Lesson 10: Composing figures

Goals

- Draw and label images of triangles under translations, rotations and reflections and then describe (orally and in writing) properties of the composite shape created by the images.
- Generalise that lengths and angle sizes are preserved under any translation, rotation or reflection.
- Identify side lengths and angles that have equivalent measurements in composite shapes and explain (orally and in writing) why they are equivalent.

Learning Targets

- I can find missing side lengths or angles using properties of translations, rotations and reflections.

Lesson Narrative

In this lesson, students create composite shapes using translations, rotations, and reflections of polygons and continue to observe that the side lengths and angle sizes do not change. They use this understanding to draw conclusions about the composite shapes. Later, they will use these skills to construct informal arguments, for example about the sum of the angles in a triangle.

When students rotate around a vertex or reflect in the side of a shape, it is easy to lose track of the centre of rotation or line of reflection since they are already part of the shape. It can also be challenging to name corresponding points, line segments, and angles when a shape and its transformation share a side. Students attend to these details carefully in this lesson.

Consider using the optional activity if you need to reinforce students' belief that translations, rotations and reflections preserve distances and angle sizes after the main activities.

Addressing

- Lines are taken to lines, and line segments to line segments of the same length.
- Angles are taken to angles of the same size.

Instructional Routines

- Stronger and Clearer Each Time
- Clarify, Critique, Correct
- Compare and Connect

•	Think Pair Share

Required Materials

Geometry toolkits

tracing paper, graph paper, coloured pencils, scissors, and an index card to use as a straightedge or to mark right angles, plus a ruler and protractor. Clear protractors with no holes and with radial lines printed on them are recommended.

Student Learning Goals

Let's use reasoning about translations, rotations and reflections to find measurements without measuring.

10.1 Angles of an Isosceles Triangle

Warm Up: 10 minutes

Isosceles triangles are triangles with (at least) one pair of congruent sides. Isosceles triangles also have (at least) one pair of congruent angles. In this warm-up, students show why this is the case using translations, rotations and reflections by exploiting the fact that translations, rotations and reflections of the plane do not change angle sizes.

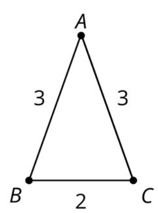
Launch

Give students 3 minutes of quiet work time followed by whole-class discussion.

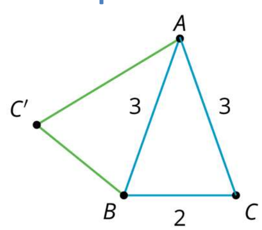
Student Task Statement

Here is a triangle.

1. Reflect triangle ABC over line AB. Label the image of C as C'.
2. Rotate triangle ABC' around A so that C' matches up with B.
3. What can you say about the sizes of angles B and C?



Student Response

1. 

2. Rotating ABC' as described takes ABC' back to the original triangle.

3. The sizes of angles B and C are the same. Neither the rotation nor the reflection changes the angle sizes, and so since these transformations take the angle at C to the angle at B, they must have the same size.

Activity Synthesis

Select students to share their images and conclusions about the sizes of angles B and C.

Time permitting, mention that it is also true that when a triangle has two angles with the same size then the sides opposite those angles have the same length (i.e., the triangle is isosceles). This can also be shown with translations, rotations and reflections. Reflect the triangle first and then line up the sides containing the pairs of angles with the same size.

10.2 Triangle Plus One

10 minutes

The purpose of this task is to use translations, rotations and reflections to describe an important picture that students have seen earlier in KS3 when they developed the formula for the area of a triangle. They first found the area of a parallelogram to be base × height and then, to find $\frac{1}{2}$base × height for the area of a triangle, they "composed" two copies of a triangle to make a parallelogram. The focus of this activity is on developing this precise language to describe a familiar geometric situation.

Students need to remember and use an important property of 180 degree rotations, namely that the image of a line after a 180 degree rotation is parallel to that line. This is what allows them to conclude that the shape they have built is a parallelogram.

Instructional Routines

- Stronger and Clearer Each Time
- Think Pair Share

Launch

Arrange students in groups of 2. Provide access to geometry toolkits. A few minutes of quiet work time, followed by sharing with a partner and a whole-class discussion.

Action and Expression: Internalise Executive Functions. To support development of organisational skills, check in with students within the first 2–3 minutes of work time. Look for students who struggle with visualising the 180 degree rotation using centre M. Consider pausing for a brief discussion to invite 1–2 pairs of students to demonstrate and explain how to do the rotation.
Supports accessibility for: Memory; Organisation

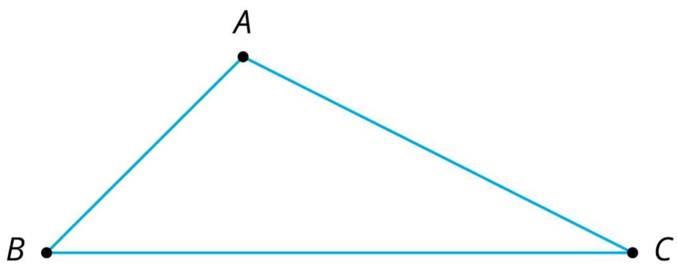
Anticipated Misconceptions

Students may struggle to see the 180° rotation using centre M. This may be because they do not understand that M is the centre of rotation or because they struggle with visualising a 180° rotation. Offer these students the rotation overlay from earlier in this unit to help them see the rotated triangle.

Student Task Statement

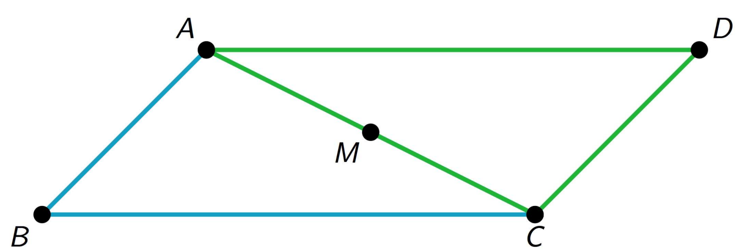
Here is triangle ABC.

1. Draw midpoint M of side AC.
2. Rotate triangle ABC 180 degrees using centre M to form triangle CDA. Draw and label this triangle.
3. What kind of quadrilateral is $ABCD$? Explain how you know.



Student Response

1.



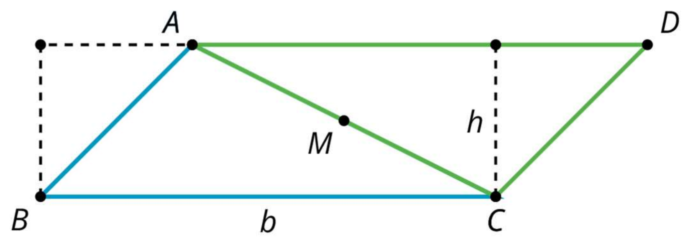
2. A parallelogram. The 180 degree rotation around M takes line AB to line CD and so these are parallel. It also takes line BC to line AD so these lines are also parallel. That means that $ABCD$ is a parallelogram.

Are You Ready for More?

In the activity, we made a parallelogram by taking a triangle and its image under a 180-degree rotation around the midpoint of a side. This picture helps you justify a well-known formula for the area of a triangle. What is the formula and how does the diagram help justify it?

Student Response

A formula for the area of a triangle is $A = \frac{1}{2}bh$, where b is a base of the triangle and h the corresponding height. In the image we have added b and h to the parallelogram, and marked off two triangles of interest with dashed lines. Since the left and right triangles have the same area, the area of the parallelogram displayed is the same as the area of a rectangle with height h and base b, namely, an area of bh. Since the area of each triangle is half the area of the parallelogram, each triangle has area $A = \frac{1}{2}bh$.



Activity Synthesis

Begin the discussion by asking, "What happens to points A and C under the rotation?" (They end up at C and A, respectively.) This type of rotation and analysis will happen several times in upcoming lessons.

Next ask, "How do you know that the lines containing opposite sides of $ABCD$ are parallel?" (They are taken to one another by a 180 degree rotation.) As seen in the previous lesson, the image of a 180° rotation of a line ℓ is parallel to ℓ. Students also saw that when 180° rotations were applied to a *pair* of parallel lines it resulted in a (sometimes) new pair of parallel lines which are also parallel to the original lines. The logic here is the same, except that only one line is being rotated 180° rather than a pair of lines. This does not need to be mentioned unless it is brought up by students.

Finally, ask students "How is the area of parallelogram $ABCD$ is related to the area of triangle ABC?" (The area of the parallelogram $ABCD$ is twice the area of triangle ABC because it is made up of ABC and CDA which has the same area as ABC.) Later in this unit, area of shapes and their images under translations, rotations and reflections will be studied further.

Writing, Speaking: Stronger and Clearer Each Time. In this routine, students are given a thought-provoking question or prompt and are asked to create a first draft response. Students meet with 2–3 partners to share and refine their response through conversation. While meeting, listeners ask questions such as, “What did you mean by . . .?” and “Can you say that another way?” Finally, students write a second draft of their response reflecting ideas from partners and improvements on their initial ideas. The purpose of this routine is to provide a structured and interactive opportunity for students to revise and refine their ideas through verbal and written means.
Design Principle(s): Optimise output (for justification)

How It Happens:

1. Use this routine to provide students a structured opportunity to refine their justification for the question asking “What kind of quadrilateral is $ABCD$? Explain how you know.” Give students 2–3 minutes to individually create first draft responses in writing.

2. Invite students to meet with 2–3 other partners for feedback.

   Instruct the speaker to begin by sharing their ideas without looking at their written draft, if possible. Listeners should press for details and clarity.

   Provide students with these prompts for feedback that will help individuals strengthen their ideas and clarify their language: “What do you mean when you say...?”, “Can you describe that another way?”, “How do you know the lines are parallel?”, and “What happens to lines under rotations?” Be sure to have the partners switch roles. Allow 1–2 minutes to discuss.

3. Signal for students to move on to their next partner and repeat this structured meeting.

4. Close the partner conversations and invite students to revise and refine their writing in a second draft. Students can borrow ideas and language from each partner to strengthen the final product.

   Provide these sentence frames to help students organise their thoughts in a clear, precise way: “Quadrilateral $ABCD$ is a ____ because....” and “Another way to verify this is....”.

   Here is an example of a second draft:

   “$ABCD$ is a parallelogram, I know this because a 180-degree rotation creates new lines that are parallel to the original lines. In this shape, the 180-degree rotation takes line AB to line CD and line BC to line AD. I checked this by copying the triangle onto tracing paper and rotating it 180 degrees. This means the new shape will have two pairs of parallel sides. Quadrilaterals that have two pairs of parallel sides are called parallelograms.”

5. If time allows, have students compare their first and second drafts. If not, have the students move on by discussing other aspects of the activity.

10.3 Triangle Plus Two

15 minutes

This activity continues the previous one, building a more complex shape this time by adding an additional copy of the original triangle. The three triangle picture in the task statement will be important later in this unit when students show that the sum of the three angles in a triangle is 180°. To this end, encourage students to notice that the points E, A, and D all lie on a line.

As with many of the lessons applying transformations to build shapes, students are constantly using their structural properties to make conclusions about their shapes. Specifically, that translations, rotations and reflections preserve angles and side lengths.

Instructional Routines

- Clarify, Critique, Correct

Launch

Keep students in the same groups. Provide access to geometry toolkits. Allow for a few minutes of quiet work time, followed by sharing with a partner and a whole-class discussion.

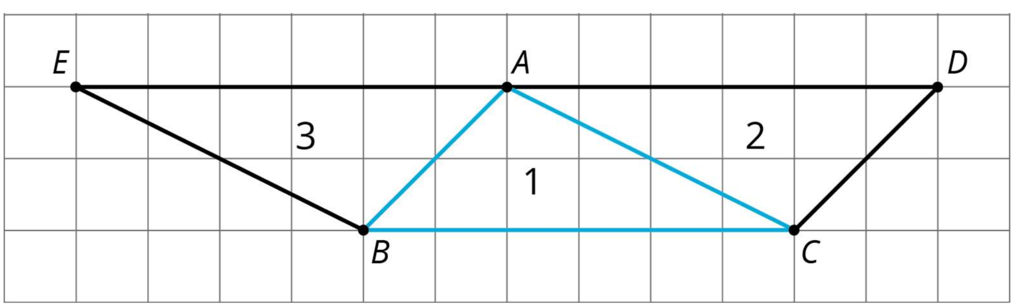
Representation: Internalise Comprehension. Demonstrate and encourage students to use colour coding and annotations to highlight connections between representations in a problem. For example, use the same colour to highlight corresponding points in different triangles and consider redrawing the triangles in the same orientation to emphasise corresponding parts.
Supports accessibility for: Visual-spatial processing

Anticipated Misconceptions

Students may have trouble understanding which pairs of points correspond in the first two questions, particularly the fact that point A in one triangle may not correspond to point A in another. Use tracing paper to create a transparency of triangle ABC, with its points labelled, and let students perform their transformation. They should see A, B, and C on top of points in the new triangle.

Student Task Statement

The picture shows 3 triangles. Triangle 2 and triangle 3 are images of triangle 1 under rotations.



1. Describe a transformation that takes triangle 1 to triangle 2. What points in triangle 2 correspond to points A, B, and C in the original triangle?

2. Describe a transformation that takes triangle 1 to triangle 3. What points in triangle 3 correspond to points A, B, and C in the original triangle?

3. Find two pairs of line segments in the diagram that are the same length, and explain how you know they are the same length.

4. Find two pairs of angles in the diagram that have the same size, and explain how you know they have the same size.

Student Response

1. Answers vary. Sample response: a 180-degree rotation using the midpoint of side AC as centre. In triangle 2, point C corresponds to A in the original, D corresponds to B in the original, and A corresponds to C in the original.

2. Answers vary. Sample responses: a 180-degree rotation using the midpoint of side AB as centre or a 180 degree rotation using the midpoint of line segment AC as centre followed by a translation taking A to E. In triangle 3, point A corresponds to B in the original, B corresponds to A in the orginal, and E corresponds to C in the original.

3. Answers vary. Sample response: line segment AE and line segment BC are the same length and line segments AB and CD are also the same length. This is true because a translation, rotation or reflection doesn't change a shape's side lengths.

4. Answers vary. Sample response: $\angle D$ and $\angle ABC$ have the same size and so do $\angle E$ and $\angle ACB$. This is true because a translation, rotation or reflection doesn't change a shape's angle sizes.

Activity Synthesis

Ask students to list as many different pairs of matching line segments as they can find. Then, do the same for angles. Record these for all to see. Students may wonder why there are fewer pairs of line segments: this is because of shared sides AB and AC. If they don't ask, there's no reason to bring it up.

If you create a visual display of these pairs, hang on to the information about angles that have the same size. The same diagram appears later in this unit and is used for a proof about the sum of the angles in a triangle.

After this activity, ask students to summarise their understanding about lengths and angle sizes under translations, rotations and reflections. If students don't say it outright, you should: "Under any translation, rotation or reflection, lengths and angle sizes are preserved."

Writing: Clarify, Critique, Correct. In this routine, students are given an incorrect or incomplete piece of mathematical work. This may be in the form of a written statement, drawing, problem-solving steps, or another mathematical representation. Pairs of students analyse, reflect on, and improve the written work by correcting errors and clarifying meaning. Typical prompts are: "Is anything unclear?" or "Are there any reasoning errors?"

The purpose of this routine is to engage students in analysing mathematical thinking that is not their own and to solidify their knowledge through communicating about conceptual errors and ambiguities in language.
Design Principle(s): Maximise meta-awareness

How It Happens:

1. In the class discussion for this activity, present this incomplete description of a transformation that takes triangle 1 to triangle 2:

   "CB and AD are the same because you turn ABC."

   Prompt students to identify the ambiguity of this response. Ask students, "What do you think this person is trying to say? What is unclear?"

2. Give students 1 minute of individual time to respond to the questions in writing, and then 3 minutes to discuss with a partner.

   As pairs discuss, provide these sentence frames for scaffolding: "I think that what the author meant by 'turn ABC' was….", "The part that is the most unclear to me is … because….", and "I think this person is trying to say….". Encourage the listener to press for detail by asking follow-up questions to clarify the intended meaning of the statement. Allow each partner to take a turn as the speaker and listener.

   Listen for students using appropriate geometry terms such as "transformation" and "rotation" in explaining why the two sides are equivalent.

3. Then, ask students to write a more precise version and explain their reasoning in writing with their partner. Improved responses should include for each step an explanation, order/time transition words (first, next, then, etc.), and/or reasons for decisions made during steps.

   Here are two sample improved responses:

   "First, I used tracing paper to create a copy of triangle ABC because I wanted to transform it onto the new triangle. Next, I labelled the vertices on my tracing paper. Then, using the midpoint of side AC as my centre, I rotated the tracing paper 180 degrees, because then it matched up on top of triangle 2. So, sides CB and AD are equivalent."

   or

   "First, I accessed my geometry technology tool to create the drawing of only triangle 1 and 2. Next, I tried different transformations of triangle 1 to make the points A, B, and C fall on top of points in triangle 2. The one that worked was rotating triangle 1 180 degrees using the midpoint of side AC. Finally, I know that sides CB and AD are equivalent because triangle 1 is exactly on top of triangle 2."

4. Ask each pair of students to contribute their improved response to a poster, the whiteboard, or digital projection. Call on 2–3 pairs of students to present their response to the whole class, and invite the class to make comparisons among the responses shared and their own responses.

   Listen for responses that identify the correct pair of equivalent sides and explain how they know. In this conversation, also allow students the opportunity to name other equivalent sides and angles.

5. Close the conversation with the generalisation that lengths and angle sizes are preserved under any translation, rotation or reflection, and then move on to the next lesson activity.

10.4 Triangle ONE Plus

Optional: 10 minutes

This activity builds upon the ideas of the previous one. This time a pattern is built from a single triangle via reflections and the goal is to study the angles and side lengths in this pattern as it grows. If they build the pattern carefully, students may notice after putting together 6 triangles, like in the previous activity, that two of the sides of the triangles (one side of the original and one side of the 6th) lie on the same line. The 6 triangle pattern can be reflected over this line to make it “complete” with 12 copies of the original triangle. Alternatively, students may notice the right angle made by 3 triangles and reason that they can complete a circle with 4 right angles. Rather than repeating the reflecting procedure 12 times, it is possible to use the structure of what they have learned along the way to accurately predict how many copies make a circle.

Instructional Routines

- Compare and Connect

Launch

Provide access to geometry toolkits. Allow for 8 minutes of work time, then a brief whole-class discussion.

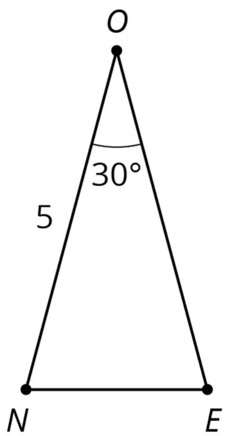
Anticipated Misconceptions

If students are stuck with the first reflection, suggest that they use tracing paper. If you need to, show them the first reflected triangle, then have them continue to answer the problems and do the next reflection on their own.

Some students may have difficulty with the length of OT, since it uses the initial information that triangle ONE is isosceles (otherwise, OT is unlabelled). Ask students what other information is given and if they can use it to out the missing length.

Student Task Statement

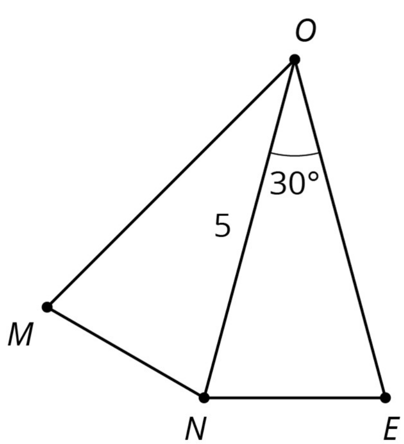
Here is isosceles triangle ONE. Its sides ON and OE have equal lengths. Angle O is 30 degrees. The length of ON is 5 units.



1. Reflect triangle ONE in line segment ON. Label the new vertex M.
2. What is the size of angle MON?
3. What is the size of angle MOE?
4. Reflect triangle MON in line segment OM. Label the point that corresponds to N as T.
5. How long is $\overline{OT}$? How do you know?
6. What is the size of angle TOE?
7. If you continue to reflect each new triangle this way to make a pattern, what will the pattern look like?

Student Response

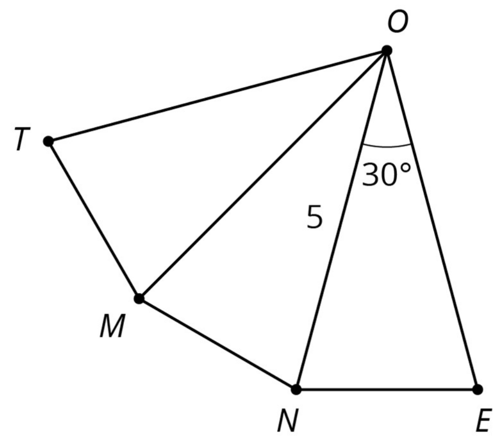
1.



2. 30° because angle sizes are the same after applying a transformation and angle NOE becomes angle NOM after reflection.

3. 60° because $\angle MOE$ is got by putting together $\angle MON$ and $\angle NOE$ and these are both 30 degree angles.

4.



5. Line segment OT measures 5 units because it is the image of line segment ON after a reflection so it has the same length as line segment ON.

6. 90°, a right angle because $\angle MOE$ measures 60° and $\angle TOM$ measures 30°. Since $\angle TOE$ is got by putting together $\angle TOM$ and $\angle MOE$ so it is a 90 degree angle.

7. Answers vary. One possible description: Eventually point O will be completely surrounded by triangles, with the 12th triangle touching $\overline{OE}$ again. There are 4 right angles in a full circle and each right angle has 3 copies of the original triangle.

Activity Synthesis

The main goal of this activity is to apply and reinforce students' belief that translations, rotations and reflections preserve distances and angle sizes. Watch and make sure students are doing well with this. If they are not, reinforce the concept. It is critical that this is understood by students before moving forward.

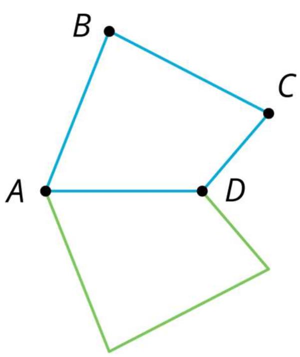
Identify a few students, each with a different response, to share their description of the pattern they saw in the last question. The way in which each student visualises and explains this shape may give insight into the different strategies used to create the final pattern.

Note: It is *not* important nor required that students know or understand how to find the base angle sizes of the isosceles triangles, or even that the base angles have the same size (though students do study this in the warm-up). Later in this unit, students will learn and prove that the sum of the angle sizes in a triangle is 180 degrees.

Engagement: Develop Effort and Persistence. Break the class into small discussion groups and then invite a representative from each group to report back to the whole class.
Supports accessibility for: Attention; Social-emotional skills Speaking: Compare and Connect. Use this routine when students describe the pattern they saw when reflecting the triangle multiple times. Ask students to consider what is the same and what is different about each description. Draw students' attention to the different ways the pattern is explained. Some students may benefit from the use of gestures to support their understanding of the descriptions. These exchanges can strengthen students' mathematical language use and reasoning to make sense of strategies used to create composite shapes.
Design Principle(s): Maximise meta-awareness

Lesson Synthesis

Briefly review the properties of translations, rotations and reflections (they preserve lengths and angle sizes). Go over some examples of corresponding sides and angles in shapes that share points. For example, talk about which sides and angles correspond if this image is found by reflecting $ABCD$ in line AD.



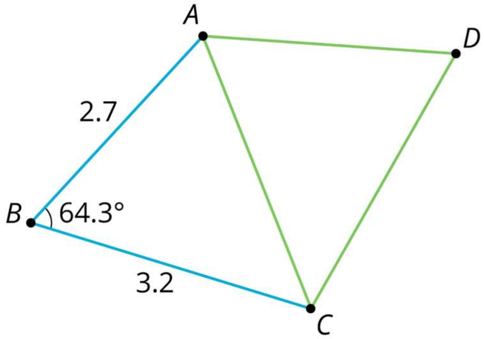
Point out to students that sides on a reflection line do not move, so they are their own image when we reflect in a side. Also, the centre of rotation does not move, so it is its own image when we rotate around it. All points move with a translation.

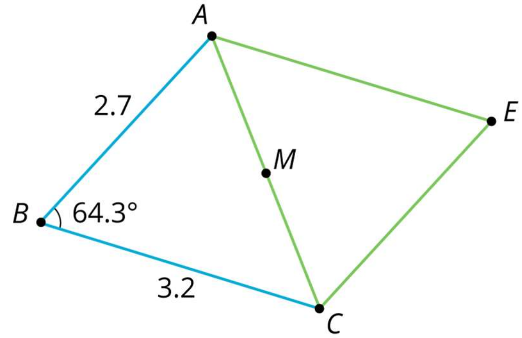
10.5 Identifying Side Lengths and Angle Sizes

Cool Down: 5 minutes

Students apply the fact that translations, rotations and reflections preserve side lengths and angles. They are presented with a shape and two transformed images, and they use what they know to find the side lengths and angles of the transformed shapes.

Student Task Statement



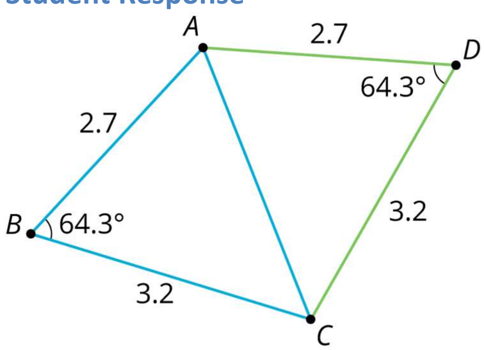


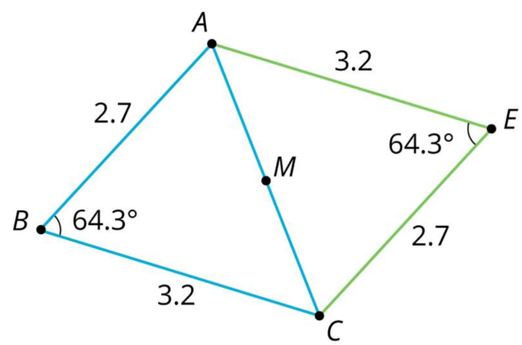
Here is a diagram showing triangle ABC and some transformations of triangle ABC.

On the left side of the diagram, triangle ABC has been *reflected* in line AC to form quadrilateral $ABCD$. On the right side of the diagram, triangle ABC has been *rotated* 180 degrees using midpoint M as a centre to form quadrilateral $ABCE$.

Using what you know about translations, rotations and reflections, side lengths and angle sizes, label as many side lengths and angle sizes as you can in quadrilaterals $ABCD$ and $ABCE$.

Student Response

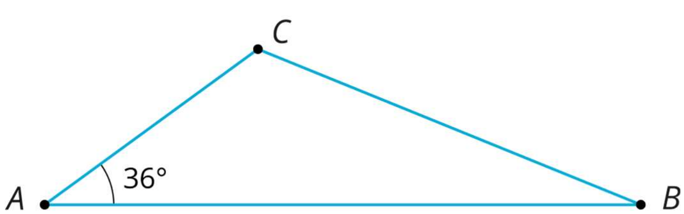




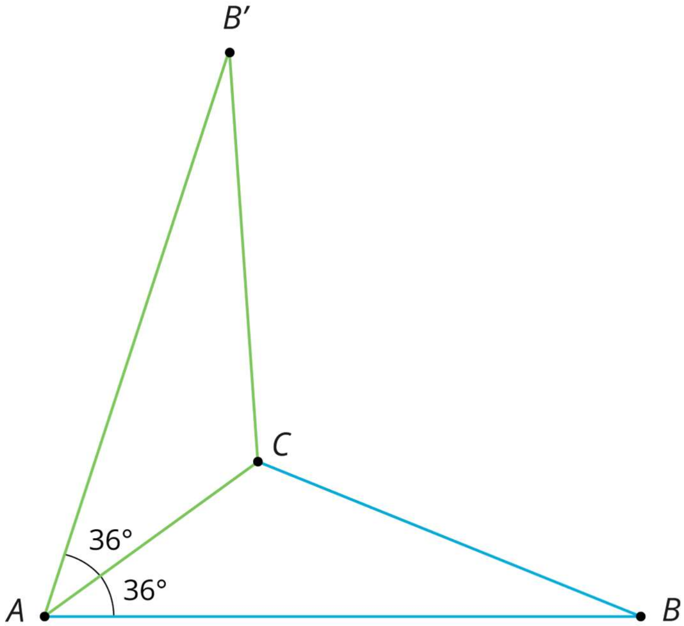
Student Lesson Summary

Earlier, we learned that if we apply a sequence of translations, rotations and reflections to a shape, then corresponding sides have equal length and corresponding angles have equal size. These facts let us figure out things without having to measure them!

For example, here is triangle ABC.



We can reflect triangle ABC in side AC to form a new triangle:



Because points A and C are on the line of reflection, they do not move. So the image of triangle ABC is $AB'C$. We also know that:

- Angle $B'AC$ measures 36° because it is the image of angle BAC.
- Line segment AB' has the same length as line segment AB.

When we construct shapes using copies of a shape made with translations, rotations and reflections, we know that the sizes of the images of line segments and angles will be equal to the sizes of the original line segments and angles.

Lesson 10 Practice Problems

1. **Problem 1 Statement**

   Here is the design for the flag of Trinidad and Tobago.

Describe a sequence of translations, rotations, and reflections that take the lower left triangle to the upper right triangle.

Solution

Answers vary. Sample response: 180 degree rotation around the centre point of the flag. Another sample response: The lower left triangle is first translated to the right so that it shares an edge with the upper right triangle. Then it's rotated 180 degrees around the midpoint of the common side.

2. **Problem 2 Statement**

Here is a picture of an older version of the flag of Great Britain. There is a transformation that takes triangle 1 to triangle 2, another that takes triangle 1 to triangle 3, and another that takes triangle 1 to triangle 4.

a. Measure the lengths of the sides in triangles 1 and 2. What do you notice?

b. What are the side lengths of triangle 3? Explain how you know.

c. Do all eight triangles in the flag have the same area? Explain how you know.

Solution

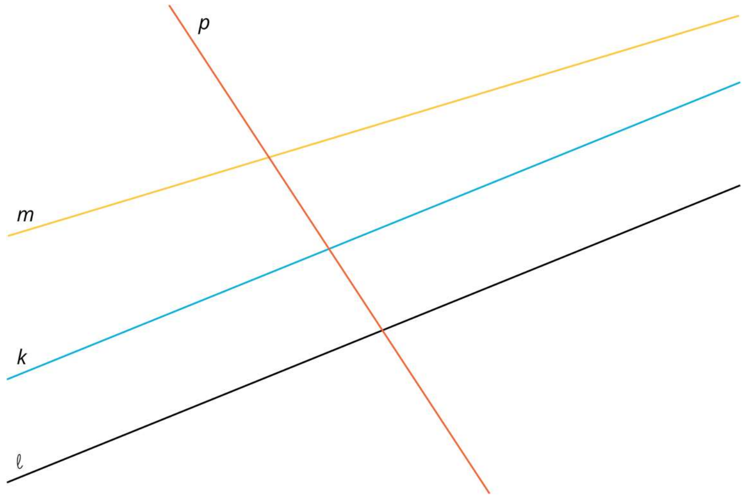
a. Answers vary. The side lengths of the two triangles are the same.

b. The side lengths will be the same as triangle 1, because there is a transformation taking triangle 1 to triangle 3.

c. No. The four triangles without number labels are larger, so they will not have the same area as the smaller labelled triangles.

3. **Problem 3 Statement**

a. Which of the lines in the picture is parallel to line ℓ? Explain how you know.



b. Explain how to translate, rotate or reflect line ℓ to obtain line k.

c. Explain how to translate, rotate or reflect line ℓ to obtain line p.

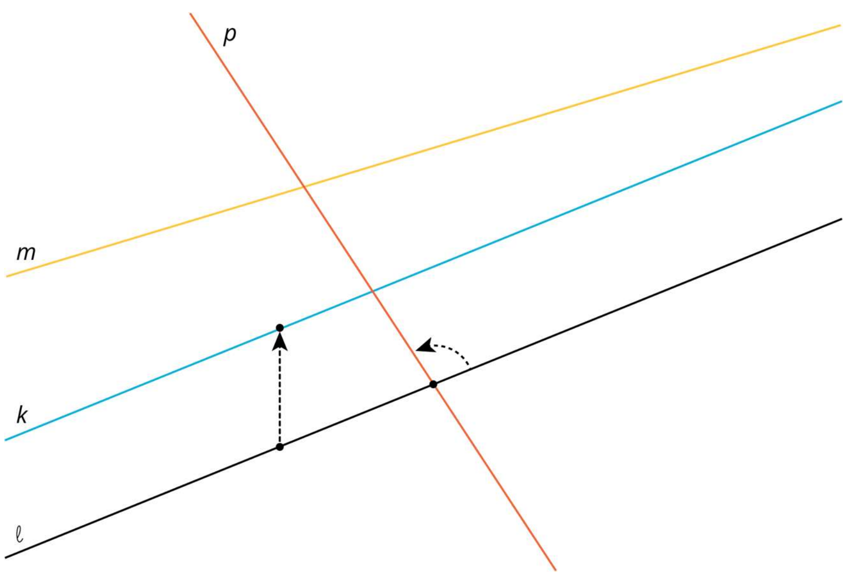
Solution

a. k. These two lines do not intersect no matter how far out they extend.

b. Line k can be obtained by translating line ℓ.

c. Line p can be obtained by rotating line ℓ.

The picture shows how to translate ℓ to get k and how to rotate ℓ to get p.



4. **Problem 4 Statement**

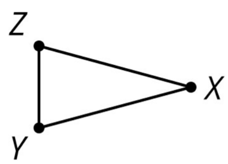
Point A has coordinates (3,4). After a translation 4 units left, a reflection in the x-axis, and a translation 2 units down, what are the coordinates of the image?

Solution

(-1,-6)

5. **Problem 5 Statement**

Here is triangle XYZ:



Draw these three rotations of triangle XYZ together.

a. Rotate triangle XYZ 90 degrees clockwise around Z.

b. Rotate triangle XYZ 180 degrees around Z.

c. Rotate triangle XYZ 270 degrees clockwise around Z.

Solution

Each rotation shares vertex Z with triangle XYZ. The four triangles together look like a pinwheel.