INTERNATIONAL BACCALAUREATE

Mathematics: analysis and approaches

MAA

EXERCISES [MAA 5.20]

INTEGRATION BY PARTS

Compiled by Christos Nikolaidis

O. Practice questions

1. [Maximum mark: 16] ***[without GDC]***

Find $I_1 = \int 2x\mathrm{e}^x \mathrm{d}x$, $I_2 = \int 2x\cos x \mathrm{d}x$, $I_3 = \int 2x\sin x \mathrm{d}x$, $I_4 = \int 2x\ln x \mathrm{d}x$

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

2. [Maximum mark: 10] ***[without GDC]***

Find $I_1 = \int (3x^2 + 4x + 1)\cos x \mathrm{d}x$, $I_2 = \int (3x^2 + 4x + 1)\ln x \mathrm{d}x$

3. [Maximum mark: 6] ***[without GDC]***

Find $I = \int e^{2x} \cos 2x dx$ by using integration by parts in two different ways:

METHOD A: by integrating e^{2x} first.

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

METHOD B: by integrating $\cos 2x$ first.

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

4. [Maximum mark: 10] ***[without GDC]***

Let $I_n = \int x^n \mathrm{e}^x \mathrm{d}x$.

(a) Find I_0. [2]

(b) Express I_n in terms of I_{n-1} by using integration by parts. [2]

(c) Find I_1, I_2 and I_3 by using the recursive relation found above. Express the results in the form $I_n = p(x)\mathrm{e}^x + c$, where $p(x)$ is a polynomial. [6]

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

5. [Maximum mark: 12] ***[without GDC]***

Consider $I = \int 4\sin x \cos x \mathrm{d}x$

(a) Find I by using

(i) the substitution $u = \sin x$.

(ii) the substitution $u = \cos x$.

(iii) the double angle formula for $\sin 2\theta$.

(iv) integration by parts. [10]

(b) Explain the difference in the results. [2]

6. [Maximum mark: 12] ***[without GDC]***

Calculate the definite integral $I = \int_0^1 (3x+2)\mathrm{e}^x \mathrm{d}x$

METHOD A: Find the definite integral first and then the definite (preferable!)

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

METHOD B: Apply integration by parts on the definite integral, keeping the limits.

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

A. Exam style questions (SHORT)

7. [Maximum mark: 6] ***[without GDC]***

Find $\int (\theta \cos\theta - \theta)\mathrm{d}\theta$.

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

8. [Maximum mark: 6] ***[without GDC]***

Find $\int \frac{\ln x}{\sqrt{x}}\mathrm{d}x$.

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

9. [Maximum mark: 6] ***[without GDC]***

Find $\int e^x \cos x \, dx$.

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

10. [Maximum mark: 6] ***[without GDC]***

Find $\int e^{2x} \sin x \, dx$.

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

11*. [Maximum mark: 7] ***[without GDC]***

Let $f(x) = x\ln x - x,\ x > 0$.

(a) Find $f'(x)$. [3]

(b) Using integration by parts, find $\int (\ln x)^2 \mathrm{d}x$. [4]

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

12. [Maximum mark: 6] ***[without GDC]***

Find $\int \arctan x \mathrm{d}x$.

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

13*. [Maximum mark: 6] ***[without GDC]***

Find $\int 2x \arctan x \mathrm{d}x$.

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

14*. [Maximum mark: 6] ***[without GDC]***

Find $\int \frac{x^2}{\mathrm{e}^{2x}} \mathrm{d}x$.

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

15. [Maximum mark: 6] ***[with / without GDC]***

(a) Use integration by parts to find $\int x^2 \ln x \mathrm{d}x$.

(b) Evaluate $\int_1^2 x^2 \ln x \mathrm{d}x$

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

16. [Maximum mark: 5] ***[without GDC]***

Calculate the exact value of $\int_1^e x^5 \ln x \mathrm{d}x$

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

17. [Maximum mark: 6] ***[without GDC]***

Show that $\int_0^{\frac{\pi}{6}} x\sin 2x\mathrm{d}x = \frac{\sqrt{3}}{8} - \frac{\pi}{24}$.

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

18. [Maximum mark: 7] ***[without GDC]***

Find $\int_0^a \arcsin x\mathrm{d}x,\ 0 < a < 1$.

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

B. Exam style questions (LONG)

19. [Maximum mark: 14] ***[without GDC]***

(a) Find $I_n = \int x^n \ln x \mathrm{d}x$ in terms of n, where $n \in R$, $n \neq -1$. [4]

(b) Find $J_n = \int \frac{(\ln x)^n}{x} \mathrm{d}x$ in terms of n, where $n \in R$. [3]

(c) **Hence** find

(i) $\int \sqrt{x} \ln x \mathrm{d}x$, (ii) $\int \frac{\sqrt{\ln x}}{x} \mathrm{d}x$, (iii) $\int \left(\frac{\ln x}{x^2} + \frac{(\ln x)^2}{x} \right) \mathrm{d}x$. [7]

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

20. [Maximum mark: 16] ***[without GDC]***

Let $A_n = \int x^n \sin x \mathrm{d}x$ and $B_n = \int x^n \cos x \mathrm{d}x$

(a) Find A_0 and B_0. [2]

(b) Express A_n in terms of B_{n-1} by using integration by parts. [2]

(c) Express B_n in terms of A_{n-1} by using integration by parts. [2]

(d) **Hence** express

(i) A_n in terms of A_{n-2}.

(ii) B_n in terms of B_{n-2}. [5]

(e) Find A_1, A_2 and A_3 by using the results above. [5]

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

.....

21. [Maximum mark: 28] ***[without GDC]***

Let $I_n = \int \sin^n x \mathrm{d}x$

(a) Find I_0 and I_1. [4]

(b) Express I_n in terms of I_{n-2} by using integration by parts. [4]

(c) Find I_2, I_3, I_4, I_5 by using the recursive relation found in (b). [4]

(d) Find I_2, I_4 by using the double angle formula $\sin^2 x = \frac{1-\cos 2x}{2}$. [8]

(e) Find I_3, I_5 by using the substitution $u = \cos x$ [8]

NOTICE

The following table is from my lecture notes.

Please make sure that you are able to solve all the examples in column 2

and the theoretical questions in column 3.

General Form	Examples	Theoretical Questions
$I_n = \int x^n e^x dx$ $I_{n,m} = \int x^n e^{mx} dx$	$\int x^3 e^x dx$, $\int x^2 e^{3x} dx$	Express I_n in terms of I_{n-1} Hence find $I_0, I_1, I_2, \ldots$
$I_n = \int x^n \cos x dx$ $I_n = \int x^n \sin x dx$	$\int x^2 \cos x dx$	Express I_n in terms of I_{n-2}
$I_{n,m} = \int x^n \cos(mx) dx$ $I_{n,m} = \int x^n \sin(mx) dx$	$\int x^2 \cos 3x dx$	
$I_n = \int x^n \ln x dx$	$\int x^5 \ln x dx$, $\int \frac{\ln x}{x^5} dx$ $\int \sqrt{x} \ln x dx$	Find a general formula for I_n
$I_{n,m} = \int e^{nx} \sin(mx) dx$ $I_{n,m} = \int e^{nx} \cos(mx) dx$	$\int e^{3x} \sin 2x dx$ $\int e^{-x} \sin 2x dx$	Find a general formula for $I_{n,m}$
$I_n = \int \cos^n x dx$ $I_n = \int \sin^n x dx$	$\int \cos^2 x dx$ $\int \cos^3 x dx$	Express I_n in terms of I_{n-2} Hence find I_2, I_4 and I_3, I_5
$I_{n,m} = \int \sin(nx) \cos(mx) dx$	$\int \sin 2x \cos 3x dx$	Find a general formula for $I_{n,m}$
$I_n = \int x^n \arctan x dx$ $I_n = \int x^n \arcsin x dx$ $I_n = \int x^n \arccos x dx$	$\int \arctan x dx$, $\int x \arctan x dx$, $\int x^2 \arctan x dx$ $\int \arcsin x dx$, $\int x^2 \arcsin x dx$	
$I_n = \int (\ln x)^n dx$	$\int (\ln x)^2 dx$, $\int (\ln x)^3 dx$	