Lesson 9: Side length quotients in similar triangles

Goals

- Calculate unknown side lengths in similar triangles using the ratios of side lengths within the triangles and the scale factor between similar triangles.
- Generalise (orally) that the quotients of pairs of side lengths in similar triangles are equal.

Learning Targets

- I can decide if two triangles are similar by looking at quotients of lengths of corresponding sides.
- I can find missing side lengths in a pair of similar triangles using quotients of side lengths.

Lesson Narrative

In prior lessons, students learned that similar triangles are the images of each other under a sequence of translations, rotations, reflections and enlargements, and that as a result, there is a scale factor that we can use to multiply all of the side lengths in one triangle to find the corresponding side lengths in a similar triangle. In this lesson, they will discover that if you determine the quotient of a pair of side lengths in one triangle, it will be equal to the quotient of the corresponding side lengths in a similar triangle. While this fact is not limited to triangles, this lesson focuses on the particular case of triangles so that students are ready to investigate the concept of gradient in upcoming lessons.

Building On

- Recognise and represent proportional relationships between quantities.

Addressing

- Understand congruence and similarity using physical models, transparencies, or geometry software.
- Understand that a two-dimensional shape is similar to another if the second can be obtained from the first by a sequence of rotations, reflections, translations, and enlargements; given two similar two-dimensional shapes, describe a sequence that exhibits the similarity between them.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Discussion Supports

Required Materials

Geometry toolkits

tracing paper, graph paper, coloured pencils, scissors, and an index card to use as a straightedge or to mark right angles, plus a ruler and protractor. Clear protractors with no holes and with radial lines printed on them are recommended.

Student Learning Goals

Let's find missing side lengths in triangles.

9.1 Two-three-four and Four-five-six

Warm Up: 5 minutes

Two sets of triangle side lengths are given that do not form similar triangles. Students should recognise that there is no single scale factor that multiplies all of the side lengths in one triangle to get the side lengths in the other triangle.

Launch

Give 2 minutes of quiet work time followed by a whole-class discussion.

Anticipated Misconceptions

Students might think that adding the same number to each side length will result in similar triangles. Drawing a picture helps students see why this is not true.

Student Task Statement

Triangle A has side lengths 2, 3, and 4. Triangle B has side lengths 4, 5, and 6. Is triangle A similar to triangle B?

Student Response

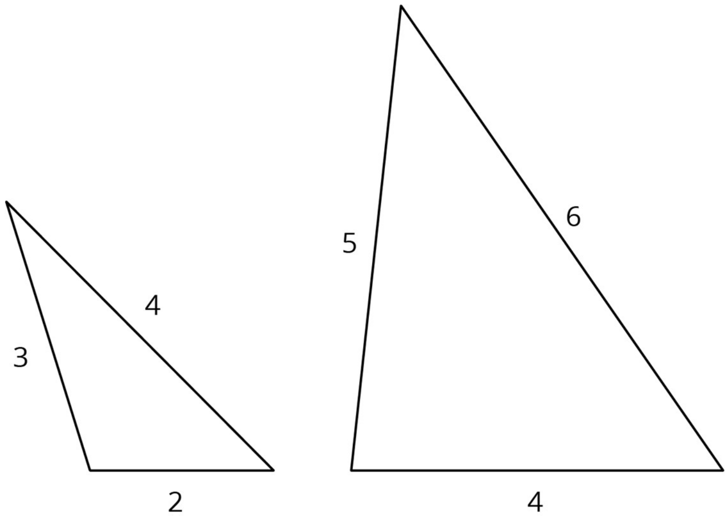
No. Sample explanations:

- If two shapes are similar, then there is a single scale factor we can multiply all of the side lengths in one to get the side lengths in the other. Since doubling 2 gives 4, and doubling 3 gives 6, the third side in the second triangle would have to be 8 for the two to be similar.
- These triangles are not similar because we double the shortest side in triangle A to get the shortest side in triangle B, but we multiply the longest side in triangle A by 1.5 to get the longest side in triangle B. So the scale factor is not the same for all side lengths.

Activity Synthesis

Ask students how they can tell without drawing a diagram. Make sure students understand that the triangles cannot be similar because you can't apply the same scale factor to each side of one triangle to get the corresponding sides of the other triangle.

Display diagrams of the triangles for visual confirmation.



9.2 Quotients of Sides Within Similar Triangles

15 minutes

In previous lessons, students have seen that corresponding side lengths of similar polygons are proportional. That is, the side lengths in one polygon can be calculated by multiplying corresponding side lengths in a similar polygon by the same scale factor. This activity explores ratios of side lengths *within* triangles and how these compare for similar triangles. If a and b are the side lengths of a triangle then the corresponding side lengths of a similar triangle have lengths sa and sb for some positive scale factor s. This means that the ratios $a : b$ and $sa : sb$ are equivalent.

As students work, circulate to make sure that students have the correct values in the table, and address any misconceptions with individual groups as needed. Also watch for students who look to explain why the internal ratios of corresponding side lengths of similar triangles are equivalent and invite them to share their thinking during the discussion.

Instructional Routines

- Discussion Supports

Launch

Arrange students in groups of 3. Assign one of the columns in the second table to one student in each group. Tell students, "Each group is going to compare side lengths in similar triangles. Work for 5 minutes by yourself. Then compare your findings with your partners."

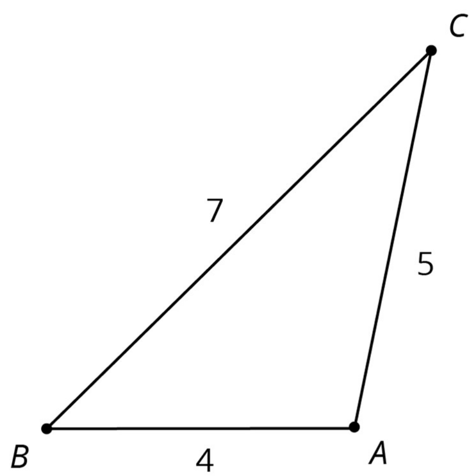
Representation: Internalise Comprehension. Activate or supply background knowledge. Allow students to use calculators to ensure inclusive participation in the activity.
Supports accessibility for: Memory; Conceptual processing

Anticipated Misconceptions

If students find quotients in fraction form, they need to recognise that the fractions are equivalent.

Student Task Statement

Triangle ABC is similar to triangles DEF, GHI, and JKL. The scale factors for the enlargements that show triangle ABC is similar to each triangle are in the table.



1. Find the side lengths of triangles DEF, GHI, and JKL. Record them in the table.

triangle	scale factor	length of short side	length of medium side	length of long side
ABC	1	4	5	7
DEF	2			
GHI	3			
JKL	$\frac{1}{2}$			

2. Your teacher will assign you one of the three columns. For all four triangles, find the quotient of the triangle side lengths assigned to you and record it in the table. What do you notice about the quotients?

triangle	(long side) ÷ (short side)	(long side) ÷ (medium side)	(medium side) ÷ (short side)
ABC	$\frac{7}{4}$ or 1.75		
DEF			
GHI			
JKL			

3. Compare your results with your partners' and complete your table.

Student Response

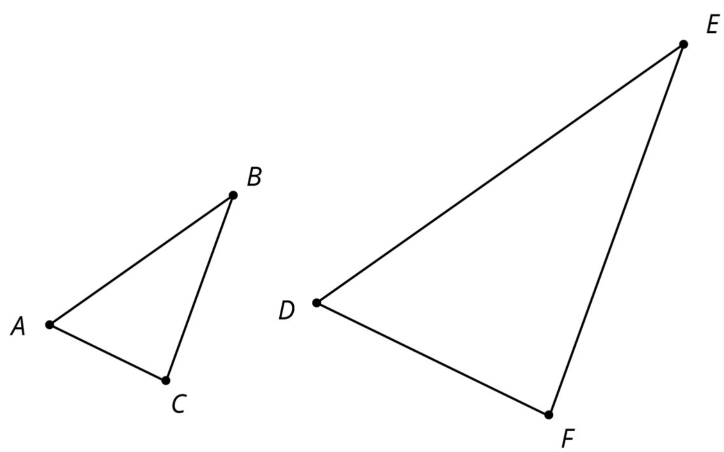
triangle	scale factor	length of short side	length of medium side	length of long side
ABC	1	4	5	7
DEF	2	8	10	14
GHI	3	12	15	21
JKL	$\frac{1}{2}$	2	2.5	3.5

triangle	(long side) ÷ (short side)	(long side) ÷ (medium side)	(medium side) ÷ (short side)
ABC	$\frac{7}{4}$ or 1.75	$\frac{7}{5}$ or 1.4	$\frac{5}{4}$ or 1.25
DEF	$\frac{14}{8}$ or 1.75	$\frac{14}{10}$ or 1.4	$\frac{10}{8}$ or 1.25
GHI	$\frac{21}{12}$ or 1.75	$\frac{21}{15}$ or 1.4	$\frac{15}{12}$ or 1.25
JKL	$\frac{3.5}{2}$ or 1.75	$\frac{3.5}{2.5}$ or 1.4	$\frac{2.5}{2}$ or 1.25

The quotients in each column are the same.

Are You Ready for More?

Triangles ABC and DEF are similar. Explain why $\frac{AB}{BC} = \frac{DE}{EF}$.



Student Response

There is a scale factor s such that $s \times AB = DE$ and $s \times BC = EF$. So $\frac{s \times AB}{s \times BC} = \frac{DE}{EF}$, and $\frac{AB}{BC} = \frac{DE}{EF}$.

Activity Synthesis

The main takeaway from this activity is that quotients of corresponding side lengths in similar triangles are equal. Ask students for the triangles examined what the value of (medium side) ÷ (long side) would be? For the original triangle, it would be $\frac{5}{7}$, and students can check that this is the same value for the other triangles.

Ask students if they think the value of (medium side) ÷ (long side) would be $\frac{5}{7}$ for *any* triangle similar to ABC. Ask them to explain why. Help them to see that a triangle similar to ABC will have side lengths $4s$, $5s$, and $7s$ for some (positive) scale factor s. The medium side divided by the long side will be $5s \div 7s = \frac{5s}{7s} = \frac{5}{7}$.

Speaking, Representing: Discussion Supports. At the end of the whole-class discussion, display this prompt for all to see, "The value of (medium side) ÷ (long side) will be ____ for any triangle similar to ABC because…". Give students 2–3 minutes to write a response. Invite students to read what they wrote to a partner as a way to rehearse what they will say when they share with the whole class. Rehearsing provides students with additional opportunities to speak and clarify their thinking. Listen for and amplify statements that use both formal and informal language such as, ratio, quotient, multiple, scale factor, reduce, simplify, and common factor. This will help students to explain their reasoning using appropriate language structure.
Design Principle(s): Optimise output (for explanation)

9.3 Using Side Quotients to Find Side Lengths of Similar Triangles

15 minutes

In this activity, students calculate side lengths of similar triangles. They can use the scale factor between the similar triangles, studied in depth in previous lessons. Or they can look at internal ratios between corresponding side lengths within the triangles, introduced in the previous lesson. Students need to think strategically about which side lengths to calculate first since there are many missing values. As they discover more side lengths, this opens up more paths for finding the remaining values.

As students work, monitor for students who:

- Use scale factors between triangles.
- Notice that the long side is twice the short side in GHI and use that to find c, d, or e.
- Notice that the long sides are equal in ABC and use that to find h, d, or e.

Select students who use different strategies to find side lengths to share during the discussion.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect

- Discussion Supports

Launch

Tell students, “There are many ways to find the values of the unknown side lengths in similar triangles. Use what you have learned so far.” Give students 5 minutes of quiet work time followed by 5 minutes small group discussion and then a whole-class discussion.

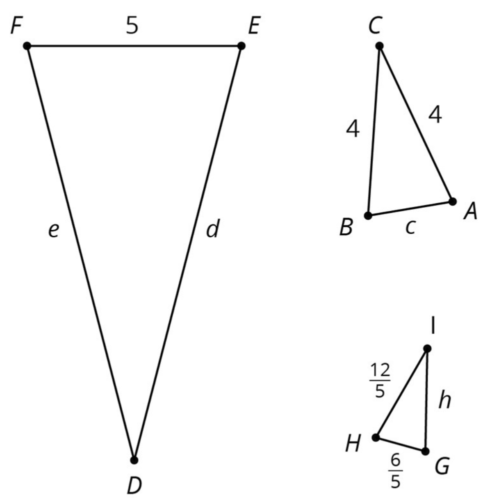
Representation: Internalise Comprehension. Demonstrate and encourage students to use colour coding and annotations to highlight connections between representations in a problem. For example, use the same colour for corresponding side lengths.
Supports accessibility for: Visual-spatial processing

Anticipated Misconceptions

If students have trouble locating corresponding sides, suggest that they use tracing paper so they can rotate and or translate them. Another technique is to colour corresponding side lengths the same colour. For example, they could colour AB, EF, and GH all red.

Student Task Statement

Triangles ABC, EFD, and GHI are all similar. The side lengths of the triangles all have the same units. Find the unknown side lengths.



Student Response

$c = 2, d = 10, e = 10, h = \frac{12}{5}$

Activity Synthesis

Ask selected students to share the following strategies:

- using (external) scale factors to move from one triangle to another

- using quotients of corresponding side lengths within the triangles (internal scale factors)

Both methods are efficient and the method to use is guided by what information is missing and the numbers involved in the calculations. For example, if h is the first missing value we find, then comparing with triangle ABC and using internal scale factors is appropriate. To find c, again we can compare ABC and GHI and internal scale factors are appropriate again because $\frac{6}{5}$ is half of $\frac{12}{5}$ whereas comparing $\frac{12}{5}$ and 4 is more involved (the scale factor is $\frac{5}{3}$ from $\triangle GHI$ to $\triangle ABC$).

Ask students to articulate how they knew which sides of the similar triangles correspond. Make sure to make the following reasoning pathways explicit for all:

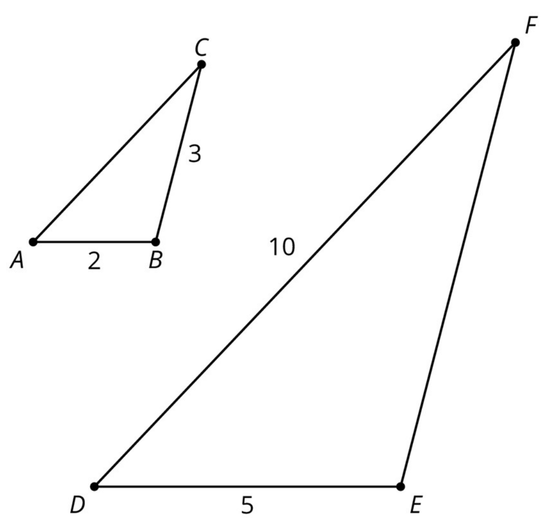
- Triangle ABC has two equal side lengths, so the other two triangles will as well. This insight is efficient for finding h.

- One side of triangle GHI is twice the length of another side, so this will be true of the other triangles as well. This insight is helpful for finding c, d, and e.

Emphasise that there are many different relationships that can be used to find side lengths of similar triangles.

Speaking: Discussion Supports: To support students to articulate how they knew which sides of the similar triangles correspond, provide sentence frames such as, “For similar triangles _____ and _____ I know that sides _____ and _____ correspond because ________."
Sentence frames invite and incentivise more student participation, conversation, and meta-awareness of language. This routine should help students reason about the ratio of corresponding sides of similar triangles and communicate their understanding.
Design Principle(s): Support sense-making, Optimise output (for comparison)

Lesson Synthesis



These two triangles are similar. Ask students what the scale factor is from $\triangle ABC$ to $\triangle DEF$. It's $\frac{5}{2}$ since sides AB and DE are corresponding sides. One way to find AC would be to divide the length of DF by the scale factor $\frac{5}{2}$ giving a length of 4. A simpler arithmetic way to do this is to notice that DF is twice the length of DE. This means that AC is twice the length of AB (scaling AC and AB *both* by $\frac{5}{2}$ does not change their quotient!).

Sometimes both methods for calculating missing side lengths are equally effective. For EF, we can notice that it is $\frac{5}{2}$ the length of the corresponding side BC so that's 7.5. Or we can notice that it is $\frac{3}{2}$ the length of DE, again 7.5 ($\frac{3}{2}$ is the quotient of the corresponding sides BC and AB in $\triangle ABC$).

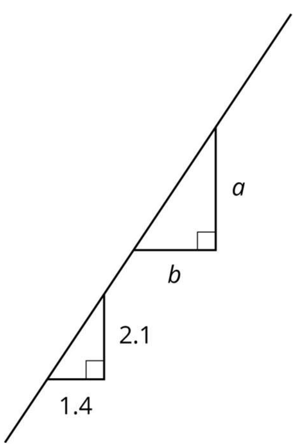
9.4 Similar Sides

Cool Down: 5 minutes

Students apply the equality of internal ratios of sides on similar triangles to find missing side lengths. These particular triangles (gradient triangles) will be a focus of study in the following lessons.

Student Task Statement

The two triangles shown are similar. Find the value of $\frac{a}{b}$.



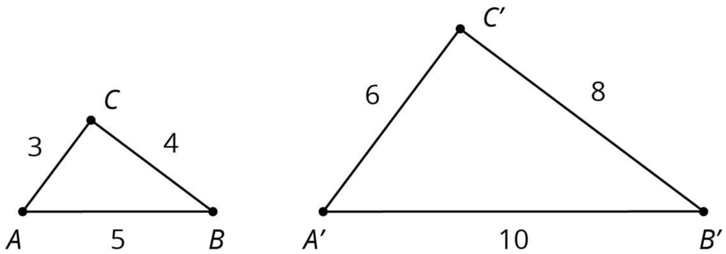
Student Response

$\frac{3}{2}$ or 1.5

Student Lesson Summary

If two polygons are similar, then the side lengths in one polygon are multiplied by the same scale factor to give the corresponding side lengths in the other polygon.

For these triangles the scale factor is 2:



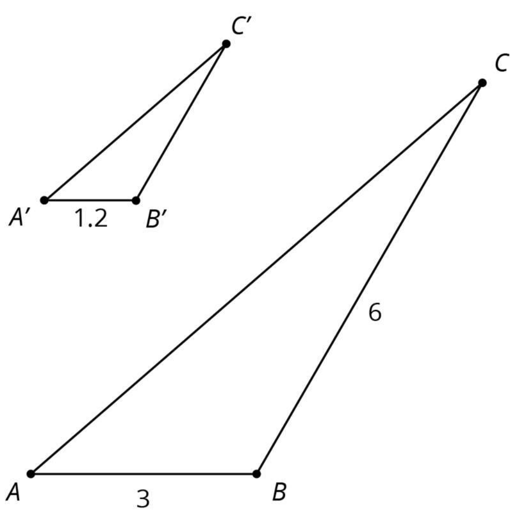
Here is a table that shows relationships between the short and medium length sides of the small and large triangle.

	small triangle	large triangle
medium side	4	8
short side	3	6
(medium side) ÷ (short side)	$\frac{4}{3}$	$\frac{8}{6}=\frac{4}{3}$

The lengths of the medium side and the short side are in a ratio of $4:3$. This means that the medium side in each triangle is $\frac{4}{3}$ as long as the short side. This is true for all similar polygons; the ratio between two sides in one polygon is the same as the ratio of the corresponding sides in a similar polygon.

We can use these facts to calculate missing lengths in similar polygons. For example, triangles $A'B'C'$ and ABC shown here are similar. Let's find the length of line segment $B'C'$.

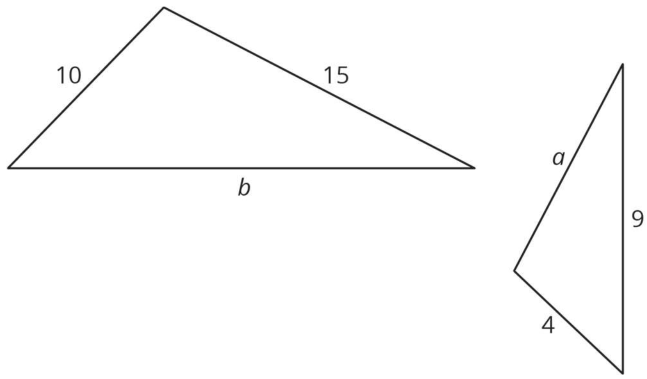
In triangle ABC, side BC is twice as long as side AB, so this must be true for any triangle that is similar to triangle ABC. Since $A'B'$ is 1.2 units long and $2 \times 1.2 = 2.4$, the length of side $B'C'$ is 2.4 units.



Lesson 9 Practice Problems

1. **Problem 1 Statement**

   These two triangles are similar. What are a and b? Note: the two shapes are not drawn to scale.

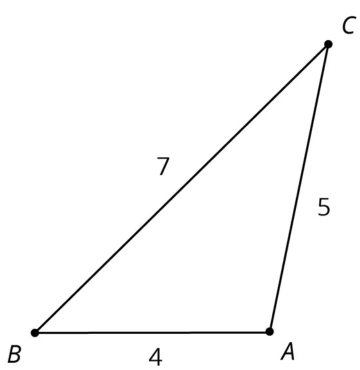
   

   Solution

   $a = 6$, $b = 22.5$ (the scale factor between the triangles is 2.5)

2. **Problem 2 Statement**

   Here is triangle ABC. Triangle XYZ is similar to ABC with scale factor $\frac{1}{4}$.

   a. Draw what triangle XYZ might look like.

   b. How do the angles of triangle XYZ compare to triangle ABC? Explain how you know.

   c. What are the side lengths of triangle XYZ?

   d. For triangle XYZ, calculate (long side) ÷ (medium side), and compare to triangle ABC.

   Solution

   a. Answers vary.

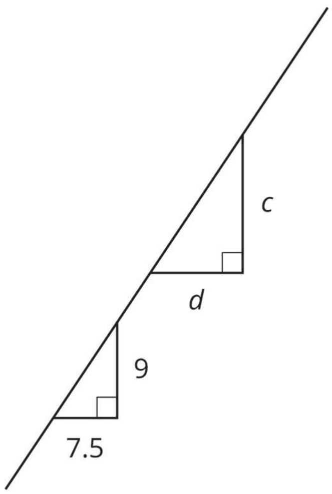
b. The angles are the same, because in similar polygons, corresponding angles are congruent.

c. The side lengths are 1, $\frac{5}{4}$, and $\frac{7}{4}$.

d. The result is $\frac{7}{5}$, the same as the corresponding result for triangle ABC.

3. **Problem 3 Statement**

The two triangles shown are similar. Find the value of $\frac{d}{c}$.

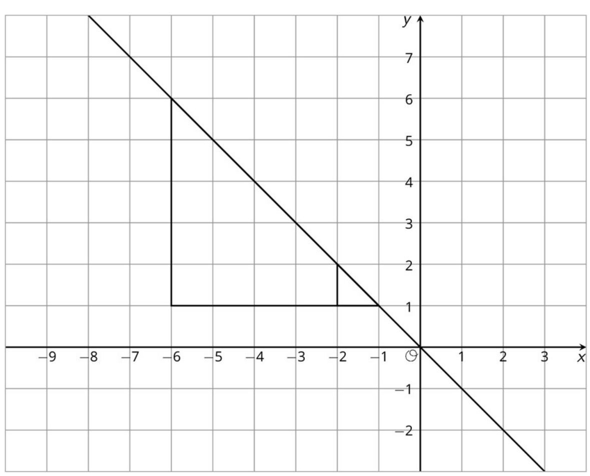


Solution

$\frac{5}{6}$ (or equivalent)

4. **Problem 4 Statement**

The diagram shows two nested triangles that share a vertex. Find a centre and a scale factor for an enlargement that would move the larger triangle to the smaller triangle.



Solution

Centre: (-1,1), scale factor: $\frac{1}{5}$

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