Lesson 7: Simulating multi-step experiments

Goals

- Coordinate (orally) a real-world situation and a chance event that could be used to simulate that situation.
- Perform a multi-step simulation, and use the results to estimate the probability of a compound event in a real-world situation (using words and other representations).

Learning Targets

- I can use a simulation to estimate the probability of a multi-step event.

Lesson Narrative

In this lesson, students see that compound events can be simulated by using multiple chance experiments. In this case, it is important to communicate precisely what represents one outcome of the simulation. For example, if we want to know the probability that a family with three children will have at least one girl, we can toss one coin to represent each child and use each set of three coin tosses to represent one family. Therefore, if we toss a coin 30 times, we will have run this simulation only 10 times.

Students continue to consider how a real-world situation can be represented using simulation.

Addressing

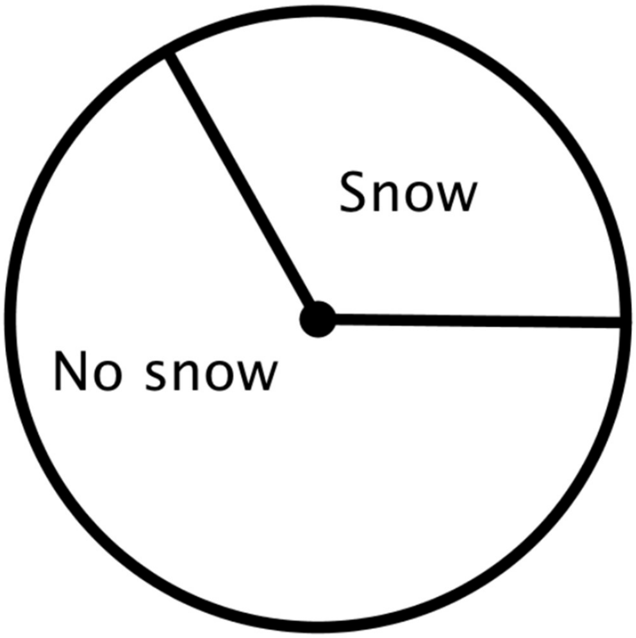
- Analyse proportional relationships and use them to solve real-world and mathematical problems.
- Design and use a simulation to generate frequencies for compound events. For example, use random digits as a simulation tool to approximate the answer to the question: If 40% of donors have type A blood, what is the probability that it will take at least 4 donors to find one with type A blood?

Instructional Routines

- Clarify, Critique, Correct
- Co-Craft Questions
- Discussion Supports
- Notice and Wonder
- Take Turns

Required Materials

Paper bags
Paper clips
Pre-printed slips, cut from copies of the blackline master



Multi-link cubes

Required Preparation

Print and cut up spinners from the Alpine Zoom blackline master. One spinner for each group of 3 students.

For the Kiran's Game activity, a paper bag containing 4 multi-link cubes (2 black and 2 white) is needed for every 3 students.

Other simulation tools (dice, bags with coloured multi-link cubes, etc.) should be available.

Student Learning Goals

Let's simulate more complicated events.

7.1 Notice and Wonder: Ski Business

Warm Up: 5 minutes

The purpose of this warm-up is to elicit ideas that will be useful in the discussions in this lesson. While students may notice and wonder many things about these images, the business side of skiing and its dependence on weather are the important discussion points.

Instructional Routines

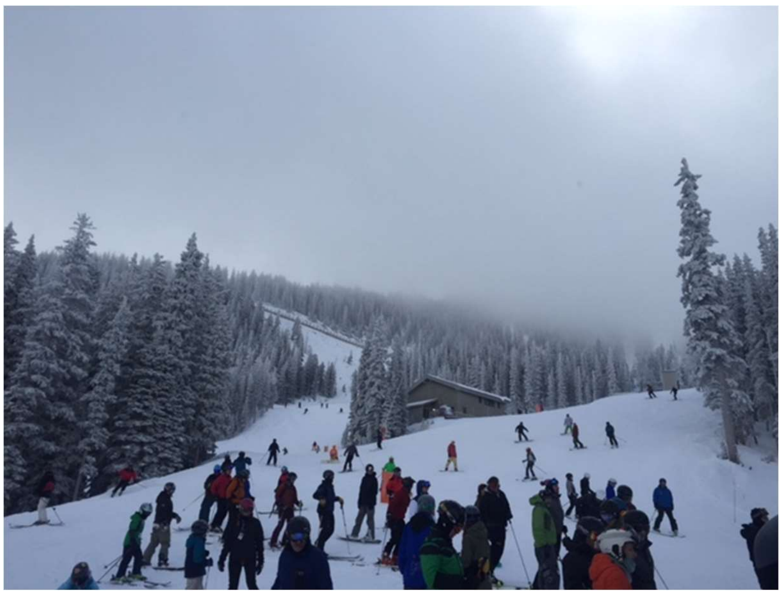
- Notice and Wonder

Launch

Arrange students in groups of 2. Tell students that they will look at two images, and their job is to think of at least one thing they notice and at least one thing they wonder. Display the images for all to see. Ask students to give a signal when they have noticed or wondered about something. Give students 1 minute of quiet think time, and then 1 minute to discuss the things they notice with their partner, followed by a whole-class discussion.

Student Task Statement

What do you notice? What do you wonder?



Student Response

Things students may notice:

- I notice that the business depends on the weather, which is not always predictable.
- I notice that they probably do not have artificial snow and rely on the weather.
- I notice there are a lot of people at this ski place.

Things students may wonder:

- I wonder what the chance of snow is.
- I wonder what they do if it doesn't snow.
- I wonder how long the skiing season is.

Activity Synthesis

Ask students to share the things they noticed and wondered. Record and display their responses for all to see. If possible, record the relevant reasoning on or near the image. After each response, ask the class whether they agree or disagree and to explain alternative ways of thinking, referring back to the images each time. If the dependence of ski businesses on weather does not come up during the conversation, ask students to discuss this idea.

7.2 Alpine Zoom

15 minutes (there is a digital version of this activity)

In this activity, students continue to model real-life situations with simulations, but now the situations have more than one part. Finding the exact probability for these situations is advanced, but simulations are not difficult to run and an estimate of the probability can be found using the long-run results from simulations. If other simulation tools are not available, you will need the blackline master.

Instructional Routines

- Co-Craft Questions

Launch

Arrange students in groups of 3. After students have had a chance to think about an experiment themselves, select groups to share their responses.
If possible, allow them to use the simulation they have suggested (rolling a dice, papers in a bag, etc.). If the simulation is not readily available, provide each group with a spinner from the blackline master. Give students 5 minutes for partner discussion, 5 minutes to run the simulation, then 5 minutes for a whole-class discussion.

Students using the digital version have an applet on which they can run up to 10 simulation trials.

Representation: Develop Language and Symbols. Eliminate barriers and provide concrete manipulatives to connect symbols to concrete objects or values. Provide access to simulation tools, such as dice, papers in bags, and spinners.
Supports accessibility for: Visual-spatial processing; Fine-motor skills Conversing: Co-Craft Questions. To begin, display only the scenario of the Alpine Zoom ski business, without revealing the questions in this activity. Ask students to write possible mathematical questions about the situation. As pairs share their questions with the class, listen for and amplify questions about the probability that Alpine Zoom will make money during spring half term. If no student asks about the probability that Alpine Zoom will make money, ask students to adapt a question to align with the learning goals of this lesson. Then reveal and ask students to work on the actual questions of the task. This routine will help develop students' meta-awareness of language as they generate questions about the probability of real-life situations.
Design Principle(s): Optimise output; Maximise meta-awareness

Anticipated Misconceptions

Students may be confused by the phrase "at least 4 days." Explain that in this context, it means 4 or more.

Student Task Statement

Alpine Zoom is a ski business. To make money over spring half term, they need it to snow at least 4 out of the 10 days. The weather forecast says there is a $\frac{1}{3}$ chance it will snow each day during the break.

1. Describe a chance experiment that you could use to simulate whether it will snow on the first day of spring half term.

2. How could this chance experiment be used to determine whether Alpine Zoom will make money?

   Pause here so your teacher can give you the supplies for a simulation.

3. Simulate the weather for 10 days to see if Alpine Zoom will make money over spring half term. Record your results in the first row of the table.

	day 1	day 2	day 3	day 4	day 5	day 6	day 7	day 8	day 9	day 10	Did they make money?
simulation 1											
simulation 2											

simulation 3											
simulation 4											
simulation 5											

4. Repeat the previous step 4 more times. Record your results in the other rows of the table.

5. Based on your group's simulations, estimate the probability that Alpine Zoom will make money.

Student Response

1. Answers vary. Sample response: Roll a dice. If it lands on a 5 or 6, it will snow on the first day of break. If it lands on anything else, it does not snow.

2. Do the chance experiment 10 times and write down whether it snows each day. If it snows on at least 4 days, then the company will make money.

3. Answers vary.

4. Answers vary. Sample response:

	day 1	day 2	day 3	day 4	day 5	day 6	day 7	day 8	day 9	day 10	Did they make money?
simulation 1	snow	no	no	no	snow	no	snow	no	no	snow	yes
simulation 2	no	no	no	snow	snow	no	no	no	no	no	no
simulation 3	snow	no	snow	no	no	no	snow	snow	no	snow	yes
simulation 4	no	no	no	no	no	snow	no	no	no	snow	no
simulation 5	no	no	snow	no	no	no	snow	no	no	snow	no

5. Answers vary. Sample response: $\frac{2}{5}$

Activity Synthesis

The purpose of this discussion is for students to understand the connection between the results of their simulation and the real-life situation.

Ask each group for the number of times Alpine Zoom made money in their simulations.

Consider asking these discussion questions:

- "Using the class's data, estimate the probability that Alpine Zoom will make money." (Theoretically, this should be close to 45%.)
- "Do you anticipate Alpine Zoom will make money this spring half term?" (It's not likely, but it's possible.)
- "Over the next 10 years, if the weather patterns continue to be the same, do you anticipate Alpine Zoom will make money over that time or not?" (This is even less likely. There is less than a 50% chance it will make money each season, so over 10 years, it will probably lose money more than make money.)
- "Is this a business you would invest in? Explain your reasoning."
(I would not invest in it because it is unlikely to make money over the years.)

7.3 Kiran's Game

Optional: 15 minutes

Since this activity is mainly included for practice and may take some additional time to complete, it is included as an optional task and including it is up to the teacher's discretion.

In this activity, students practise doing many trials of multi-step situations to estimate the probability of an event. In the discussion following the activity, students construct arguments about how changes to the game might affect the probability of winning.

Instructional Routines

- Clarify, Critique, Correct

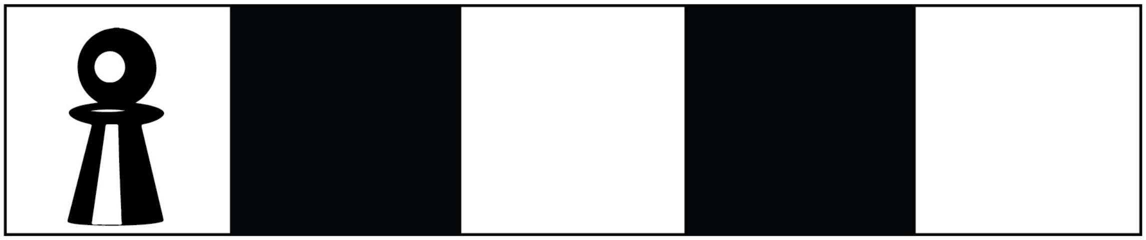
Launch

Arrange students in groups of 3. Provide each group with a paper bag containing 2 black blocks and 2 white blocks inside. If black and white blocks are not available, instruct students on their colour equivalents. Give students 5 minutes to run the simulation, 5 minutes for partner discussion, then have a whole-class discussion.

Action and Expression: Internalise Executive Functions. Provide students with a graphic organiser for recording their results and calculations of probability of winning Kiran's game.
Supports accessibility for: Language; Organisation

Student Task Statement

Kiran invents a game that uses a board with alternating black and white squares. A playing piece starts on a white square and must advance 4 squares to the other side of the board within 5 turns to win the game.



For each turn, the player draws a block from a bag containing 2 black blocks and 2 white blocks. If the block colour matches the colour of the next square on the board, the playing piece moves onto it. If it does not match, the playing piece stays on its current square.

1. Take turns playing the game until each person in your group has played the game twice.
2. Use the results from all the games your group played to estimate the probability of winning Kiran's game.
3. Do you think your estimate of the probability of winning is a good estimate? How could it be improved?

Student Response

Answers vary. Sample response:

1. No response needed.
2. Since nobody won in all 6 games played, the probability of winning should be low. Since it's possible to win, though, I don't think the probability should be 0. The probability is probably between 0 and $\frac{1}{6}$.
3. I don't think this is a very good estimate of the probability of winning since we only played 6 times and the chances of winning are so low.
   It could be improved by playing the game a lot more times.

Are You Ready for More?

How would each of these changes, on its own, affect the probability of winning the game?

1. Change the rules so that the playing piece must move 7 spaces within 8 moves.
2. Change the board so that all the spaces are black.
3. Change the blocks in the bag to 3 black blocks and 1 white block.

Student Response

1. It would be harder to win. You can still only get 1 wrong block, but now you must get it right 7 times instead of only 4.

2. This would not affect the chances of winning. There is still a probability of 0.5 to move each time.

3. This would make moving onto black squares easier, but harder to move on to white squares. This game is slightly more difficult to win than the original.

Activity Synthesis

The purpose of the discussion is for students to think about how changing the rules of the game might change the probability of winning.

Collect the data from the class for the number of wins and display the results for all to see.

Consider asking these discussion questions:

- "Based on the class's data, estimate the probability of winning the game." (The theoretical probability of winning is $\frac{3}{16} \approx 0.19$.)
- "Does the game seem too easy or hard to win? If so, how could Kiran change the game slightly to make it harder or easier?" (If it is too hard, move the pawn closer to the end or allow more than 5 moves to win.)
- "The bag contained 2 black and 2 white blocks. If the bag had 4 blocks of each colour, would that make it easier or harder to win?" (Neither. It would be the same difficulty since there is still an equal chance to get each colour.)
- "Do you think the estimate from the class's data is a better estimate than the one you got on your own?" (Since there is more data from the entire class, it should be a better estimate than one individual's.)

Reading, Writing, Speaking: Clarify, Critique, Correct. Before students share their estimated probability of winning Kiran's game, present an incorrect solution based on a misconception that arises when conducting a few trial simulations. For example: "The probability of winning Kiran's game is zero because nobody in our group won in all of the six games played." Ask students to identify the error, critique the reasoning, and revise the original statement. As students discuss in partners, listen for students who state that estimating the probability of winning based on only six trials will not result in an accurate estimate. Therefore, more trials must be conducted in order to improve the estimate of the probability of winning. This routine will engage students in meta-awareness as they critique and correct the language used to estimate the probability of winning a game.
Design Principles(s): Cultivate conversation; Maximise meta-awareness

7.4 Simulation Nation

10 minutes

In this activity, students practise what they have learned about simulations by matching real-life scenarios to simulations. In the discussion, students are asked to explain their reasoning for their choices and think about other valid choices that could be made.

Instructional Routines

- Discussion Supports
- Take Turns

Launch

Keep students in groups of 3. Give students 5 minutes of small-group time to take turns matching the items and discussing their reasoning, followed by whole-class discussion.

Conversing: Discussion Supports. Invite students to take turns matching a situation to a simulation. Display sentence frames to help students explain their reasoning. For example, "Situation ___ matches with simulation ___ because . . ." Listen for the connection between the numerical quantities in the simulation and the situation. Encourage students to challenge each other when they disagree. This will help students justify how all the parts of the simulation can be used to match accordingly with the situation.
Design Principle(s): Support sense-making; Cultivate conversation

Anticipated Misconceptions

Students may not see the connection between the standard dice and the situation with 3 doors. Remind students it is important that the probabilities match, but not necessarily the outcomes. Since the simulation matches 2 of the outcomes to one door, the probabilities will match.

Student Task Statement

Match each situation to a **simulation**.

Situations:

1. In a small lake, 25% of the fish are female. You capture a fish, record whether it is male or female, and toss the fish back into the lake. If you repeat this process 5 times, what is the probability that at least 3 of the 5 fish are female?
2. Elena makes about 80% of her free throws. Based on her past successes with free throws, what is the probability that she will make exactly 4 out of 5 free throws in her next basketball game?
3. On a game show, a contestant must pick one of three doors. In the first round, the winning door has a holiday. In the second round, the winning door has a car. What is the probability of winning a holiday and a car?
4. Your choir is singing in 4 concerts. You and one of your classmates both learned the solo. Before each concert, there is an equal chance the choir director will select you or

the other student to sing the solo. What is the probability that you will be selected to sing the solo in exactly 3 of the 4 concerts?

Simulations:

A. Toss a standard dice 2 times and record the outcomes. Repeat this process many times and find the proportion of the simulations in which a 1 or 2 appeared both times to estimate the probability.

B. Make a spinner with four equal sections labelled 1, 2, 3, and 4. Spin the spinner 5 times and record the outcomes. Repeat this process many times and find the proportion of the simulations in which a 4 appears 3 or more times to estimate the probability.

C. Toss a fair coin 4 times and record the outcomes. Repeat this process many times, and find the proportion of the simulations in which exactly 3 heads appear to estimate the probability.

D. Place 8 blue chips and 2 red chips in a bag. Shake the bag, select a chip, record its colour, and then return the chip to the bag. Repeat the process 4 more times to obtain a simulated outcome. Then repeat this process many times and find the proportion of the simulations in which exactly 4 blues are selected to estimate the probability.

Student Response

1. Simulation B
2. Simulation D
3. Simulation A
4. Simulation C

Activity Synthesis

The purpose of this discussion is for students to articulate the reasons they chose to match the items they did.

For each situation, select students to explain why the simulation should go with it. Although some students may have just looked at a portion of the situation and simulation, encourage students to explain all of the parts of the simulation. Consider the problem with fish; 25% is mentioned and the spinner is the only option that also has a 25% chance associated with it. Prompt students for more details by asking,

- “Why do we need to spin the spinner 5 times?” (A fish is selected from the lake 5 times.)
- “Why does the number need to show up 3 or more times?” (We want a probability that three or more fish are female.)

- "What do the numbers 1 through 4 represent when doing a trial with the spinner?" (Each section represents a $\frac{1}{4}$ probability. The section labelled '4' is the 25% chance that a fish will be female, while sections labelled 1–3 are the 75% chance that a fish will not be female.)
- "Could the spinner have 8 sections? If so, how would you label the sections? What would each label represent?" (Yes, labels vary. Sample response: Label sections 1–8, where sections 7–8 represent the 25% chance that a fish will be female, while sections labelled 1–6 are the 75% chance that a fish will not be female.)

For each of the scenarios, ask students if any part of it could be changed and still result in the simulation working. For example, there could be 4 blue chips and 1 red chip in the bag for simulation D. For simulation C, we could count the fraction of times when 3 tails appear rather than heads.

Representation: Internalise Comprehension. Use colour and annotations to illustrate connections between representations. As students share their reasoning for matching situations with simulations, use colour and annotations to scribe their thinking on a display of each problem so that it is visible for all students.
Supports accessibility for: Visual-spatial processing; Conceptual processing

Lesson Synthesis

Consider asking these discussion questions:

- "How are the simulations in this lesson different from the simulations in the previous lesson?" (These have multiple parts for each experiment. Also, it would be difficult to calculate the exact probability, so simulations seem more necessary.)
- "The chance that it will be cloudy on a single day is simulated by rolling a standard dice twice. How many times will the dice need to be rolled to simulate a week?" (14 times. It is rolled twice for each day and there are 7 days in a week, so 14 rolls are needed.)
- "Each day, a student randomly reaches into a bowl of fruit and picks one for their lunch that day. To simulate the situation, he creates a spinner with 4 equal sections labelled: apple, orange, watermelon, and peach. Why might this simulation not represent the situation very well?" (Usually watermelons are much larger than the other 3 fruits listed, so there is probably not an equal chance of that being selected, so the spinner should probably have a larger wedge for watermelons.)

7.5 Battery Life

Cool Down: 5 minutes

Student Task Statement

The probability of a certain brand of battery going dead within 15 hours is $\frac{1}{3}$. Noah has a toy that requires 4 of these batteries. He wants to estimate the probability that at least one battery will die before 15 hours are up.

1. Noah will simulate the situation by putting marbles in a bag. Drawing one marble from the bag will represent the outcome of one of the batteries in the toy after 15 hours. Red marbles represent a battery that dies before 15 hours are up, and green marbles represent a battery that lasts longer.

   How many marbles of each colour should he put in the bag? Explain your reasoning.

2. After doing the simulation 5 times, Noah has the following results. What should he use as an estimate of the probability that at least one battery will die within 15 hours?

trial	result
1	GGRG
2	GRGR
3	GGGG
4	RGGG
5	GGGR

Student Response

1. 1 red marble and 2 green marbles (or some multiple of these). Based on the probability of each battery dying, $\frac{1}{3}$ of the marbles should be red.

2. $\frac{4}{5}$ or equivalent.

Student Lesson Summary

The more complex a situation is, the harder it can be to estimate the probability of a particular event happening. Well-designed simulations are a way to estimate a probability in a complex situation, especially when it would be difficult or impossible to determine the probability from reasoning alone.

To design a good simulation, we need to know something about the situation. For example, if we want to estimate the probability that it will rain every day for the next three days, we could look up the weather forecast for the next three days. Here is a table showing a weather forecast:

	today (Tuesday)	Wednesday	Thursday	Friday
probability of rain	0.2	0.4	0.5	0.9

We can set up a simulation to estimate the probability of rain each day with three bags.

- In the first bag, we put 4 slips of paper that say “rain” and 6 that say “no rain.”
- In the second bag, we put 5 slips of paper that say “rain” and 5 that say “no rain.”
- In the third bag, we put 9 slips of paper that say “rain” and 1 that says “no rain.”

Then we can select one slip of paper from each bag and record whether or not there was rain on all three days. If we repeat this experiment many times, we can estimate the probability that there will be rain on all three days by dividing the number of times all three slips said “rain” by the total number of times we performed the simulation.

Lesson 7 Practice Problems

Problem 1 Statement

Priya’s cat is pregnant with a litter of 5 kittens. Each kitten has a 30% chance of being chocolate brown. Priya wants to know the probability that at least two of the kittens will be chocolate brown.

To simulate this, Priya put 3 white cubes and 7 green cubes in a bag. For each trial, Priya pulled out and returned a cube 5 times. Priya conducted 12 trials.

Here is a table with the results.

trial number	outcome
1	ggggg
2	gggwg
3	wgwgw
4	gwggg
5	gggwg
6	wwggg
7	gwggg
8	ggwgw
9	wwwgg
10	ggggw
11	wggwg
12	gggwg

a. How many successful trials were there? Describe how you determined if a trial was a success.

b. Based on this simulation, estimate the probability that *exactly* two kittens will be chocolate brown.

c. Based on this simulation, estimate the probability that *at least* two kittens will be chocolate brown.

d. Write and answer another question Priya could answer using this simulation.

e. How could Priya increase the accuracy of the simulation?

Solution

a. 5 of the 12 trials were successful. Drawing two or more white blocks (w's) counted as a success.

b. $\frac{3}{12}$ (or equivalent, or nearby approximation)

c. $\frac{5}{12}$ (or equivalent, or nearby approximation)

d. Answers vary. Sample response: What is the probability that none of the kittens will be chocolate brown?

e. Priya could conduct more trials to increase the accuracy of the simulation.

Problem 2 Statement

A team has a 75% chance to win each of the 3 games they will play this week. Clare simulates the week of games by putting 4 pieces of paper in a bag, 3 labelled "win" and 1 labelled "lose." She draws a paper, writes down the result, then replaces the paper and repeats the process two more times. Clare gets the result: win, win, lose. What can Clare do to estimate the probability the team will win at least 2 games?

Solution

She needs to repeat the process many more times to get a good estimate of the probability. She has only done it once right now. After she has repeated the simulation of the week many times, she could count the fraction of simulated weeks that included at least 2 wins and use that as an estimate for the probability.

Problem 3 Statement

a. List the sample space for selecting a letter at random from the word "PINEAPPLE."

b. A letter is randomly selected from the word "PINEAPPLE." Which is more likely, selecting "E" or selecting "P?" Explain your reasoning.

Solution

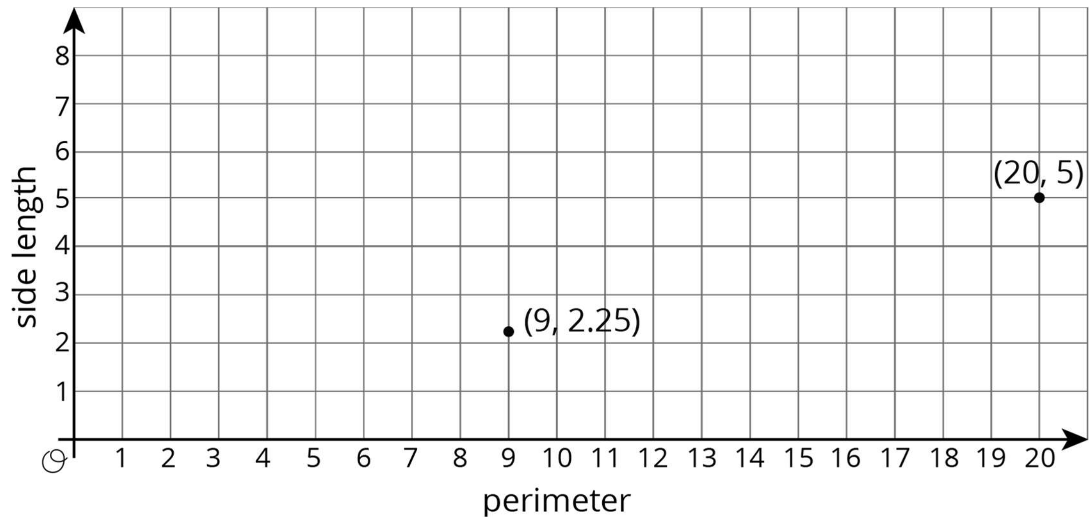
a. P, I, N, E, A, L (or equivalent)

b. Selecting the letter "P" is more likely because there are 3 Ps in the word "pineapple," but there are only 2 Es.

Problem 4 Statement

On a graph of side length of a square vs. its perimeter, a few points are plotted.

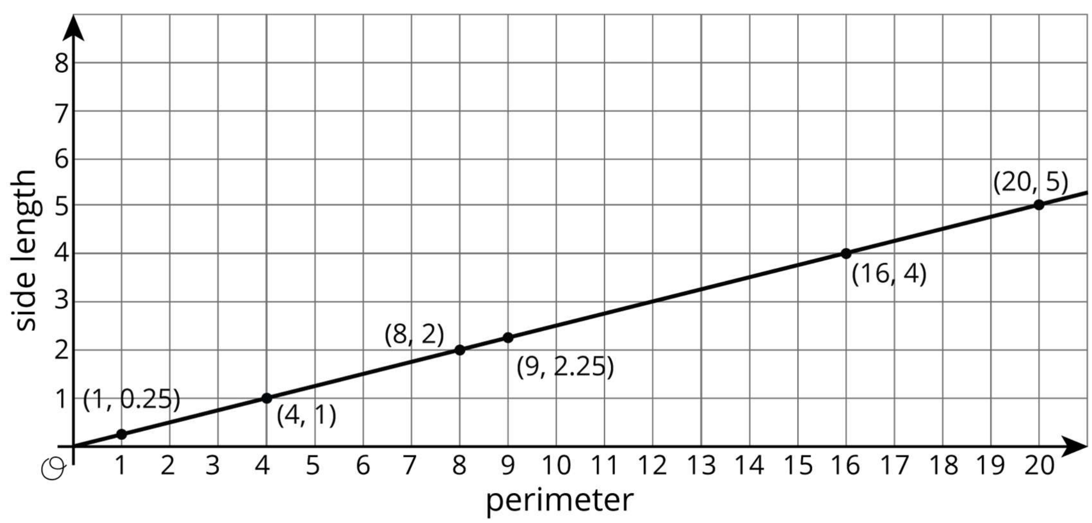
a. Add at least two more ordered pairs to the graph.



b. Is there a proportional relationship between the perimeter and side length? Explain how you know.

Solution

a.



b. There is a proportional relationship between side length and perimeter. When graphed, the ordered pairs lie on a line that passes through the origin.