Lesson 3: Balanced moves

Goals
- Compare and contrast (orally and in writing) solution paths to solve an equation in one variable by performing the same operation on each side.
- Correlate (orally and in writing) changes on balance diagrams with moves that create equivalent equations.

Learning Targets
- I can add, subtract, multiply, or divide each side of an equation by the same expression to get a new equation with the same solution.

Lesson Narrative

In this lesson students move from using balances to using equations in order to represent a problem. In the warm-up they match a series of balances with the corresponding series of equations. They see how moves that maintain the balance of a balance correspond to moves that maintain the equality of an equation, such as halving the value of each side or subtracting the same unknown value from each side. In the next activity students match pairs of equations with the corresponding equation move—performing the same operation on each side—that produces the second from the first. In the activity after that, they compare different choices of moves that lead to the same solution. In this activity the solution is negative, which would not have been representable with balances. Students can check that it is a solution by substituting into the equation, reinforcing the idea that a solution is a number that makes the equality in an equation true, and that different moves maintain the equality. Students reason about why the steps in solving an equation maintain the equality and compare different solution methods.

Addressing
- Analyse and solve linear equations and pairs of simultaneous linear equations.
- Solve linear equations in one variable.

Instructional Routines
- Collect and Display
- Clarify, Critique, Correct
- Compare and Connect
- Think Pair Share

Required Materials
Pre-printed cards, cut from copies of the blackline master

1 $3x + 7 = 5x$ $7 = 2x$	A Multiply each side by $-\frac{1}{3}$
2 $12x + 3 = 6$ $4x + 1 = 2$	B Add $-3x$ to each side
3 $10 - 6x = 4 + 5x$ $7 - 6x = 1 + 5x$	C Add $3x$ to each side
4 $\frac{5x}{-3} = \frac{12}{1}$ $5x = -36$	D Add -3 to each side
5 $-3(4x - 3) = -15$ $4x - 3 = 5$	E Multiply each side by $\frac{1}{3}$
6	F Multiply each side by -3

Required Preparation

Print and cut up the Matching Equation Moves blackline master for the matching activity. Prepare one set of cards for every 2 students.

Student Learning Goals

Let's rewrite equations while keeping the same solutions.

3.1 Matching Balances

Warm Up: 10 minutes

The purpose of this warm-up is for students to revisit ideas they learned in the previous lesson about balanced balances:

- You can add or subtract the same thing on each side and the balance stays in balance.
- You can divide each side by the same number and the balance stays in balance.

Instructional Routines

- Collect and Display

Launch

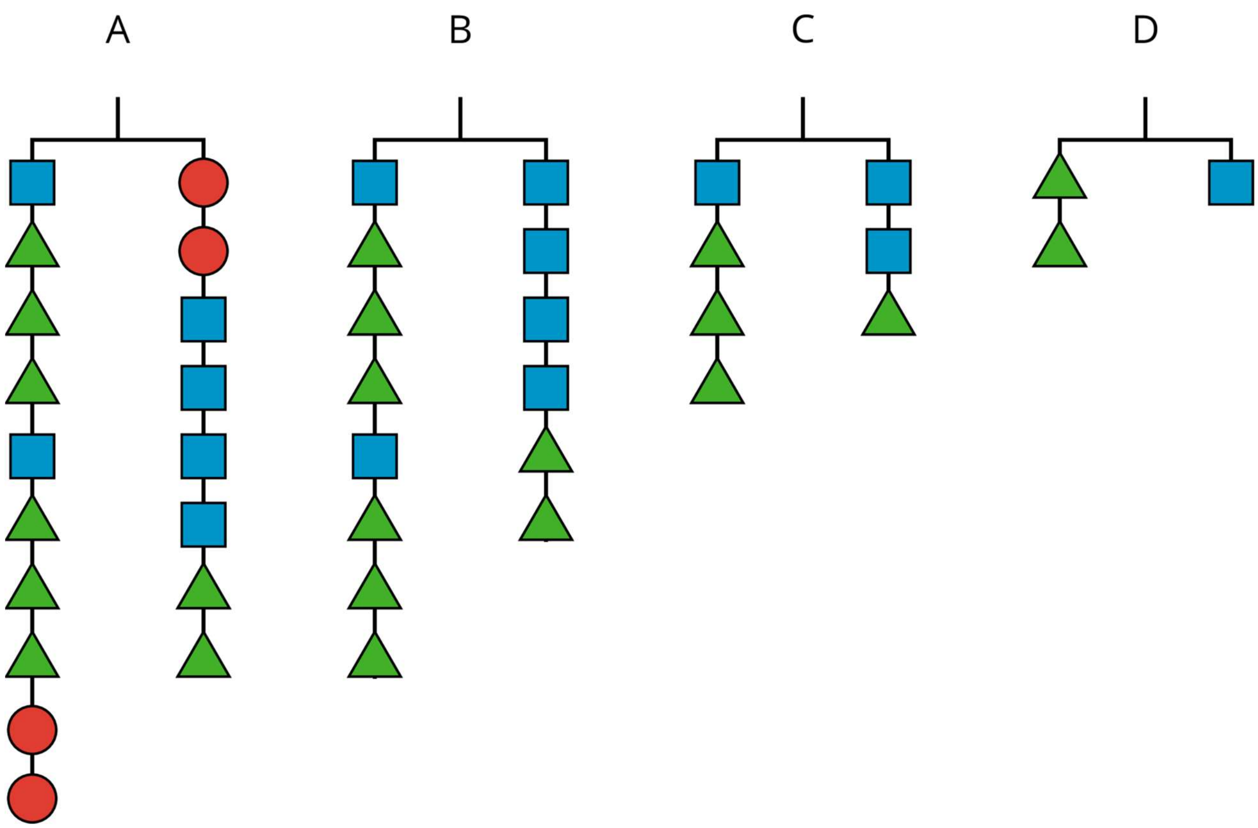
Give students 2 minutes of quiet work time followed by a whole-class discussion.

Anticipated Misconceptions

Some students may think a variable stands for more than one object. Tell these students that a variable only stands for one object, as it also only represents one number.

Student Task Statement

Figures A, B, C, and D show the result of simplifying the balance in Figure A by removing equal weights from each side.



Here are some equations. Each equation represents one of the balance diagrams.

$2(x + 3y) = 4x + 2y$
$2y = x$
$2(x + 3y) + 2z = 2z + 4x + 2y$
$x + 3y = 2x + y$

1. Write the equation that goes with each figure:

    A:

    B:

    C:

    D:

2. Each variable (x, y, and z) represents the weight of one shape. Which goes with which?

3. Explain what was done to each equation to create the next equation. If you get stuck, think about how the balances changed.

Student Response

1.
    A: $2(x + 3y) + 2z = 2z + 4x + 2y$
    B: $2(x + 3y) = 4x + 2y$
    C: $x + 3y = 2x + y$
    D: $2y = x$

2. x is the blue square. y is the green triangle. z is the red circle.

3. The same type and number of objects were removed from each side.

Activity Synthesis

Ask students to explain how they decided on the matching equation. As students discuss the final questions, highlight responses that emphasise the same objects being removed from each side creates the next figure in line. The exception to this is the move from Balance B to Balance C, where the number of objects on each side is halved. Ask students to explain why this is an okay move, even though different objects are being removed from each side (1 square and 3 triangles on the left, 2 squares and 1 triangle on the right).

3.2 Matching Equation Moves

15 minutes

In this activity, students match a card with two equations to another card describing the move that turns the first equation into the second. The goal is to help students think about equations the same way they have been thinking about balances: objects where equality is maintained so long as the same move is made on each side. Additionally, this is the first activity where students encounter equation moves involving negative numbers, which is not possible when using balances.

Instructional Routines

- Clarify, Critique, Correct

Launch

Review with students what we know about equations based on reasoning about balances:

- We can add the same quantity to each side, and the equation is still true (the balance is still in balance).
- We can subtract the same quantity from each side, and the equation is still true.
- We can double or triple or halve or third the things that appear on each side, and the equation is still true. More generally, we can multiply the number of things on each side by the same number.

Tell students that balance diagrams are really only useful for reasoning about positive numbers, but the processes above also work for negative numbers. Negative numbers are just numbers, and they have to follow the same rules as positive numbers. In fact, if we allow negative numbers into the mix, we can express any manoeuvre with one of two types of moves:

- Add the same thing to each side. (The “thing” could be negative.)
- Multiply each side by the same thing. (The “thing” could be a fraction less than 1.)

Arrange students in groups of 2. Give each group 12 pre-cut slips from the blackline master. Give 3–4 minutes for partners to match the numbered slips with the lettered slips then 1–2 minutes to trade places with another group and review each other’s work. Ask partners who finish early to write down on a separate sheet of paper what the next move would be for each of the numbered cards if the goal were to solve for x. Follow with a whole-class discussion.

Representation: Internalise Comprehension. Chunk this task into more manageable parts to differentiate the degree of difficulty or complexity by beginning with fewer cards. For example, give students a subset of the cards to start with and introduce the remaining cards once students have completed their initial set of matches.
Supports accessibility for: Conceptual processing; Organisation

Student Task Statement

Your teacher will give you some cards. Each of the cards 1 through 6 show two equations. Each of the cards A through E describe a move that turns one equation into another.

1. Match each number card with a letter card.

2. One of the letter cards will not have a match. For this card, write two equations showing the described move.

Student Response

1. B
2. E
3. D
4. F
5. A
6. C. Answers vary. Possible response: $5 - 3x = 2x + 8$, $5 = 5x + 8$.

Activity Synthesis

The goal of this discussion is to get students using the language of equations and describing the changes happening on each side when solving. Ask:

- “What is a move you could do to the equation $7 = 2x$ on card 1 that would result in an equation of the form $x = \_$? What is another move that would also work?” (Multiply each side by $\frac{1}{2}$. Divide each side by 2.)
- “Which numbered card was the most challenging to match?” (Card 2, because at first I only looked at the x-terms and thought the move involved a change of $8x$.)
- “Does anyone have a value for x that would solve one of the numbered cards? How did you figure it out?” ($x = 2$ is a solution for card 5. I added 3 to each side and then multiplied each side by $\frac{1}{4}$.)

End the discussion by inviting groups to share the equations they wrote for card 6 and describe how they match the move "add $3x$ to each side."

Speaking: Clarify, Critique, Correct. Display the statements: “When we add to both sides, it is the same.” and “When we multiply both sides, it stays the same.” Ask students to clarify or improve these statements in a way that is more specific. Prompt students to think about positive and negative numbers as well as fractions. This will help students to use the language of equations to explain why you can add (or subtract) and multiply (or divide) each side of an equation by an expression involving rational numbers and still have an equivalent equation.
Design Principle(s): Optimise output (for generalisation)

3.3 Keeping Equality

10 minutes

The purpose of this activity is to get students thinking about strategically solving equations by paying attention to their structure. Expanding brackets first versus dividing first is a common point of divergence for students as they start solving.

Identify students who choose different solution paths to solve the last two problems.

Instructional Routines

- Compare and Connect
- Think Pair Share

Launch

Arrange students in groups of 2. Give students 2 minutes quiet think time for problem 1, then 3–5 minutes partner time to discuss problem 1 and complete the other problems. Follow with a whole-class discussion.

Conversing: Compare and Connect. Display Noah's and Lin's solution methods side by side. Once students have determined that they are both correct, ask students to explain the differences between the approaches. Amplify mathematical language students use to distinguish between expanding or dividing. This will help students reflect on and produce mathematical language to explain why they choose to take either approach when answering the remaining questions.
Design Principle(s): Maximise meta-awareness; Cultivate conversation

Anticipated Misconceptions

Some students may not expand brackets or collect like terms before performing the same operation on each side.

Student Task Statement

1. Noah and Lin both solved the equation $14a = 2(a - 3)$.

   Do you agree with either of them? Why?

   Noah's solution:

   $$\begin{aligned} 14a &= 2(a-3) \\ 14a &= 2a - 6 \\ 12a &= -6 \\ a &= -\frac{1}{2} \end{aligned}$$

   Lin's solution:

$$14a = 2(a-3)$$
$$7a = a-3$$
$$6a = \text{-}3$$
$$a = \text{-}\frac{1}{2}$$

2. Elena is asked to solve $15 - 10x = 5(x+9)$. What do you recommend she does to each side first?

3. Diego is asked to solve $3x - 8 = 4(x+5)$. What do you recommend he does to each side first?

Student Response

1. Both Noah and Lin have correct solutions. Explanations vary. Sample response: Both Noah and Lin followed valid solution paths. Substituting $a = \text{-}\frac{1}{2}$ into the original equation yields a true statement, so their solutions are correct.

2. Answers vary. There are at least two solution paths to this equation: you can divide each side by 5 first, then collect like terms, or you can expand brackets and collect like terms, then continue to solve.

3. Answers vary. There are still two solution paths to this equation, but one is much simpler than the other. Since not all the terms are multiples of 4, dividing first by 4 will give a fractional coefficient of x on one side. Therefore, expanding first and then collecting like terms and solving is the simpler solution path.

Are You Ready for More?

In a cryptarithmetic puzzle, the digits 0–9 are represented with letters of the alphabet. Use your understanding of addition to find which digits go with the letters A, B, E, G, H, L, N, and R.

HANGER + HANGER + HANGER = ALGEBRA

Student Response

A:2, B:8, E:1, G:6, H:9, L:7, N:0, R:4

Activity Synthesis

Have previously identified groups share the different solution paths they chose for solving the last two questions.

To highlight the different strategies, ask:

- "What are the advantages of choosing to multiply out first? To divide first?" (Answers vary. Multiplying out first eliminates confusion about which terms can be subtracted from each side. Dividing first makes the numbers smaller and easier to mentally calculate.)

- “What makes it easier to multiply out versus divide first on the last question?” (Dividing by 4 before multiplying out will result in non-integer terms, which can be harder to add and subtract mentally.)
- “Is one path more ‘right’ than another?” (No. As long as we follow valid steps, like adding or multiplying the same thing to each side of an equation, the steps are right and will give a correct solution.)

Lesson Synthesis

Display the equation $6x + 12 = 10x - 4$ for all to see. Tell students to think of three different things they could do to each side of the question but still maintain equality. Invite students to share their moves. Possible responses include:

- subtract $6x$ from each side
- add 4 to each side
- divide each side by 2

Ask students, "If you made a mistake when solving this equation and thought that $x = 2$, how would you be able to tell?" (If I put 2 into the equation, I would get that $24 = 16$, which isn't true.)

3.4 More Matching Moves

Cool Down: 5 minutes

Student Task Statement

Match these equation balancing steps with the description of what was done in each step.

Step 1:

$$\begin{aligned} 12x - 6 &= 10 \\ 6x - 3 &= 5 \end{aligned}$$

Step 2:

$$\begin{aligned} 6x - 3 &= 5 \\ 6x &= 8 \end{aligned}$$

Step 3:

$$\begin{aligned} 6x &= 8 \\ x &= \frac{4}{3} \end{aligned}$$

Descriptions to match with each step:

A: Add 3 to both sides

B: Multiply both sides by $\frac{1}{6}$

C: Divide both sides by 2

Student Response

In Step 1, we did C, divide both sides by 2.

In Step 2, we did A, add 3 to both sides.

In Step 3, we did B, multiply both sides by $\frac{1}{6}$.

Student Lesson Summary

An equation tells us that two expressions have equal value. For example, if $4x + 9$ and $\text{-}2x - 3$ have equal value, we can write the equation

$4x + 9 = \text{-}2x - 3$

Earlier, we used balances to understand that if we add the same positive number to each side of the equation, the sides will still have equal value. It also works if we add *negative numbers*! For example, we can add -9 to each side of the equation.

$$\begin{array}{lll} 4x + 9 + \text{-}9 & = \text{-}2x - 3 + \text{-}9 & \text{add -9 to each side} \\ 4x & = \text{-}2x - 12 & \text{combine like terms} \end{array}$$

Since expressions represent numbers, we can also add *expressions* to each side of an equation. For example, we can add $2x$ to each side and still maintain equality.

$$\begin{array}{lll} 4x + 2x & = \text{-}2x - 12 + 2x & \text{add } 2x \text{ to each side} \\ 6x & = \text{-}12 & \text{combine like terms} \end{array}$$

If we multiply or divide the expressions on each side of an equation by the same number, we will also maintain the equality (so long as we do not divide by zero).

$$6x \times \frac{1}{6} = \text{-}12 \times \frac{1}{6} \qquad \text{multiply each side by } \frac{1}{6}$$

or

$$6x \div 6 = \text{-}12 \div 6 \qquad \text{divide each side by 6}$$

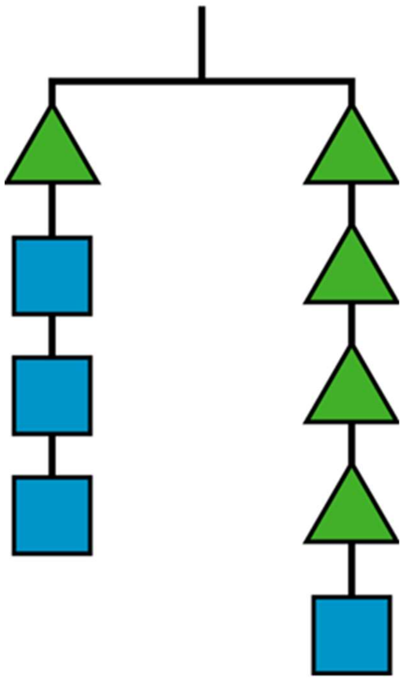
Now we can see that $x = \text{-}2$ is the solution to our equation.

We will use these moves in systematic ways to solve equations in future lessons.

Lesson 3 Practice Problems

1. **Problem 1 Statement**

In this balance, the weight of the triangle is x and the weight of the square is y.



a. Write an equation using x and y to represent the balance.

b. If x is 6, what is y?

Solution

a. $x + 3y = 4x + y$

b. $y = 9$

2. **Problem 2 Statement**

Andre and Diego were each trying to solve $2x + 6 = 3x - 8$. Describe the first step they each make to the equation.

a. The result of Andre's first step was $-x + 6 = -8$

b. The result of Diego's first step was $6 = x - 8$.

Solution

a. Andre subtracted $3x$ from each side.

b. Diego subtracted $2x$ from each side.

3. **Problem 3 Statement**

a. Complete the table with values for x or y that make this equation true: $3x + y = 15$.

b. Create a graph, plot these points, and find the gradient of the line that goes through them.

Solution

a.

x	2	4	6	0	3	5	$\frac{7}{3}$
y	9	3	-3	15	6	0	8

b. Gradient = -3

4. **Problem 4 Statement**

Match each set of equations with the move that turned the first equation into the second.

a. $6x + 9 = 4x - 3$
$2x + 9 = \text{-}3$

b. $\text{-}4(5x - 7) = \text{-}18$
$5x - 7 = 4.5$

c. $8 - 10x = 7 + 5x$
$4 - 10x = 3 + 5x$

d. $\frac{\text{-}5x}{4} = 4$
$5x = \text{-}16$

e. $12x + 4 = 20x + 24$
$3x + 1 = 5x + 6$

1. Multiply both sides by $\frac{\text{-}1}{4}$
2. Multiply both sides by -4
3. Multiply both sides by $\frac{1}{4}$
4. Add $\text{-}4x$ to both sides
5. Add -4 to both sides

Solution

- A: 4
- B: 1
- C: 5
- D: 2
- E: 3

5. **Problem 5 Statement**

Select **all** the situations for which only zero or positive solutions make sense.

a. Measuring temperature in degrees Celsius at an Arctic outpost each day in January.

b. The height of a candle as it burns over an hour.

c. The height above sea level of a hiker descending into a canyon.

d. The number of students remaining in school after 6:00 p.m.

e. A bank account balance over a year.

f. The temperature in degrees Fahrenheit of an oven used on a hot summer day.

Solution ["B", "D", "F"]