Sample response: 2 pounds of courgettes and 7 pounds of broccoli together cost £17 because (2,7) is on the £17 line, but they only weigh 9 pounds.

c. 6 pounds of courgettes, and 4 pounds of broccoli

4. **Problem 4 Statement**

The temperature in degrees Fahrenheit, F, is related to the temperature in degrees Celsius, C, by the equation $F = \frac{9}{5}C + 32$

a. In the Sahara desert, temperatures often reach 50 degrees Celsius. How many degrees Fahrenheit is this?

b. In parts of Alaska, the temperatures can reach -60 degrees Fahrenheit. How many degrees Celsius is this?

c. There is one temperature where the degrees Fahrenheit and degrees Celsius are the same, so that $C = F$. Use the expression from the equation, where F is expressed in terms of C, to solve for this temperature.

Solution

a. 122 degrees Fahrenheit

b. $-51\frac{1}{9}$ degrees Celsius

c. $C = \frac{9}{5}C + 32$, $C = -40$