Lesson 6: Scaling and area

Goals

- Calculate and compare (orally and in writing) the areas of multiple scaled copies of the same shape.
- Generalise (orally) that the area of a scaled copy is the product of the area of the original shape and the “square” of the scale factor.
- Recognise that a two-dimensional attribute, like area, scales at a different rate than one-dimensional attributes, like length and distance.

Learning Targets

- I can describe how the area of a scaled copy is related to the area of the original shape and the scale factor that was used.

Lesson Narrative

This lesson is optional. In this lesson, students are introduced to how the area of a scaled copy relates to the area of the original shape. Students build on their Year 7 work with exponents to recognise that the area increases by the square of the scale factor by which the sides increased. Students will continue to work with the area of scaled shapes later in this unit and in later units in this course. Although the lesson is optional, it will be particularly helpful for students to have already had this introduction when they study the area of circles in a later unit.

In two of the activities in this lesson, students build scaled copies using pattern blocks as units of area. This work with manipulatives helps accustom students to a pattern that many find counterintuitive at first. (It is a common but false assumption that the area of scaled copies increases by the same scale factor as the sides.) After that, students calculate the area of scaled copies of parallelograms and triangles to apply the patterns they discovered in the hands-on activities.

Building On

- Find the area of right-angled triangles, other triangles, special quadrilaterals, and polygons by composing into rectangles or decomposing into triangles and other shapes; apply these techniques in the context of solving real-world and mathematical problems.

Addressing

- Solve problems involving scale drawings of geometric shapes, including computing actual lengths and areas from a scale drawing and reproducing a scale drawing at a different scale.

- Solve real-world and mathematical problems involving area, volume and surface area of two- and three-dimensional objects composed of triangles, quadrilaterals, polygons, cubes, and right prisms.

Building Towards

- Know the formulas for the area and circumference of a circle and use them to solve problems; give an informal derivation of the relationship between the circumference and area of a circle.
- Solve real-world and mathematical problems involving area, volume and surface area of two- and three-dimensional objects composed of triangles, quadrilaterals, polygons, cubes, and right prisms.
- Decide whether two quantities are in a proportional relationship, e.g., by testing for equivalent ratios in a table or graphing on a coordinate plane and observing whether the graph is a straight line through the origin.

Instructional Routines

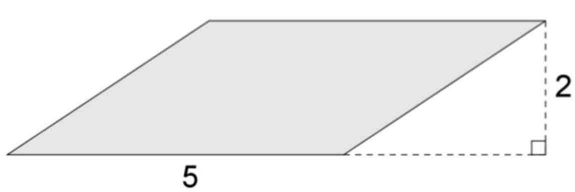
- Anticipate, Monitor, Select, Sequence, Connect
- Compare and Connect
- Discussion Supports
- Think Pair Share

Required Materials

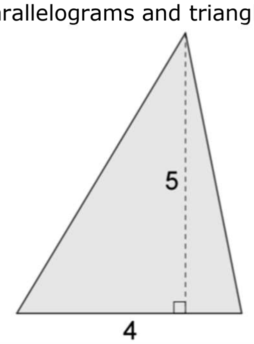
Geometry toolkits
tracing paper, graph paper, coloured pencils, scissors, and an index card to use as a straightedge or to mark right angles, plus a ruler and protractor. Clear protractors with no holes and with radial lines printed on them are recommended.

Pattern blocks
Pre-printed slips, cut from copies of the blackline master

Area of scaled parallelograms and triangles



Area of scaled parallelograms and triangles



Required Preparation

Prepare to distribute the pattern blocks, at least 16 blue rhombuses, 16 green triangles, 10 red trapeziums, and 7 yellow hexagons per group of 3–4 students.

Copy and cut up the Area of Scaled Parallelograms and Triangles blackline master so each group of 2 students can get 1 of the 2 shapes.

Student Learning Goals

Let's build scaled shapes and investigate their areas.

6.1 Scaling a Pattern Block

Warm Up: 10 minutes (there is a digital version of this activity)

By now, students understand that lengths in a scaled copy are related to the original lengths by the scale factor. Here they see that the area of a scaled copy is related to the original area by the *square* of the scale factor.

Students build scaled copies of a single pattern block, using blocks of the same shape to do so. They determine how many blocks are needed to create a copy at each specified scale factor. Each pattern block serves as an informal unit of area. Because each original shape has an area of 1 block, the $(\text{scale factor})^2$ pattern for the area of a scaled copy is easier to recognise.

Students use the same set of scale factors to build copies of three different shapes (a rhombus, a triangle, and a hexagon). They notice regularity in their repeated reasoning and use their observations to predict the number of blocks needed to build other scaled copies.

Launch

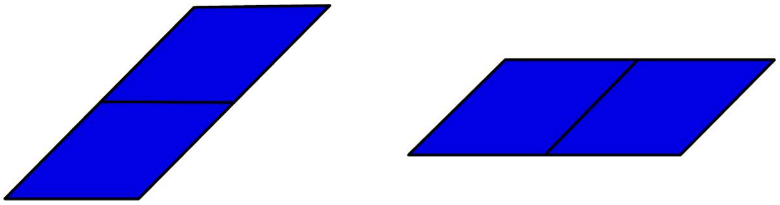
Arrange students in groups of 3–4. Distribute pattern blocks and ask students to use them to build scaled copies of each shape as described in the task. Each group would need at most 16 blocks each of the green triangle, the blue rhombus, and the red trapezium. If there are not enough for each group to have a full set with 16 each of the green, blue, and red blocks, consider rotating the blocks of each colour through the groups, or having students start with 10 blocks of each and ask for more as needed.

Give students 6–7 minutes to collaborate on the task and follow with a whole-class discussion. Make sure all students understand that “twice as long” means “2 times as long.”

Using real pattern blocks is preferred, but the Digital Activity can replace the manipulatives if they are unavailable.

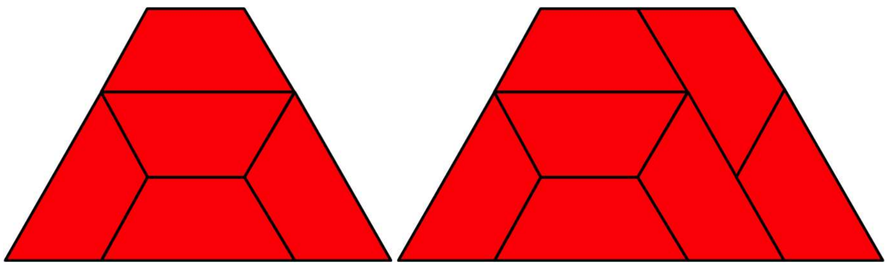
Anticipated Misconceptions

Some students may come up with one of these arrangements for the first question, because they assume the answer will take 2 blocks to build:



You could use one pattern block to demonstrate measuring the lengths of the sides of their shape, to show them which side they have not doubled.

Students may also come up with:

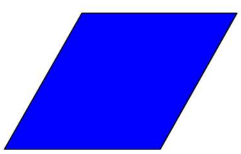


for tripling the trapezium, because they triple the height of the scaled copy but they do not triple the length. You could use the process described above to show that not all side lengths have tripled.

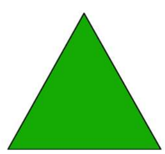
Student Task Statement

Your teacher will give you some pattern blocks. Work with your group to build the scaled copies described in each question.

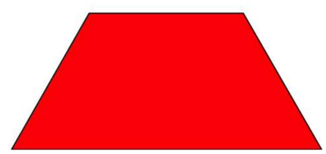
A



B



C



1. How many blue rhombus blocks does it take to build a scaled copy of shape A:
    a. Where each side is twice as long?
    b. Where each side is 3 times as long?
    c. Where each side is 4 times as long?
2. How many green triangle blocks does it take to build a scaled copy of shape B:
    a. Where each side is twice as long?
    b. Where each side is 3 times as long?
    c. Using a scale factor of 4?
3. How many red trapezium blocks does it take to build a scaled copy of shape C:

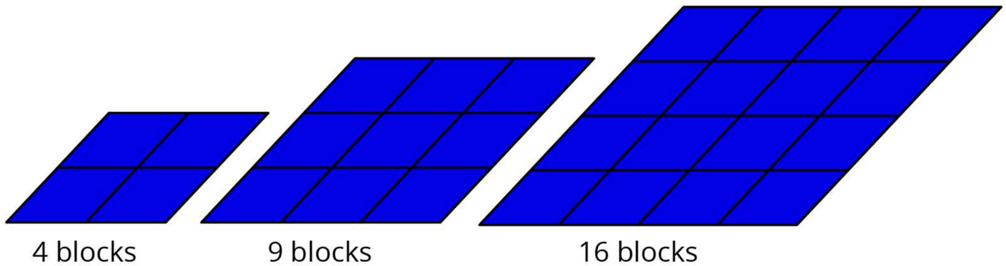
a. Using a scale factor of 2?

b. Using a scale factor of 3?

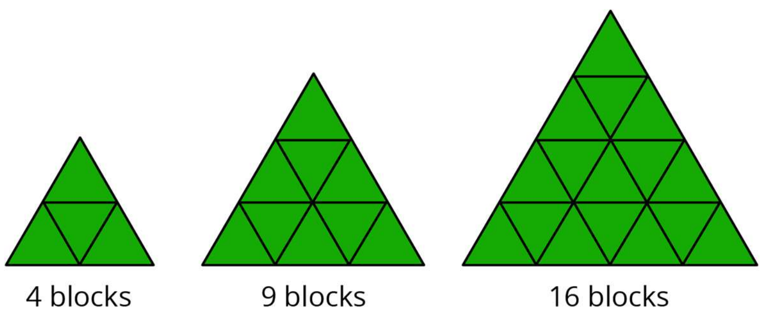
c. Using a scale factor of 4?

Student Response

1.



2.



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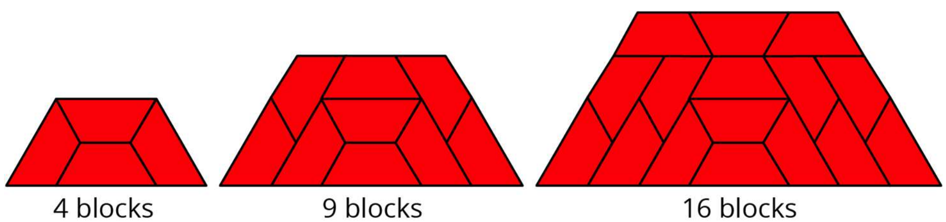


Activity Synthesis

Display a table with only the column headings filled in. For the first four rows, ask different students to share how many blocks it took them to build each shape and record their answers in the table.

scale factor	number of blocks to build shape A	number of blocks to build shape B	number of blocks to build shape C
1			
2			

3			
4			
5			
10			
s			
$\frac{1}{2}$			

To help student notice, extend, and generalise the pattern in the table, guide a discussion using questions such as these:

- In the table, how is the number of blocks related to the scale factor? Is there a pattern?
- How many blocks are needed to build scaled copies using scale factors of 5 or 10? How do you know?
- How many blocks are needed to build a scaled copy using any scale factor s?
- If we want a scaled copy where each side is half as long, how much of a block would it take? How do you know? Does the same rule still apply?

If not brought up by students, highlight the fact that the number of blocks it took to build each scaled shape equals the scale factor times itself, regardless of the shape (look at the table row for s). This rule applies to any factor, including those that are less than 1.

6.2 Scaling More Pattern Blocks

Optional: 10 minutes (there is a digital version of this activity)

This activity extends the conceptual work of the previous one by adding a layer of complexity. Here, the original shapes are comprised of more than 1 block, so the number of blocks needed to build their scaled copies is not simply $(\text{scale factor})^2$, but rather $n \times (\text{scale factor})^2$, where n is the number of blocks in the original shape. Students begin to think about how the scaled area relates to the original area, which is no longer 1 area unit. They notice that the pattern $(\text{scale factor})^2$ presents itself in the factor by which the original number of blocks has changed, rather than in the total number of blocks in the copy.

As in the previous task, students observe regularity in repeated reasoning, noticing that regardless of the shapes, starting with n pattern blocks and scaling by s uses ns^2 pattern blocks.

Also as in the previous task, the shape composed of trapeziums might be more challenging to scale than those composed of rhombuses and triangles. Prepare to support students scaling the red shape by offering some direction or additional time, if feasible.

As students work, monitor for groups who notice that the pattern of squared scale factors still occurs here, and that it is apparent if the original number of blocks is taken into account. Select them to share during class discussion.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Discussion Supports

Launch

Keep students in the same groups, or form combined groups if there are not enough blocks. Assign one shape for each group to build (or let groups choose a shape, as long as all 3 shapes are equally represented). To build a copy of each given shape using a scale factor of 2, groups will need 12 blue rhombuses, 8 red trapeziums, or 16 green triangles. To completely build a copy of each given shape with a scale factor of 3, they would need 27 blue rhombuses, 18 red trapeziums, and 36 green triangles; however, the task prompts them to stop building when they know what the answer will be.

Give students 6–7 minutes to build their shapes and complete the task. Remind them to use the same blocks as those in the original shape and to check the side lengths of each built shape to make sure they are properly scaled.

Using real pattern blocks is preferred, but the Digital Activity can replace the manipulatives if they are unavailable.

Action and Expression: Internalise Executive Functions. Chunk this task into more manageable parts to support students who benefit from support with organisation in problem solving. For example, present one question at a time and monitor students to ensure they are making progress throughout the activity.
Supports accessibility for: Organisation; Attention

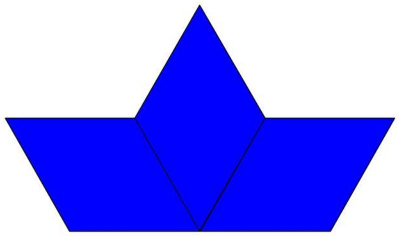
Anticipated Misconceptions

Students may forget to check that the lengths of all sides of their shape have been scaled and end with an inaccurate count of the pattern blocks. Remind them that all line segments must be scaled by the same factor.

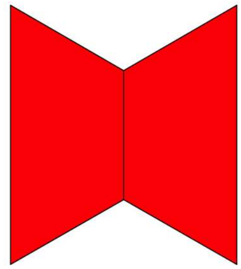
Student Task Statement

Your teacher will assign your group one of these shapes.

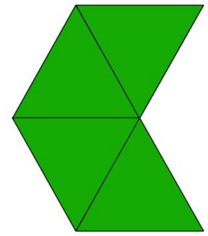
D



E



F



1. Build a scaled copy of your assigned shape using a scale factor of 2. Use the same shape blocks as in the original shape. How many blocks did it take?

2. Your classmate thinks that the scaled copies in the previous problem will each take 4 blocks to build. Do you agree or disagree? Explain you reasoning.

3. Start building a scaled copy of your assigned shape using a scale factor of 3. Stop when you can tell for sure how many blocks it would take. Record your answer.

4. How many blocks would it take to build scaled copies of your shape using scale factors 4, 5, and 6? Explain or show your reasoning.

5. How is the pattern in this activity the same as the pattern you saw in the previous activity? How is it different?

Student Response

1. Answers vary based on the original shape: 12 blue rhombuses, 8 red trapeziums, or 16 green triangles.

2. Each block in the pattern is replaced by 4 blocks in the scaled copy. There is more than one block in each pattern so the scaled copies of the patterns require more than 4 blocks.

3. Answers vary based on the original shape: 27 blue rhombuses, 18 red trapeziums, or 36 green triangles.

4. Blue rhombuses needed for scaled copies of shape D: $3 \times 4^2 = 48$, $3 \times 5^2 = 75$, $3 \times 6^2 = 108$. Red trapeziums needed for scaled copies of shape E: $2 \times 4^2 = 32$, $2 \times 5^2 = 50$, $2 \times 6^2 = 72$. Green triangles needed for scaled copies of shape F: $4 \times 4^2 = 64$, $4 \times 5^2 = 100$, $4 \times 6^2 = 144$.

5. At first glance, the pattern does not seem the same because the answers are not 4 and 9. However, each individual block still scales by 4 and then 9, so you have to multiply that by the number of blocks in the original shape to get the number of blocks in the scaled copy.

Are You Ready for More?

1. How many blocks do you think it would take to build a scaled copy of one yellow hexagon where each side is twice as long? Three times as long?

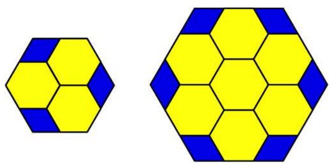
2. Figure out a way to build these scaled copies.

3. Do you see a pattern for the number of blocks used to build these scaled copies? Explain your reasoning.

Student Response

1. It should take 4 blocks and 9 blocks following the pattern for the other shapes.

2. Answers vary. Sample response: here is a way to build the scaled copies using yellow hexagons and blue rhombuses:



3. The pattern does not work if you only count the number of blocks; however, it does work if you consider the size of each block being used. The first hexagon took 6 blocks to build: 3 yellow hexagons and 3 blue rhombuses, but 3 blue rhombuses cover the same area as 1 yellow hexagon, so the size of the scaled copy is equivalent to 4 yellow hexagons, because $3 + \frac{3}{3} = 4$. Similarly, the total size of the scaled copy with scale factor 3 is equivalent to 9 yellow hexagons.

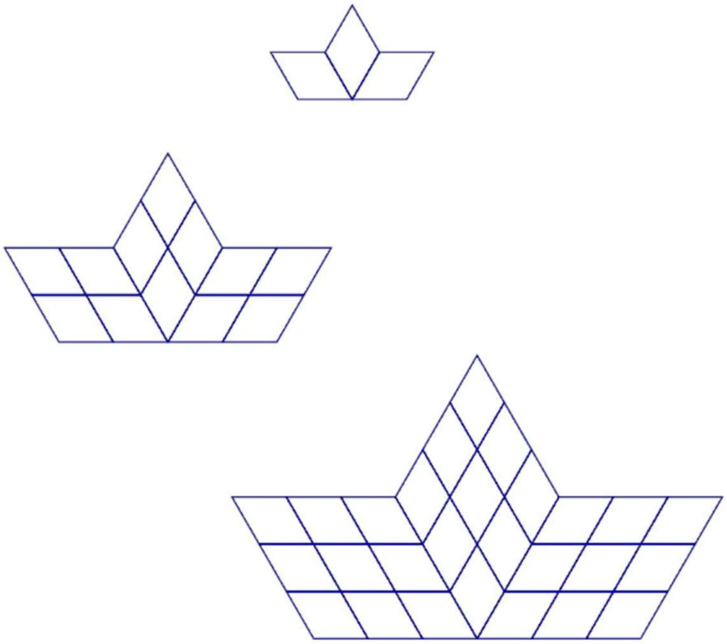
Activity Synthesis

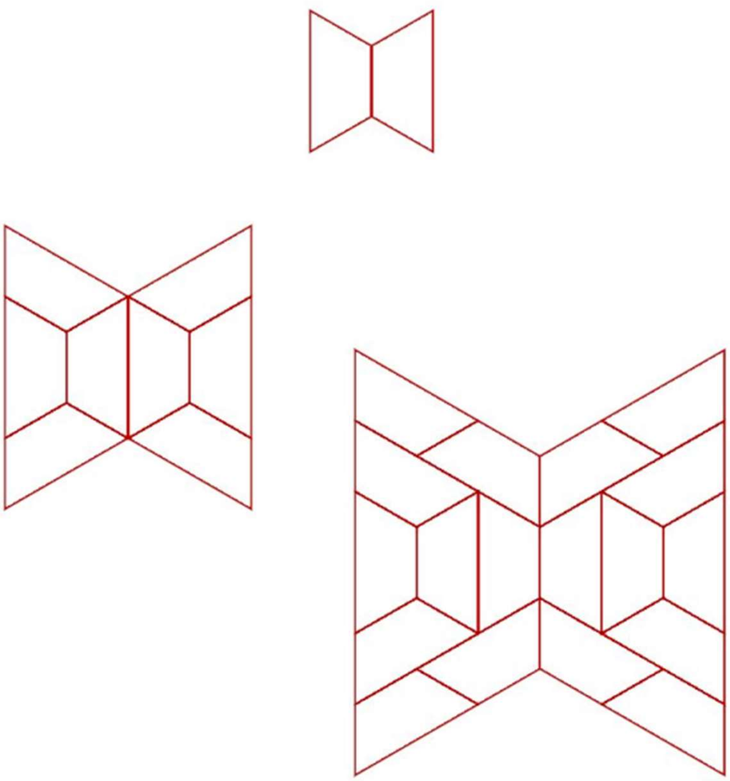
The goal of this discussion is to ensure that students understand that the pattern for the number of blocks in the scaled copies depends *both* on the scale factor and on the number of blocks in the pattern.

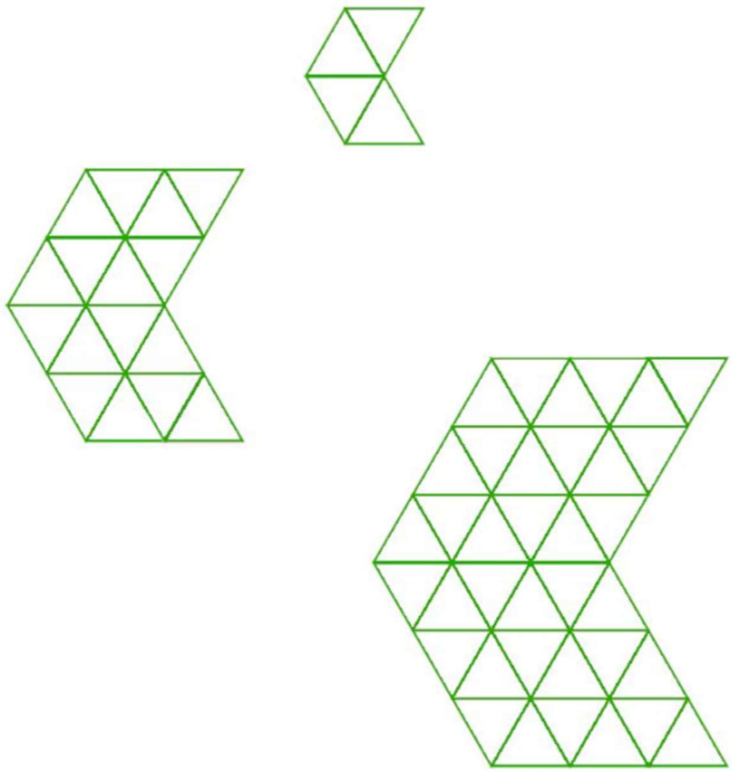
Display a table with only the column headings filled in. Poll the class on how many blocks it took them to build each scaled copy using the factors of 2 and 3. Record their answers in the table.

scale factor	number of blocks to build shape D	number of blocks to build shape E	number of blocks to build shape F
1	3	2	4
2			
3			
4			
5			
6			
s			

Consider displaying the built shapes or pictures of them for all to see.







Invite selected students to share the pattern that their groups noticed and used to predict the number of blocks needed for copies with scale factors 4, 5, and 6. Record their predictions in the table. Discuss:

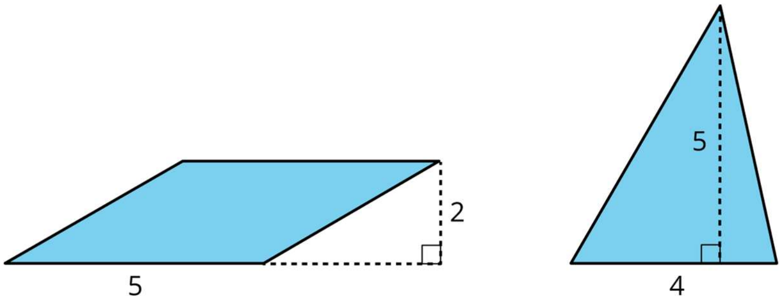
- How does the pattern for the number of blocks in this activity compare to the pattern in the previous activity? Are they related? How?
- For each shape, how many blocks does it take to build a copy using any scale factor s?

Speaking: Discussion Supports. Give students additional time to make sure that everyone in the group can describe the patterns they noticed and the ways they predicted the number of blocks needed for copies with scale factors 4, 5, and 6. Vary who is called on to represent the ideas of each group. This routine will provide students additional opportunities to prepare for and share their thinking publicly.
Design Principle(s): Optimise output (for explanation)

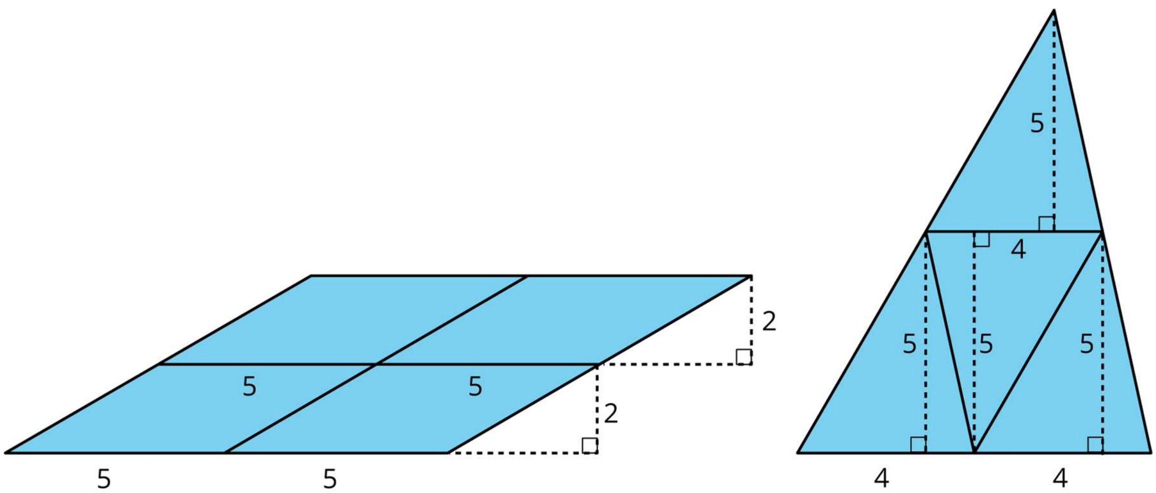
6.3 Area of Scaled Parallelograms and Triangles

Optional: 15 minutes

In this activity, students transfer what they learned with the pattern blocks to calculate the area of other scaled shapes. In groups of 2, students draw scaled copies of either a parallelogram or a triangle and calculate the areas. Then, each group compares their results with those of a group that worked on the other shape. They find that the scaled areas of two shapes are the same (even though the starting shapes are different and have different measurements) and attribute this to the fact that the two shapes had the same original area and were scaled using the same scale factors.



While students are not asked to reason about scaled areas by tiling (as they had done in the previous activities), each scaled copy can be tiled to illustrate how length measurements have scaled and how the original area has changed. Some students may choose to draw scaled copies and think about scaled areas this way.



As students find the areas of copies with scale factors 5 and $\frac{3}{5}$ without drawing (for the last question), monitor for these methods, depending on their understanding of or comfort with the (scale factor)^2 pattern:

- Scaling the original base and height and then multiplying to find the area
- Multiplying the original area by the square of the scale factor

Select students using each approach. Invite them to share their reasoning, sequenced in this order, during the discussion.

You will need the Area of Scaled Parallelograms and Triangles blackline master for this activity.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect

- Compare and Connect
- Think Pair Share

Launch

Arrange students in groups of 2. Provide access to geometry toolkits.

Distribute slips showing the parallelogram to half the groups and the triangle to the others. Give students 1 minute of quiet work time for the first question, and then time to complete the rest of the task with their partner.

Engagement: Develop Effort and Persistence. Connect a new concept to one with which students have experienced success. For example, draw students attention to the warm-up and remind them how to build scaled copies of the rhombus with scale factors of 2, 3, and 4. Ask students how they can use this technique to draw scaled copies of the parallelogram or triangle with scale factors of 2, 3, and 5.
Supports accessibility for: Social-emotional skills; Conceptual processing

Anticipated Misconceptions

Students may not remember how to calculate the area of parallelograms and triangles. Make sure that they have the correct area of 10 square units for their original shape before they calculate the area of their scaled copies.

When drawing their scaled copies, some students might not focus on making corresponding angles equal. As long as they scale the base and height of their polygon correctly, this will not impact their area calculations. If time permits, however, prompt them to check their angles using tracing paper or a protractor.

Some students might focus unnecessarily on measuring other side lengths of their polygon, instead of attending only to base and height. If time is limited, encourage them to scale the base and height carefully and check or measure the angles instead.

Student Task Statement

1. Your teacher will give you a shape with measurements in centimeters. What is the **area** of your shape? How do you know?

2. Work with your partner to draw scaled copies of your shape, using each scale factor in the table. Complete the table with the measurements of your scaled copies.

scale factor	base (cm)	height (cm)	area (cm^2)
1			
2			
3			
$\frac{1}{2}$			

$\frac{1}{3}$			

3. Compare your results with a group that worked with a different shape. What is the same about your answers? What is different?

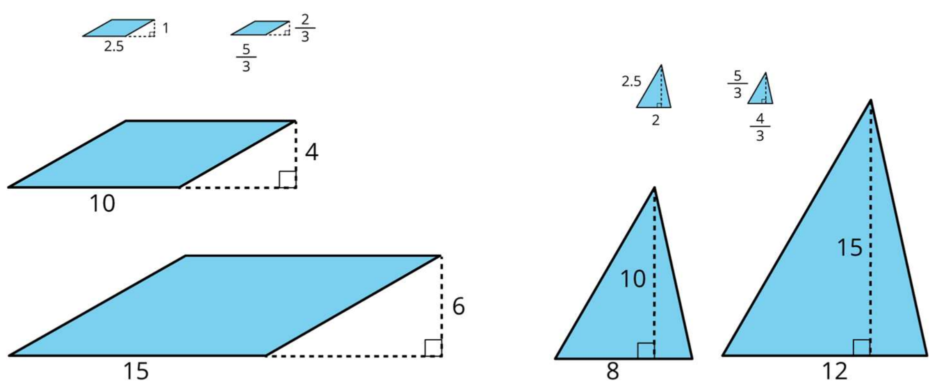
4. If you drew scaled copies of your shape with the following scale factors, what would their areas be? Discuss your thinking. If you disagree, work to reach an agreement. Be prepared to explain your reasoning.

scale factor	area (cm^2)
5	
$\frac{3}{5}$	

Student Response

1. The area of either shape is 10 square units, because $5 \times 2 = 10$ and $\frac{1}{2} \times 4 \times 5 = 10$.

2.



For the parallelogram:

scale factor	base	height	area
1	5	2	10
2	10	4	40
3	15	6	90
$\frac{1}{2}$	2.5	1	2.5
$\frac{1}{3}$	$\frac{5}{3}$	$\frac{2}{3}$	$\frac{10}{9}$

For the triangle:

scale factor	base	height	area
1	4	5	10
2	8	10	40
3	12	15	90
$\frac{1}{2}$	2	2.5	2.5
$\frac{1}{3}$	$\frac{4}{3}$	$\frac{5}{3}$	$\frac{10}{9}$

3. The areas are the same for each scale factor, even though the dimensions are different. Specifically, the bases of the parallelograms are equal to the heights of the triangles.

scale factor	area
5	250
$\frac{3}{5}$	3.6

Activity Synthesis

Invite selected students to share their solutions. Then focus class discussion on two themes: how the values in the tables for the two shapes compare, and how students determined the scaled areas for the scale factors 5 and $\frac{3}{5}$. Ask questions such as:

- What did you notice when you compared your answers with another group that worked with the other shape? (When the scale factors are the same, the scaled areas are the same, though the bases and heights are different.)
- How did you find the scaled areas for scale factors of 5 and $\frac{3}{5}$? (By scaling the original base and height and multiplying the scaled measurements; by multiplying the original area by $(\text{scale factor})^2$.)
- How is the process for finding scaled area here the same as and different than that in the previous activities with pattern blocks? (The area units are different; the pattern of squaring the scale factor is the same.)

Highlight the connection between the two ways of finding scaled areas. Point out that when we multiply the base and height each by the scale factor and then multiply the results, we are essentially multiplying the original lengths by the scale factor two times. The effect of this process is the same as multiplying the original area by $(\text{scale factor})^2$.

Representing, Speaking, and Listening: Compare and Connect. Invite students to prepare a display that shows their approach to finding the areas for scale factors of 5 and $\frac{3}{5}$. Ask students to research how other students approached the problem, in search of a method

that is different from their own. Challenge students to describe why the different approaches result in the same answers. During the whole-class discussion, emphasise the language used to explain the different strategies, especially phrases related to "squaring" and "multiplying a number by itself." This will strengthen students' mathematical language use and reasoning based on the relationship between scale factors and area.
Design Principle(s): Maximise meta-awareness

Lesson Synthesis

- If all the dimensions of a scaled copy are twice as long as in the original shape, will the area of the scaled copy be twice as large? (No)
  - Why not? (Both the length and the width get multiplied by 2, so the area gets multiplied by 4.)
- If the scale factor is 5, how many times larger will the scaled copy's area be? (25 times larger)

6.4 Enlarged Areas

Cool Down: 5 minutes

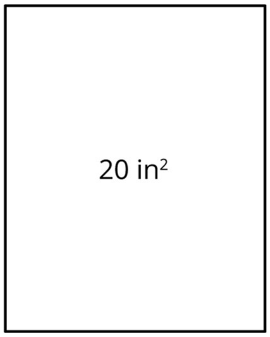
The first question gives students only the area of the original shape—but not the dimensions—to encourage them to find the area of the scaled copy by multiplying by the $(\text{scale factor})^2$; however, students can also choose a length and a width for the rectangle that would give the correct original area, and then scale those dimensions by the scale factor to calculate the area. The second question only asks students to find the $(\text{scale factor})^2$, but not to multiply by it.

Anticipated Misconceptions

Some students may multiply the original shape's area by just the scale factor, instead of by the $(\text{scale factor})^2$, getting 80 in^2. Students who do not understand the generalised rule for how scaling affects area might still be able to answer the first question correctly. They could assume some dimensions for the original rectangle that would give it an area of 20 in^2, scale those dimensions by the given scale factor, and then multiply those scaled dimensions to find the new area.

Student Task Statement

1. Lin has a drawing with an area of 20 in^2. If she increases all the sides by a scale factor of 4, what will the new area be?



2. Noah enlarged a photograph by a scale factor of 6. The area of the enlarged photo is how many times as large as the area of the original?

Student Response

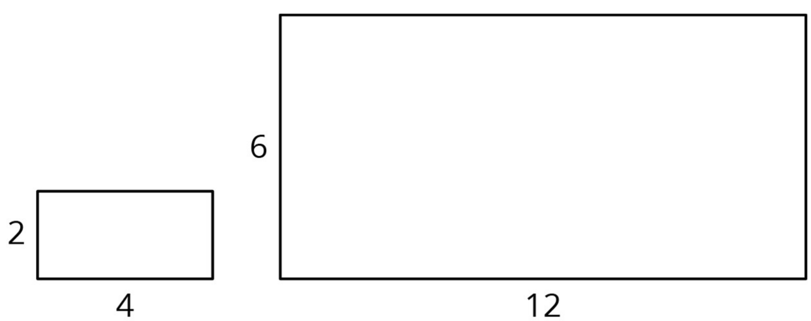
1. 320 in^2, Possible strategies:

   - $20 \times 4^2 = 320$
   - If the rectangle is 4 inches by 5 inches, the scaled copy will be 4×4 inches by 4×5 inches and $(4 \times 4) \times (4 \times 5) = 16 \times 20 = 320$.
   - If the rectangle is 2 inches by 10 inches, the scaled copy will be 4×2 inches by 4×10 inches and $(4 \times 2) \times (4 \times 10) = 8 \times 40 = 320$.

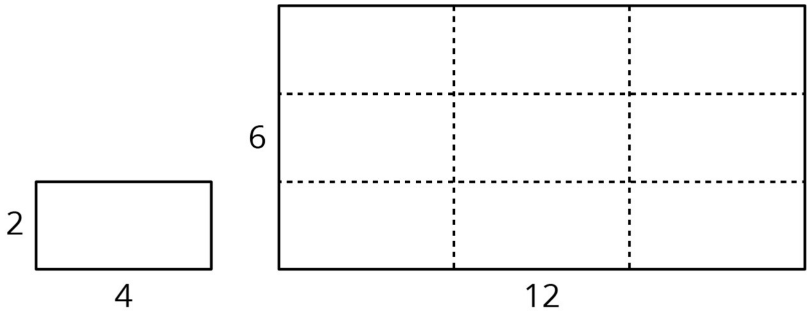
2. 36 times as large, because $6^2 = 36$.

Student Lesson Summary

Scaling affects lengths and areas differently. When we make a scaled copy, all original lengths are multiplied by the scale factor. If we make a copy of a rectangle with side lengths 2 units and 4 units using a scale factor of 3, the side lengths of the copy will be 6 units and 12 units, because $2 \times 3 = 6$ and $4 \times 3 = 12$.



The area of the copy, however, changes by a factor of (scale factor)^2. If each side length of the copy is 3 times longer than the original side length, then the area of the copy will be 9 times the area of the original, because 3×3, or 3^2, equals 9.



In this example, the area of the original rectangle is 8 units^2 and the area of the scaled copy is 72 units^2, because $9 \times 8 = 72$. We can see that the large rectangle is covered by 9 copies of the small rectangle, without gaps or overlaps. We can also verify this by multiplying the side lengths of the large rectangle: $6 \times 12 = 72$.

Lengths are one-dimensional, so in a scaled copy, they change by the scale factor. Area is two-dimensional, so it changes by the *square* of the scale factor. We can see this is true for a rectangle with length l and width w. If we scale the rectangle by a scale factor of s, we get a rectangle with length $s \times l$ and width $s \times w$. The area of the scaled rectangle is $A = (s \times l) \times (s \times w)$, so $A = (s^2) \times (l \times w)$. (Commutative and associative properties.) The fact that the area is multiplied by the square of the scale factor is true for scaled copies of other two-dimensional shapes too, not just for rectangles.

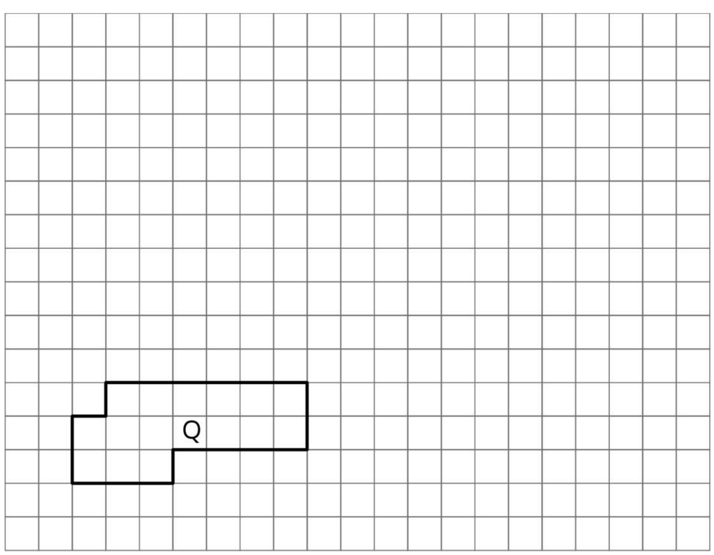
Glossary

- area

Lesson 6 Practice Problems

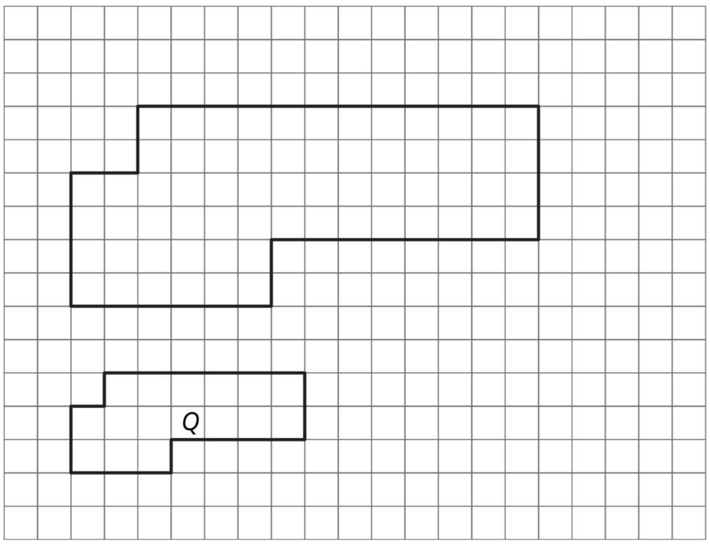
Problem 1 Statement

On the grid, draw a scaled copy of polygon Q using a scale factor of 2. Compare the perimeter and area of the new polygon to those of Q.



Solution

The perimeter of Q is 20 units, and the area of Q is 16 square units. The perimeter of the scaled copy is 40 units, and its area is 64 square units. The perimeter is multiplied by the scale factor of 2, and the area is multiplied by the square of the scale factor, which is 4.



Problem 2 Statement

A right-angled triangle has an area of 36 square units.

If you draw scaled copies of this triangle using the scale factors in the table, what will the areas of these scaled copies be? Explain or show your reasoning.

scale factor	area (units^2)
1	36
2	
3	
5	
$\frac{1}{2}$	
$\frac{2}{3}$	

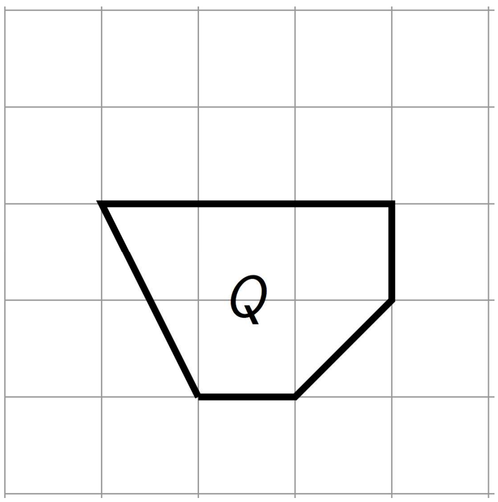
Solution

The area of each scaled triangle is the same as the area of the original triangle (36 square units) multiplied by the square of the scale factor:

scale factor	area (units^2)
1	36
2	144
3	324
5	900
$\frac{1}{2}$	9
$\frac{2}{3}$	16

Problem 3 Statement

Diego drew a scaled version of a polygon P and labelled it Q.



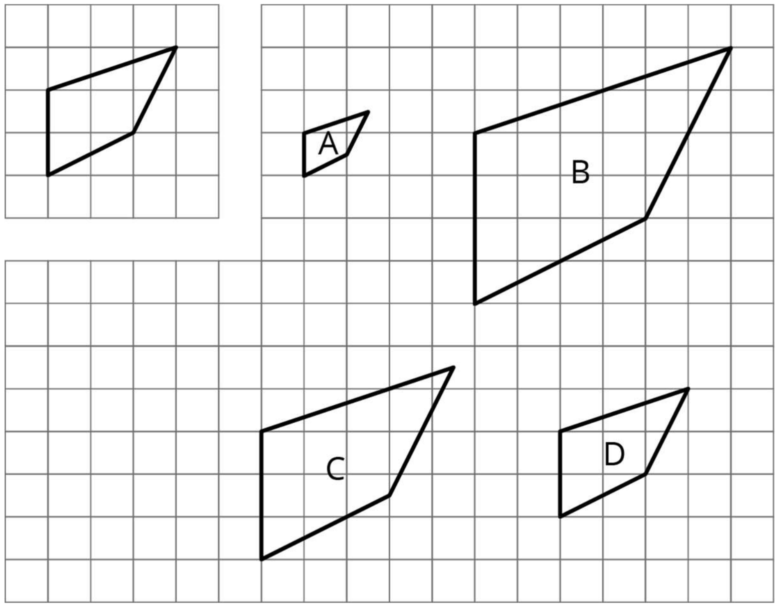
If the area of polygon P is 72 square units, what scale factor did Diego use to go from P to Q? Explain your reasoning.

Solution

$\frac{1}{4}$. The area of Q is 4.5 square units (3 whole square units, one 2 unit by 1 unit right-angled triangle, and one 1 unit by 1 unit right-angled triangle). This area is $\frac{1}{16}$ of the area of P. This means the scale factor is $\frac{1}{4}$ because $\frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$.

Problem 4 Statement

Here is an unlabelled polygon, along with its scaled copies polygons A–D. For each copy, determine the scale factor. Explain how you know.



Solution

a. $\frac{1}{2}$ because the vertical side on the copy is $\frac{1}{2}$ the length of the vertical side on the original

b. 2 because the vertical side on the copy is twice the length of the vertical side on the original

c. $\frac{3}{2}$ because the vertical side on the copy is $\frac{3}{2}$ the length of the vertical side on the original

d. 1 because the original and the copy have the same size

Problem 5 Statement

Solve each equation mentally.

a. $\frac{1}{7} \times x = 1$

b. $x \times \frac{1}{11} = 1$

c. $1 \div \frac{1}{5} = x$

Solution

a. $x = 7$

b. $x = 11$

c. $x = 5$