Focus East Asia

A Classic from China: The *Nine Chapters*

by Prof. Randy Schwartz

The Beijing Olympics last August riveted the planet's attention on China as a rising power. Not only were the athletic events themselves exciting to watch, but the opening and closing ceremonies were also dramatic, giving people a sense of China's immensely long history and its big role in the world. Schoolcraft's International Institute, through its Focus East Asia project this year, has organized speakers, films, and other activities to help us learn more about this important region.

In mathematics and other sciences, China has also played an advanced world role. This is not a new development. In fact, the Needham Research Institute in Cambridge, England has been working for decades to summarize the rich history of China's contributions to science and technology. So far, the summary occupies over 20 large published volumes; volume 3 is on mathematics.

This article is about the most important mathematical work in China's long history, the *Jiuzhang Suanshu* ("Nine Chapters on the Art of Calculation"). The book was used throughout China for centuries, and it also circulated in Korea and Japan, influencing mathematics there. The author(s) and date of the work are no longer known, but clues in the text— including the units used in story problems— indicate that it was probably written shortly after 200 BC. The original version of the *Nine Chapters* presented rules and algorithms but without formal proof or derivation. Later, in the year 263 AD, the mathematician Liu Hui provided a written commentary that included justification for the techniques used.

Looking over portions of the *Nine Chapters*, and solving some of the story problems in it, is a good way to see how the development of mathematics in Asia was shaped by how life and society were organized there. Life in the West and in the East have had similarities and differences, so we can expect that the mathematics of these two cultures will also have some similarities and some differences.

Below, I'll provide some background about the book and then present 10 story problems from it. I challenge you, the reader, to solve as many of the problems as you can (see Challenge to Our Readers, page 12). All 10 problems are drawn from the edited translation by Shen Kangshen et al. (1999).

The *Nine Chapters* versus the *Elements*

Of the works considered China's Ten Mathematical Classics, the *Nine Chapters* is the oldest and the most influential. Like Euclid's *Elements* in the West, it was used as a basic textbook for mathematics from ancient times all the way to 1600 AD and even later.

Comparing these two works, we can't help but notice some differences between the roots of Eastern and Western mathematics—

- The *Nine Chapters* was focused more on practical problem-solving than on theory. It is a how-to manual consisting of 246 exemplary problems and their solutions. After a few problems of a given type were solved, the general method of solution was summarized. By contrast, the *Elements* was a rigorous development of the structure of geometry and some related fields, with theorems built up in a careful manner from a foundation consisting of basic "elements" (axioms and postulates). The style of the *Nine Chapters* was inductive; the *Elements*, deductive.
- In ancient times, Asians were far more adept at arithmetic than were Westerners. The *Nine Chapters* used decimal place-value arithmetic at a time when Europeans were still using Roman numerals or other cumbersome systems. The ancient Chinese were also the first to use negative numbers, a practice not adopted in Europe until the 1400's. The Chinese word for mathematics, *suanshu*, that appears in the title of the *Nine Chapters*, literally means "the art of calculation".
- The study of prime numbers, factorization, and other topics in number theory, which was an important part of the *Elements* and of ancient Greek mathematics, was not taken up in traditional Chinese mathematics. For example, in China the greatest common divisor of two numbers was found not by factoring, but by a process of repeated subtraction that was also described by Euclid.
- Centuries before other peoples, the Chinese developed algorithms for solving linear problems, including matrix methods and techniques of excess and deficit.

How much Chinese mathematics was rooted in practical problem-solving is reflected in the nine chapter subjects of the *Jiuzhang Suanshu*. The chapters correspond to the nine arithmetical arts of Chinese tradition:

Chapt. 1 "Field Measurement"— calculating the areas of various shapes of farming plots such as triangles, rectangles, and circles (using an accurate approximation of pi), plus arithmetical rules for fractions and greatest common divisors.

Chapt. 2 "Millet and Rice"— using ratios and proportions in the commercial exchange of different kinds of grains and other products.

Chapt. 3 "Distribution by Proportion"— subdividing quantities by direct, inverse, or compound proportion, plus a discussion of arithmetic and geometric progressions.

Chapt. 4 "What Width?"— calculating an unknown dimension of a rectangle or rectangular solid if the area or volume is known, including how to find reciprocals , square roots, and cube roots of various types of numbers.

Chapt. 5 "Construction Consultations"— finding the areas and volumes of shapes and solids used in designing buildings and other structures.

Chapt. 6 "Fair Levies"— proportionally distributing wages and taxes, continuing Chapter 3.

Chapt. 7 "Excess and Deficit"— solving linear equations with arithmetical algorithms and without algebra.

Chapt. 8 "Rectangular Grids"— using matrices to solve simultaneous linear equations.

Chapt. 9 "Right Triangles"— solving practical geometric problems using properties of right triangles, and quadratic equations by an adaptation of the square-root algorithm.

For centuries, the *Nine Chapters* was used to train civil servants in the prestigious imperial bureaucracy. This was consistent with Chinese philosophical traditions, in which intellectuals were supposed to use their skills to benefit society in practical ways. For example, a line of hereditary state officials called *chouren* was specifically charged with "reading the heavens" (studying astronomy) and doing the calculations needed to make calendars, which were so important in this agrarian society. The *chouren*'s routine need for various kinds of approximations stimulated the development of algorithms that utilized fractional and signed numbers. The word *chouren* came to mean both "astronomers" and "mathematicians". (Yǎn and Shírán, pp. 22-24, 32, 48-49, 232)

Decimal Numbers

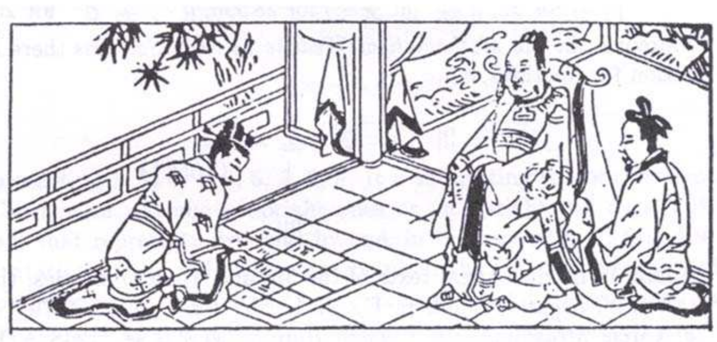
One of the first things that leaps out at you when you examine the *Nine Chapters* is that although it's an ancient work, it uses a "modern" decimal number system.

Decimal numeration arose, along with the earliest writing of Chinese characters, during the transition from slave-ownership to feudalism. In China this transition occurred during the Spring and Autumn (770-476 BC) and the Warring States (475-221 BC) periods, a time when the most advanced societies in Europe were those of Greece and Rome still based on slavery. Later, in medieval times, decimal place-value numeration diffused outward from Asia, spreading from India through the Middle East and North Africa to Europe.

The Chinese counting board is a good example of how a technological invention can influence how science develops, and even how people think. The counting board, in use by 400 BC, was made of polished wood and had rulings that formed a grid of square cells (see illustration on this page). Since Chinese characters are written in columns, proceeding from right to left, it was natural to adapt such columns to represent numbers according to their units, tens, hundreds, etc., from right to left. Each of the digits 1 through 9 was represented by its own Chinese character made with tally-like strokes. These digit characters were formed easily with counting rods (*chousuan*), which were small bamboo sticks having a square or triangular cross-section, and varnished with lacquer. Rods with a red dot were used for positive numbers, and those with a black dot for negative numbers. To "write" a number on the counting board, its digit characters could be placed, one per cell, on one row of the grid. (A blank cell stood for what we would call the zero digit.) To do an arithmetic problem, two or more numbers could be placed on neighboring rows, and the results calculated on successive rows of the board, much as we might do today in working a lengthy addition or multiplication on paper.

You can see how a merchant or a scholar who started doing arithmetic on such a board would quickly acquire the mental habit of breaking quantities into their components column-by-column. This became ingrained into Chinese culture. The abacus, which wasn't widely used in East Asia until medieval times, retained the decimal place-value system in its columns of beads.

For more permanent records, numerals and other characters could also be written directly on plain strips of bamboo or on bone, tortoise shell, silk fabric, or paper, the latter invented in China in the First Century AD. (Paper, silk fabric, and lacquer were all Chinese inventions.)



No old drawings of counting boards survive from China, but this drawing from 18th-Century Japan shows that the boards were introduced into that country. Drawing from Shen et al., p. 14.

System of Units

The decimal number system also influenced Chinese units of measurement.

In 221 BC the ruler Shi Huangdi, based in the Wei Valley, had quickly conquered several nearby warring states and unified the region, ushering in China's first empire, the Qin Dynasty (221-201 BC). The Great Wall was completed under the Qin dynasty. To help ensure the administrative efficiency and cultural unity needed in the empire, the government imposed a standard system of characters for writing as well as a standard system of units for measurement. Thus, China's uniform system of units was due to the early centralization of its political rule.

The units adopted in the Qin and later dynasties included those for length, based on the *zhang*, approximately 7½ feet:

1 *zhang* = 10 *chi* = 100 *cun*;

for longer distances, the *li* or "Chinese mile" was used, equal to 180 *zhang*.

Units of area were applied particularly for measuring farm plots, where a man's "pace" or *bu* (60 *cun*) was a handy reference. The units of area were as follows:

1 *qing* = 100 *mu* = 24,000 square *bu*;

the *qing* was equivalent to about 11.4 of our acres.

Units of volume:

1 *hu* = 10 *dou* = 100 *sheng*;

the *hu* was equivalent to about 5.3 of our gallons.

Units of weight:

1 *jin* = 16 *liang* = 384 *zhu*;

the *jin* was equivalent to a little over 7 of our ounces.

Ratio and Proportion

Many of the problems in the early chapters of *Jiuzhang Suanshu* are simple ones involving ratios and proportions. They were designed to give the reader practice with the system of units, the use of basic arithmetic (including fractions), and the Rule of Three. The latter rule, versions of which were used in many world regions in ancient and medieval times, allows an unknown quantity to be calculated if it stands in a proportion with three known quantities. The following sample problems, drawn from Chapters 2 and 3, respectively, deal with quantities of lacquer and woven silk, classic products of China.

continued on next page

The *Nine Chapters* continued from page 9

> Now pay 5785 coins to purchase 1 *hu* 6 *dou* 7 2/3 *sheng* of lacquer. Tell: how much is 1 *dou*? (Shen et al., p. 151) **(1)**

> Now given 1 *jin* of silk costs 240 coins. Tell: given 1328 coins, how many *jin* of silk are obtained? (Shen et al., p. 169) **(2)**

Your answer to Problem 2 should be given exactly, in units of *jin*, *liang*, *zhu*, and fractions of a *zhu*.

Small Plots and Huge Granaries

In feudal times, agricultural production was the main source of wealth. The vast majority of the people were poor peasants. They paid for things not with coins but with portions of their harvest, especially quantities of grain.

In China, millet was the most common grain, especially in the north. Rice was a comparative luxury enjoyed mostly by the upper classes, for it "takes a whole village" to tend the young shoots of rice, keep the paddies flooded, and manage the harvest.

Most peasants worked small plots of land owned by a landlord. In exchange, they were required every year to give the landlord a certain portion of the grain harvest, known as *rent*. This practice is reflected in another problem from Chapter 3 involving proportions:

> Now given a field of 1 *mu*, 6 2/3 *sheng* of millet is collected [as rent]. Tell: given 1 *qing* 26 *mu* 159 [square] *bu* of field how much millet is collected? (Shen et al., p. 171) **(3)**

Millet or other grains were also used to pay taxes to the central authorities every year. These taxes went to enrich the emperor and his court in the capital; to the network of local officials and military troops stationed across the empire and on its frontiers; and to government granaries that were maintained for times of famine. The sheer task of transporting tax grains to central repositories was a gargantuan one, but it was meted out proportionally to the various localities according to a system called *junshun* ("fair levies"). Localities might be levied in direct proportion to their number of households, in inverse proportion to their distance to the central bureau, and/or in inverse proportion to the local cost of grain. In solving problems such as the following, which is the first one in Chapter 6, rounding of the answers is necessary, since the carts used to transport grain taxes were always filled to capacity.

> Now given the task of transporting tax millet is distributed among four counties. County A, 8 days from the tax bureau, has 10,000 households; County B, 10 days from the bureau, has 9,500 households; County C, 13 days from the bureau, has 12,350 households; County D, 20 days from the bureau, has 12,200 households. The total tax millet is 250,000 *hu* needing 10,000 carts. Assume the task is to be distributed in accordance with the distance from the bureau and the number of households. Tell: how much millet should each county transport? How many carts does each county employ? (Shen et al., p. 310) **(4)**

The following problem from the same chapter deals with shipments of millet to Taicang, the national granary. This granary was located in Ch'ang-an (now called Xi'an), the first capital of China, completed during the Han Dynasty (206 BC – 220 AD). The source of the millet is Shanglin, the old Imperial Garden of the Qin Dynasty, lying to the west of Ch'ang-an. The problem amounts to finding the travel distance between these two sites.

> Now someone transports provisions between two posts. An unloaded cart travels 70 *li* a day and a loaded one 50 *li* a day. Transporting millet from the National Granary to Shanglin. One makes 3 round trips in 5 days, how far is the distance between the two posts? (Shen et al., p. 325) **(5)**

Too Much and Not Enough

Chapter 7 of the *Nine Chapters* is devoted to the use of a technique called *ying bu tsu shu* (literally "the rule of too much and not enough"), often translated as "the method of excess and deficit". The technique amounts to a way of dealing with linear relationships without the use of algebra. Chinese mathematicians were ingenious in applying it to solve many different types of problems, even nonlinear problems solved by linear approximation.

The thinking behind *ying bu tsu shu* was influenced by observations based on work with fractions. In problems such as those involving the addition or subtraction of fractions, the denominators have to be reconciled first:

$$\frac{8}{3}+\frac{7}{4}=\frac{4(8)}{4(3)}+\frac{3(7)}{3(4)}$$
$$=\frac{4(8)+3(7)}{4(3)}$$
$$=\frac{53}{12}.$$

The Chinese used the term *tong* ("uniformization") for the process of creating a common denominator, such as 4(3) = 3(4). They used the term *qi* ("homogenization") for the cross-multiplication that is needed to compare, and in this case add, the numerators: 4(8) + 3(7).

To see how the Chinese adapted homogenization to solve for unknown quantities in linear problems, consider the first example from Chapter 7:

> Now an item is purchased jointly; everyone contributes 8 [coins], the excess is 3; everyone contributes 7, the deficit is 4. Tell: The number of people, the item price, what is each? (Shen et al., p. 358)

In ancient times, when algebra as we know it did not exist, this *joint purchase* problem was not at all easy to solve. Here is how the Chinese viewed its solution, based on Liu Hui's written commentary from 263 AD. We are given:

8 coins per person → 1 item and 3 more coins
7 coins per person and 4 more coins → 1 item.

By quadrupling or tripling, respectively, we get:

4(8) coins per person → 4 items and 4(3) more coins
3(7) coins per person and 3(4) more coins → 3 items.

Adding the results,

4(8) + 3(7) coins per person → 4 + 3 items

Taking the ratio,

$$\frac{4(8)+3(7)}{4+3} \text{ coins per person} \rightarrow 1 \text{ item}$$

53/7 coins per person → 1 item.

So, to afford the purchase, each person must contribute 53/7 the value of one coin. Note the cross-multiplication used in the calculation. This was a pattern that the Chinese committed to memory for use as a shortcut, much as in the problem 8/3 + 7/4.

Once the true cost per person is known, the solution is easily finished. Comparing each person's true cost of 53/7 coins to the 8 = 56/7 coins per person that led to an excess of 3 coins, we see that those 3 coins resulted from an overpayment of 3/7 coin per person. Thus, there must be 7 buyers involved. Finally, 7 buyers each contributing 53/7 coins implies a total price of 53 coins for the item.

The British scholar Joseph Needham (1900-1995), one of the leading historians of Chinese mathematics and science, pointed out that the rationale of balancing excess and deficit found in the *Nine Chapters* seems to reflect one of the key doctrines of Confucius (c. 551 – 479 BC), whereby *yin* and *yang* must be balanced to achieve harmony (Needham, p. 119). In medieval times, the Arabs developed, and introduced to Europe, a technique called *double false position* that is somewhat similar to the Chinese method of excess and deficit. However, double false position was based on Greek theories of ratio and proportion, rather than on Confucian doctrines of homogenization and balance.

Here is the next problem from Chapter 7. Can you solve it?

> Now chickens are purchased jointly; everyone contributes 9, the excess is 11; everyone contributes 6, the deficit is 16. Tell: The number of people, the chicken price, what is each? (Shen et al., p. 358) **(6)**

Matrices

The Chinese counting board, with its grid of square cells, was also useful for storing and manipulating rows and columns of numbers, rather than rows and columns of single digits. Such a numerical grid was called a *fangcheng*, literally "divided rectangle". Much later in the West, this would be called a *matrix*.

The Chinese were many centuries in advance of the rest of the world in using matrices to solve systems of linear equations. The coefficients of each equation were stored in one column, and the columns were filled from right to left. The numbers were then manipulated using the same types of operations described above for problems of excess and deficit: multiplying or dividing a column by a given number, adding or subtracting two columns, etc. To solve a system of linear equations, the rectangle of coefficients was reduced to triangular and then diagonal form, in a process identical to what Europeans would later call Gaussian Elimination.

The first problem in Chapter 8 involves the harvest of three different grades of rice:

> Now given 3 bundles of top grade paddy, 2 bundles of medium grade paddy, [and] 1 bundle of low grade paddy. Yield: 39 *dou* of grain. 2 bundles of top grade paddy, 3 bundles of medium grade paddy, [and] 1 bundle of low grade paddy, yield 34 *dou*. 1 bundle of top grade paddy, 2 bundles of medium grade paddy, [and] 3 bundles of low grade paddy, yield 26 *dou*. Tell: how much [*dou*] does one bundle of each grade yield? (Shen et al., p. 399) **(7)**

The illustration at the top of the next column shows how the above problem would be solved on a traditional Chinese counting board. Chapter 8 includes other problems with as many as six equations and six unknowns.

Geometric Design and Surveying

In the Far East, just as in the West, land surveying and construction were major stimuli for the development of geometry.

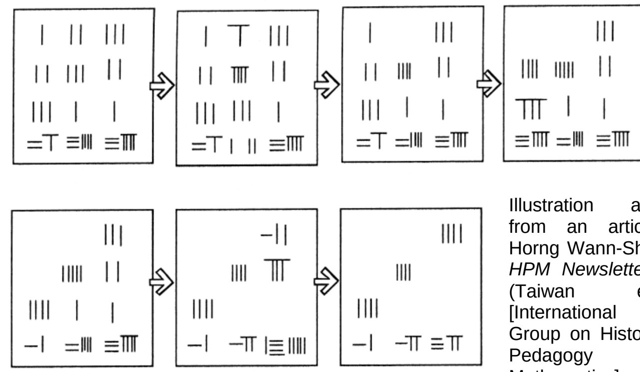
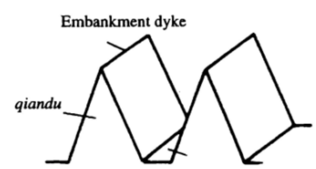


Illustration adapted from an article by Horng Wann-Sheng in *HPM Newsletter* 4:2b (Taiwan edition) [International Study Group on History and Pedagogy of Mathematics].

The first major steps in building roads and canals in China were taken by the emperors of the Qin and Han dynasties. A system of roads built around 100 BC made it possible to journey from Beijing to Canton (Guangzhou) on horseback in only 32 days, a distance of about 1200 miles. The Grand Canal, nearly 200 miles long, was built to connect the two largest river systems, the Yangtze and the Huang Ho. Such roads and canals played a crucial role in conveying troops, traders, crops, and other things, helping to unify the country. Crop irrigation also required extensive networks of canals. Each year, in the intervals between seasonal farming tasks, every able-bodied man was required to donate about one month digging canals or toiling on other public-works projects for his feudal lord, a practice called *corvée labor*. Chinese technology during the Han period was the most advanced in the world.

Building a canal was a significant feat of mathematics and engineering. Plane and solid geometry were needed in designing and surveying the canal and in many phases of its construction. For instance, dirt dug from the canal bed was routinely used to build embankment dykes of packed mud on either side; calculations were done to ensure that the dyke design required exactly the volume of earth that would be excavated. The following problem from Chapter 5 asks for the volume of a certain *qiandu*, a word that originally meant "embankment dyke" but came to mean "triangular prism". Chinese mathematicians had discovered formulae for the volumes of a wide variety of solids, including the *qiandu*.



Drawing: Shen et al., p. 268.

> Now given a *qiandu* with a lower breadth of 2 *zhang*, a length of 18 *zhang* 6 *chi* and an altitude of 2 *zhang* 5 *chi*. Tell: what is the volume? (Shen et al., pp. 267-8) **(8)**

Your answer to the above problem should be given in cubic *chi*.

Chapter 9 of the *Nine Chapters* is focused mostly on right triangles and their use in various types of problems. The Chinese were masters at this. The shorter leg of a right triangle was called the *gōu*; the longer leg, *gŭ*; the hypotenuse, *xián*, literally "bowstring"; the right triangle itself, *gōugŭ*; and the Gōugŭ Rule or theorem was what we call the Pythagorean Theorem.

The author(s) of the *Zhou Shadow-Gauge Manual*, the second of the Ten Mathematical Classics (c. 100 BC), gave a ringing endorsement of the importance of the Gōugŭ Rule (Pythagorean Theorem) by writing,

> Emperor Yŭ quells floods, he deepens rivers and

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The *Nine Chapters* *continued from page 11*

> streams, observes the shape of mountains and valleys, surveys the high and low places, relieves the greatest calamities and saves the people from danger. He leads the floods east into the sea and ensures no flooding or drowning. This is made possible because of the Gōugŭ theorem. (Yăn and Shírán, pp. 29-30)

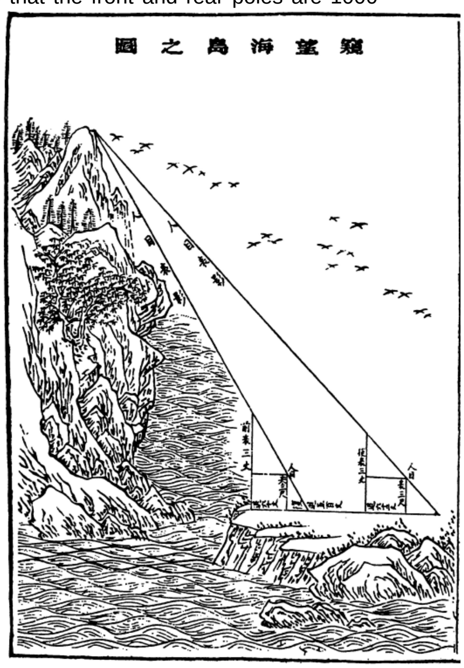
The *Zhou Shadow-Gauge Manual* was an astronomical treatise describing uses of the shadow-gauge or *gnomon*, a right-angled ruler. The treatise included a proof of the Gōugŭ theorem by the *dissection* method. In proof by dissection, which was especially common in traditional Asian geometry, a figure is imagined to be cut into shapes that are rearranged to make clear the relationships among the areas involved.

Here is a typical problem from the *Nine Chapters* that was solved using the Gōugŭ theorem:

> Now given a circular [i.e., cylindrical] log of unknown size buried in a wall. When sawn 1 *cun* deep, it shows a breadth of 1 *chi*. Tell: what is the diameter of the log? (Shen et al., p. 473) **(9)**

In his written commentary on the *Nine Chapters*, Liu Hui also provided an appendix consisting of surveying problems. The appendix was later separated off as a treatise in its own right. It became known as the *Sea Island Mathematical Manual* as a result of its first problem:

> Now survey a sea island. Erect two poles of the same height, 3 *zhang*, so that the front and rear poles are 1000 *bu* apart. They are aligned with the summit of the island. Move backwards 123 *bu* from the front pole, sighting at ground level, and find that the summit of the island coincides with the tip of the pole. Move backwards 127 *bu* from the rear pole, sighting at ground level, and find that the summit of the island also coincides with the tip of the pole. Tell: what are the height of the island and its distance from the [front] pole? (Shen et al., p. 539; illustration, p. 542) **(10)**



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Challenge to Our Readers

The above article includes 10 story problems from the Chinese textbook known as the *Nine Chapters*. Read the article, and solve as many of the problems as you can.

Submit your findings to editor Randy Schwartz (room BTC-510 or by postal or electronic mail: see page 2). If you are a math student, your instructor might be able to give you some extra-credit points for your work.

References

Needham, Joseph, *Science and Civilization in China*, Vol. 3, "Mathematics and the Sciences of the Heavens and the Earth" (Cambridge: Cambridge University Press, 1959).

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Students Explore the *Nine Chapters* from China

We were pleased when our article "A Classic from China: The *Nine Chapters*" (*The Right Angle*, October 2008) was greeted with praise from many different mathematics instructors. They and their students really got into the Reader Challenge: to solve 10 selected story problems from this ancient Chinese manuscript.

Thirteen students, enrolled in a variety of mathematics courses at Schoolcraft, submitted responses to our Challenge:

- Lisa Mundy, a student in Math 53 (Beginning Algebra) taught by Matt Cooper
- Jackie Blasius, Christie Potter, and Linda Wroblewski, all students in Math 113 (Intermediate Algebra) taught by Tanya Reynolds
- James Dickerson, Laura Houdeshell, Peter McCrary, Aaron Rogers, and Dena Sana, all students in Math 122 (Elementary Statistics) taught by Dr. Reynolds
- Allison Lebeau, a student in Math 122 (Elementary Statistics) taught by Randy Schwartz
- Michael Thomas, a student in Math 145 (Calculus for Business and Social Science) taught by Dennis Smith
- Ed Clancy and Sindhuja Sunder, both students in Math 252 (Differential Equations) taught by Prof. Schwartz.

Jim Probelski and Mardell Sitzler were among the other instructors who encouraged their students to tackle these questions.

Below, we print correct solutions to all 10 challenge problems.

Problem 1. Now pay 5785 coins to purchase 1 *hu* 6 *dou* 7 2/3 *sheng* of lacquer. Tell: how much is 1 *dou*?

Jackie wrote:

5785 coins → 1 *hu* 6 *dou* 7 2/3 *sheng*

5785 coins → 10 *dou* + 6 *dou* + (7 2/3)/10 *dou*

5785 coins → 16.76666… *dou*

5785 ÷ 16.76666… coins → 1 *dou*

approx 345 coins → 1 *dou*

[By using fractions, we can get the exact answer, 345 15/503.]

Problem 2. Now given 1 *jin* of silk costs 240 coins. Tell: given 1328 coins, how many *jin* of silk are obtained? (Your answer should be given exactly, in units of *jin*, *liang*, *zhu*, and fractions of a *zhu*.)

Michael wrote:

Since 1 *jin* of silk cost 240 coins, then 1328 coins will buy

1328 ÷ 240 *jin* = 5.5333… *jin*
= 384(5.5333…) *zhu*
= 2124 4/5 *zhu*.

Now subtract 5 *jin* = 384(5) *zhu* = 1920 *zhu*, leaving

2124 4/5 – 1920 *zhu* = 204 4/5 *zhu*
= 204 4/5 ÷ 24 *liang*
= 204 4/5 ÷ 24 *liang*
= 8.5333… *liang*.

Now subtract 8 *liang* = 24(8) *zhu* = 192 *zhu*, leaving

204 4/5 – 192 *zhu* = 12 4/5 *zhu*.

Answer: 5 *jin* 8 *liang* 12 4/5 *zhu* of silk.

Problem 3. Now given a field of 1 *mu*, 6 2/3 *sheng* of millet is collected [as rent]. Tell: given 1 *qing* 26 *mu* 159 [square] *bu* of field how much millet is collected?

Peter wrote:

1 *qing* 26 *mu* 159 bu^2 = 100 *mu* + 26 *mu* + 159 bu^2
= 126 *mu* + 159 bu^2
= 240(126) bu^2 + 159 bu^2
= 30240 bu^2 + 159 bu^2
= 30399 bu^2
= 30399 ÷ 240 *mu*
= 126.6625 *mu*.

So the rent would be 126.6625(6 2/3 *sheng*) = approx 844.416 *sheng*.

[By using fractions, we can get the exact answer, 844 5/12 *sheng*.]

Problem 4. Now given the task of transporting tax millet is distributed among four counties. County A, 8 days from the tax bureau, has 10,000 households; County B, 10 days from the bureau, has 9,500 households; County C, 13 days from the bureau, has 12,350 households; County D, 20 days from the bureau, has 12,200 households. The total tax millet is 250,000 *hu* needing 10,000 carts. Assume the task is to be distributed in accordance with the distance from the bureau and the number of households. Tell: how much millet should each county transport? How many carts does each county employ?

Allison wrote:

A: 10,000 houses ÷ 8 days = 1250
B: 9,500 houses ÷ 10 days = 950
C: 12,350 houses ÷ 13 days = 950
D: 12,200 houses ÷ 20 days = 610
Total 3760

A: (1250 ÷ 3760)250,000 ≈ 83,111.70 *hu* of millet
B: (950 ÷ 3760)250,000 ≈ 63,164.89 *hu* of millet
C: (950 ÷ 3760)250,000 ≈ 63,164.89 *hu* of millet
D: (610 ÷ 3760)250,000 ≈ 40,558.51 *hu* of millet
Total 250,000 *hu* of millet

250,000 *hu* needs 10,000 carts, so 1 cart holds 250,000 ÷ 10,000 = 25 *hu*.

A: 83,111.70 ÷ 25 ≈ 3324 carts
B: 63,164.89 ÷ 25 ≈ 2527 carts
C: 63,164.89 ÷ 25 ≈ 2527 carts
D: 40,558.51 ÷ 25 ≈ 1622 carts
Total 10,000 carts.

Problem 5. Now someone transports provisions between two posts. An unloaded cart travels 70 *li* a day and a loaded one 50 *li* a day. Transporting millet from the National Granary to Shanglin. One makes 3 round trips in 5 days, how far is the distance between the two posts?

Let the unknown distance in *li* be called *d*. Since time = distance ÷ speed, the time needed for one round trip is $\frac{d}{70}+\frac{d}{50}$ days. Thus, the time needed for three round trips is $3\left(\frac{d}{70}+\frac{d}{50}\right)$ days. But we were told this is 5 days, so:

$$3\left(\frac{d}{70}+\frac{d}{50}\right)=5$$

$$3\frac{d}{10}\left(\frac{1}{7}+\frac{1}{5}\right)=5$$

$$\frac{3}{10}d\left(\frac{12}{35}\right)=5$$

$$d=5\left(\frac{35}{12}\right)\left(\frac{10}{3}\right)=\frac{875}{18}=48\frac{11}{18}\,li.$$

Problem 6. Now chickens are purchased jointly; everyone contributes 9, the excess is 11; everyone contributes 6, the deficit is 16. Tell: The number of people, the chicken price, what is each?

Ed wrote:

16(9) coins per person →16 items and 16(11) coins excess
11(6) coins per person →11 items and 11(16) coins deficit
16(9) + 11(6) coins per person →27 items

$\frac{16(9)+11(6)}{27}$ coins per person →1 item

$\frac{70}{9}$ coins per person →1 item

So the item costs $\frac{70}{9}$ coins per person. But when each person paid $9=\frac{81}{9}$ coins, the excess was 11 coins. So, when each person overpays $\frac{81}{9}-\frac{70}{9}=\frac{11}{9}$ coins, the excess was 11 coins. Thus, there are 9 persons, and the item price is $\frac{70}{9}\times 9=70$ coins.

Problem 7. Now given 3 bundles of top grade paddy, 2 bundles of medium grade paddy, [and] 1 bundle of low grade paddy. Yield: 39 *dou* of grain. 2 bundles of top grade paddy, 3 bundles of medium grade paddy, [and] 1 bundle of low grade paddy, yield 34 *dou*. 1 bundle of top grade paddy, 2 bundles of medium grade paddy, [and] 3 bundles of low grade paddy, yield 26 *dou*. Tell: how much [*dou*] does one bundle of each grade yield?

Linda and Sindhuja used a matrix method similar to Gaussian Elimination. Jackie and Christie used a matrix method known in Europe as Cramer's Rule of Determinants. Ed and Dena used the algebraic method called Simple Elimination.

Dena wrote:

$$3x+2y+z=39 \qquad (1)$$
$$2x+3y+z=34 \qquad (2)$$
$$x+2y+3z=26 \qquad (3)$$

Subtract (2) from (1):

$$x-y=5$$
$$x=y+5 \qquad (4)$$

Triple (2):

$$6x+9y+3z=102 \qquad (5)$$

Subtract (3) from (5):

$$5x+7y=76$$
$$5x=76-7y$$
$$x=\frac{76-7y}{5} \qquad (6)$$

Equate (4) and (6):

continued on next page

Nine Chapters

continued from page 7

$$y + 5 = \frac{76 - 7y}{5}$$
$$5y + 25 = 76 - 7y$$
$$12y = 51$$
$$y = 4.25 \ dou$$
$$x = y + 5 = 9.25 \ dou$$
$$z = 39 - (3x + 2y) = 39 - (27.75 + 8.5) = 2.75 \ dou.$$

Problem 8. Now given a *qiandu* with a lower breadth of 2 *zhang*, a length of 18 *zhang* 6 *chi* and an altitude of 2 *zhang* 5 *chi*. Tell: what is the volume? (Your answer should be given in cubic *chi*.)

Linda wrote:

1 *zhang* = 10 *chi*
2 *zhang* = 20 *chi*
18 *zhang* 6 *chi* = 186 *chi*
2 *zhang* 5 *chi* = 25 *chi*.

The cross-section is a triangle, whose area is

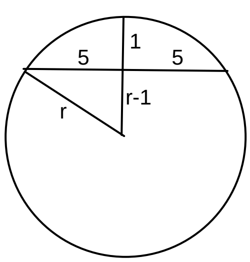
$$A = \tfrac{1}{2}bh$$
$$A = \tfrac{1}{2}(20\,chi)(25\,chi) = 250 \text{ square } chi$$

The volume of the prism is the area times the length,

$$V = AL$$
$$V = (250 \text{ square } chi)(186\,chi)$$
$$V = 46{,}500 \text{ cubic } chi\,.$$

Problem 9. Now given a circular [i.e., cylindrical] log of unknown size buried in a wall. When sawn 1 *cun* deep, it shows a breadth of 1 *chi*. Tell: what is the diameter of the log?

Recall that 1 *chi* = 10 *cun*, so the exposed breadth extends 5 *cun* to each side of the midpoint.



By the Gōugŭ theorem,

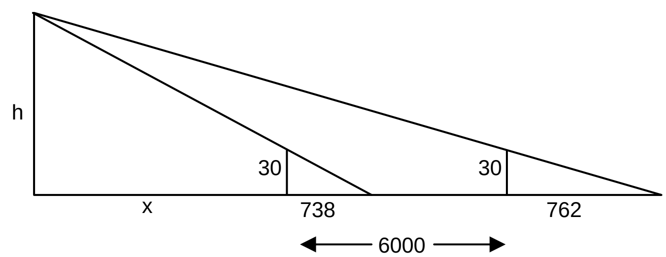
$$r^2 = 5^2 + (r-1)^2$$
$$r^2 = 25 + r^2 - 2r + 1$$
$$2r = 26$$
$$r = 13 \ cun$$
$$\text{diameter} = 2r = 26 \ cun = 2 \ chi \ 6 \ cun.$$

Problem 10. Now survey a sea island. Erect two poles of the same height, 3 *zhang*, so that the front and rear poles are 1000 *bu* apart. They are aligned with the summit of the island. Move backwards 123 *bu* from the front pole, sighting at ground level, and find that the summit of the island coincides with the tip of the pole. Move backwards 127 *bu* from the rear pole, sighting at ground level, and find that the summit of the island also coincides with the tip of the pole. Tell: what are the height of the island and its distance from the [front] pole?

Sindhuja wrote:

Use similar triangles!

[She converted all lengths to *chi* in her figure, below.]



By the similarity of the pairs of triangles,

$$\frac{h}{x+738} = \frac{30}{738} \text{ and } \frac{h}{x+6762} = \frac{30}{762}.$$

Thus,

$$30(x+738) = 738h \text{ and } 30(x+6762) = 762h$$
$$30x + 22140 = 738h \text{ and } 30x + 202860 = 762h$$
$$30x = 738h - 22140 \text{ and } 30x = 762h - 202860$$
$$738h - 22140 = 762h - 202860$$
$$202860 - 22140 = 762h - 738h$$
$$180720 = 24h$$
$$h = \frac{180720}{24} = 7530.$$

But $30(x+738) = 738h = 738(7530)$

so $x + 738 = 738(251)$

$x = 738(250) = 184500$

The height of the island is

$$h = 7530 \ chi = 1255 \ bu = 4 \ li \ 55 \ bu.$$

Its distance from the front pole is

$$x = 184500 \ chi = 30750 \ bu = 102 \ li \, 150 \ bu. \quad \Delta$$