Lesson 17: Squares and cubes

Goals

- Generalise a process for finding the volume of a cube, and justify (orally) why this can be abstracted as s^3.
- Include appropriate units (orally and in writing) when reporting lengths, areas, and volumes, e.g. cm, cm^2, cm^3.
- Interpret and write expressions with exponents ^{2} and ^{3} to represent the area of a square or the volume of a cube.

Learning Targets

- I can write and explain the formula for the volume of a cube, including the meaning of the exponent.
- When I know the edge length of a cube, I can find the volume and express it using appropriate units.

Lesson Narrative

In this lesson, students learn about perfect squares and perfect cubes. They see that these names come from the areas of squares and the volumes of cubes with whole-number side lengths. Students find unknown side lengths of a square given the area or unknown edge lengths of a cube given the volume. To do this, they make use of the structure in expressions for area and volume.

Students also use **exponents** of 2 and 3 and see that in this geometric context, exponents help to efficiently express multiplication of the side lengths of squares and cubes. Students learn that expressions with exponents of 2 and 3 are called **squares** and **cubes**, and see the geometric motivation for this terminology. (The term "exponent" is deliberately *not* defined more generally at this time. Students will work with exponents in more depth in a later unit.)

In working with length, area, and volume throughout the lesson, students must attend to units. In order to write the formula for the volume of a cube, students look for and express regularity in repeated reasoning.

Note: Students will need to bring in a personal collection of 10–50 small objects ahead of time for the first lesson of the next unit. Examples include rocks, seashells, trading cards, or coins.

Building On

- Apply the area and perimeter formulas for rectangles in real world and mathematical problems. For example, find the width of a rectangular room given the area of the flooring and the length, by viewing the area formula as a multiplication equation with an unknown factor.

- Find the volume of a cuboid with whole-number side lengths by packing it with unit cubes, and show that the volume is the same as would be found by multiplying the edge lengths, equivalently by multiplying the height by the area of the base. Represent threefold whole-number products as volumes, e.g., to represent the associative property of multiplication.

Addressing

- Apply and extend previous understandings of arithmetic to algebraic expressions.
- Write and evaluate numerical expressions involving whole-number exponents.

Building Towards

- Write and evaluate numerical expressions involving whole-number exponents.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Clarify, Critique, Correct
- Discussion Supports
- Think Pair Share

Required Materials

Multi-link cubes

Required Preparation

Prepare sets of 32 multi-link cubes for each group of 2 students.

Student Learning Goals

Let's investigate perfect squares and perfect cubes.

17.1 Perfect Squares

Warm Up: 5 minutes

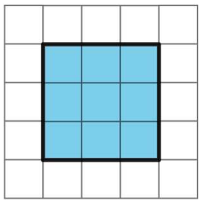
This activity introduces the concept of perfect squares. It also includes opportunities to practice using units of measurement, which offers insights on students' knowledge from preceding lessons.

Provide access to square tiles, if available. Some students may benefit from using physical tiles to reason about perfect squares.

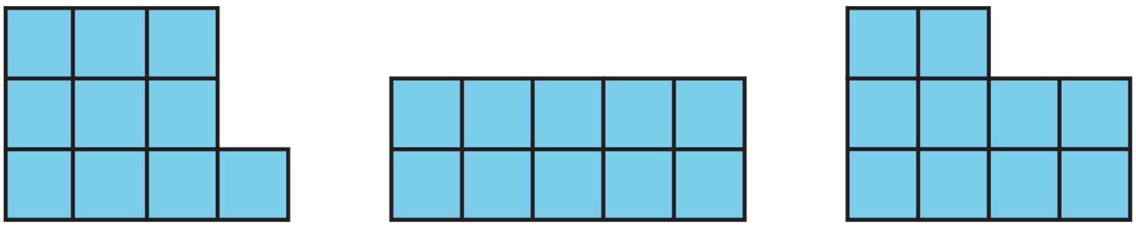
As students work, notice whether they use appropriate units for the second and third questions.

Launch

Tell students, “Some numbers are called perfect **squares**. For example, 9 is a perfect square. Nine copies of a small square can be arranged into a large square.” Display a square like this for all to see:



Explain that 10, however, is not a perfect square. Display images such as these below, emphasising that 10 small squares cannot be arranged into a large square (the way 9 small squares can).



Tell students that in this warm-up they will find more numbers that are perfect squares. Give students 2 minutes of quiet think time to complete the activity.

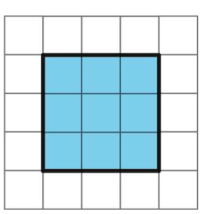
Anticipated Misconceptions

If students do not recall what the abbreviations km, cm, and sq stand for, provide that information.

Students may divide 64 by 2 for the third question. If students are having trouble with this, ask them to check by working backwards, i.e. by multiplying the side lengths to see if the product yields the given area measure.

Student Task Statement

1. The number 9 is a perfect **square**. Find four numbers that are perfect squares and two numbers that are not perfect squares.
2. A square has side length 7 in. What is its area?
3. The area of a square is 64 sq cm. What is its side length?



Student Response

1. Answers vary. For example, here are some squares: 9, 25, 4, 49, 100 and non-squares: $\frac{1}{2}$, 2, 3, 10.
2. The square has an area of 49 square inches.
3. The side length is 8 centimetres.

Activity Synthesis

Invite students to share the examples and non-examples they found for perfect squares. Solicit some ideas on how they decided if a number is or is not a perfect square.

If a student asks about 0 being a perfect square, wait until the end of the lesson, when the exponent notation is introduced. 0 *is* a perfect square because $0^2 = 0$.

Briefly discuss students' responses to the last two questions, the last one in particular. If not already uncovered in discussion, highlight that because the area of a square is found by multiplying side lengths to each other, finding the side lengths of a square with a known area means figuring out if that area measure is a product of two of the same number.

17.2 Building with 32 Cubes

Optional: 15 minutes (there is a digital version of this activity)

This activity gives students a concrete way to review the work on volume from KS2. It prompts students to recall that the volume of a cuboid can be calculated in two different ways: by counting unit cubes that can be packed into the cuboid, and by multiplying the edge lengths of the cuboid. Students also become familiar with two perfect cubes, 27 and 64, before the next activity introduces this term.

As students work, monitor the different routes they take to find the volume of the built cube. They may count all of the multi-link cubes individually, count the number of multi-link cubes per layer and then multiply that by the number of layers, or simply multiply edge lengths. Select a student who uses each method to share later.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect

Launch

Arrange students in groups of 2. Give 32 multi-link cubes to each group. If centimetre cubes are available, have students work in centimetres instead of the generic units listed here. Give students 8–10 minutes to build the largest cube they can with 32 cubes and to answer the questions.

For groups who finish early, consider asking them to combine their cubes and build the largest cube they can with 64 cubes. Then, ask them to answer the same four questions as shown in the problem statement.

Students in digital classrooms can use the applet to build the cube with 32 cubes. For students who finish early, another applet with 64 cubes can be found in Digital Extension.

Anticipated Misconceptions

Students may neglect to write units for length or area and may need a reminder to do so.

When determining area, students may multiply a side by two instead of squaring it. When determining volume, they may multiply a side by three instead of cubing it. If this happens, ask them to count individual squares so that they can see that there is an error in their reasoning.

Student Task Statement

Your teacher will give you 32 multi-link cubes. Use them to build the largest cube you can. Each small cube has an edge length of 1 unit.

1. How many multi-link cubes did you use?

2. What is the edge length of the cube you built?

3. What is the area of each face of the built cube? Be prepared to explain your reasoning.

4. What is the volume of the built cube? Be prepared to explain your reasoning.

Student Response

1. 27

2. 3 units

3. 9 square units

4. 27 cubic units

Activity Synthesis

Focus the whole-class discussion on the ways students calculated the volumes of the two cubes they built. Select previously identified students to share their approaches starting from the less efficient (counting individual cubes) to the most efficient (multiplying side lengths).

Highlight how the side length of a cube determines its volume, and specifically that the number 27 is $3 \times 3 \times 3$. If any group built a cube with 64 multi-link cubes, point out that the number 64 is $4 \times 4 \times 4$. These observations prepare students to think about perfect cubes in the next activity and about a general expression for the volume of a cube later in the lesson.

17.3 Perfect Cubes

10 minutes

Earlier, students looked at examples and non-examples of perfect squares. In this activity, they think about examples and non-examples of perfect **cubes** and find the volumes of cubes given their edge lengths. Students see that the edge length of a cube determines its volume, notice the numerical expressions that can be written when calculating volumes, and write a general expression for finding the volume of a cube.

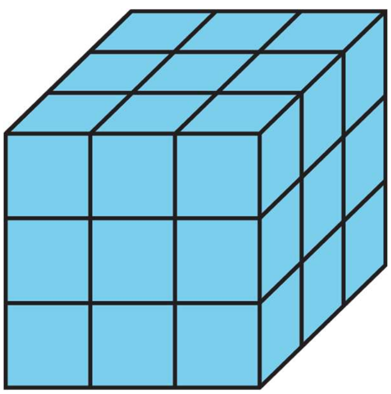
Some students may feel uncomfortable writing the answer to the last question symbolically because it involves a variable and may prefer writing a verbal explanation. This is fine; the exponential notation that follows will help greatly.

Instructional Routines

- Discussion Supports
- Think Pair Share

Launch

Tell students, "Some numbers are called perfect **cubes**. For example, 27 is a perfect cube." Display a cube like this for all to see:



Arrange students in groups of 2. Give students a few minutes of quiet think time, and another minute to discuss their responses with their partner.

Representation: Develop Language and Symbols. Use virtual or concrete manipulatives to connect symbols to concrete objects or values. Provide multi-link cubes or blocks to help students analyse examples and nonexamples of perfect cubes, paying close attention to the relationships between side lengths and volume.
Supports accessibility for: Conceptual processing Conversing, Representing: Discussion Supports. As students work in pairs to make sense of perfect cubes, encourage students to press for details as peers share their ideas. Provide sentences frames for students to use, such as "How do you know. . .?", "Tell me more about. . .", and "I agree/disagree because. . ."
Design Principle(s): Support sense-making; Maximise meta-awareness

Anticipated Misconceptions

Watch for students using square units instead of cubic units. Remind them that volume is a measure of the space inside the cube and is measured in cubic units.

Students may multiply by 3 when finding the volume of a cube instead of multiplying three edge lengths (which happen to be the same number). Likewise, they may think a perfect cube is a number times 3. Suggest that they sketch or build a cube with that edge length and count the number of unit cubes. Or ask them to think about how to find the volume of a cuboid when the edge lengths are different (e.g., a cuboid that is 1 unit by 2 units by 3 units).

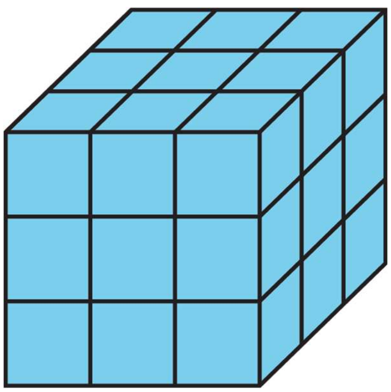
Student Task Statement

1. The number 27 is a perfect **cube**. Find four other numbers that are perfect cubes and two numbers that are *not* perfect cubes.

2. A cube has side length 4 cm. What is its volume?

3. A cube has side length 10 inches. What is its volume?

4. A cube has side length s units. What is its volume?



Student Response

1. Answers vary: 1, 8, 64, 125, 216, 1 000. Non-cubes: 2, 3, 4.

2. $4 \times 4 \times 4$ or 64 cubic cm

3. $10 \times 10 \times 10$ or 1 000 cubic inches

4. $s \times s \times s$ cubic units

Activity Synthesis

After partner discussions, invite students to share how they thought about the first question and decided if a number is or is not a perfect cube. Highlight the idea that multiplying three edge lengths allows us to determine volume efficiently, and that determining if a number is a perfect cube involves thinking about whether it is a product of three of the same number.

If a student asks about 0 being a perfect cube, wait until the end of the lesson, when exponent notation is introduced. 0 *is* a perfect cube because $0^3 = 0$.

Make sure students see the answers to the last three questions written as expressions: $4 \times 4 \times 4$, $10 \times 10 \times 10$, $s \times s \times s$

17.4 Introducing Exponents

15 minutes

This activity introduces students to the exponents of 2 and 3 and the language we use to talk about them. Students use and interpret this notation in the context of geometric squares and their areas, and geometric cubes and their volumes. Students are likely to have seen exponent notation for 10^3 in their work on place values in KS2. That experience would be helpful but is not necessary.

Note that the term “exponent” is deliberately *not* defined more generally at this time. Students will work with exponents in more depth in a later unit.

As students work, observe how they approach the last two questions. Identify a couple of students who approach the fourth question differently so they can share later. Also notice whether students include appropriate units, written using exponents, in their answers.

Instructional Routines

- Clarify, Critique, Correct

Launch

Ask students if they have seen an expression such as 10^3 before. Tell students that in this expression, the 3 is called an **exponent**. Explain the use of exponents of 2 and 3:

- “When we multiply two of the same number together, such as 5×5, we say we are **squaring** the number. We can write the expression as: 5^2 Because 5×5 is 25, we can write $5^2 = 25$, and we say, ‘5 squared is 25.’ We can also say that 25 is a perfect square. The raised 2 in 5^2 is called an exponent.”

- “When we multiply three of the same number together like $4 \times 4 \times 4$, we say we are **cubing** the number. We can write it like this: 4^3 Because $4 \times 4 \times 4$ is 64, we can write $4^3 = 64$, and we say, ‘4 cubed is 64.’ We also say that 64 is a perfect cube. The raised 3 in 4^3 is called an exponent.”

Explain that we can also use exponents as a shorthand for the units used for area and volume:

- A square with side length 5 inches has area of 25 square inches, which we can write as 25 in^2.

- A cube with edge length 4 centimetres has a volume of 64 cubic centimetres, which we can write as 64 cm^3.

Ask students to read a few areas and volumes in different units (e.g. 100 ft^2 is read “100 square feet” and 125 yd^3 is read “125 cubic yards”).

Keep students in groups of 2. Give students 3–4 minutes of quiet time to complete the activity and a minute to discuss their response with their partner. Ask partners to note any disagreements so they can be discussed.

Anticipated Misconceptions

Upon seeing 6^3 in the fourth question, some students may neglect to interpret the question, automatically calculate $6 \times 6 \times 6$, and conclude that the edge length is 216 cm. Ask them to check their answer by finding the volume of a cube with edge length 216 cm.

Student Task Statement

Make sure to include correct units of measure as part of each answer.

1. A square has side length 10 cm. Use an **exponent** to express its area.
2. The area of a square is 7^2 sq in. What is its side length?
3. The area of a square is 81 m^2. Use an exponent to express this area.
4. A cube has edge length 5 in. Use an exponent to express its volume.
5. The volume of a cube is 6^3 cm^3. What is its edge length?
6. A cube has edge length s units. Use an exponent to write an expression for its volume.

Student Response

1. 10^2 cm^2
2. 7 inches
3. 9^2 m^2
4. 5^3 in^3
5. 6 cm
6. s^3 units^3 or s^3 cubic units

Are You Ready for More?

The number 15 625 is both a perfect square and a perfect cube. It is a perfect square because it equals 125^2. It is also a perfect cube because it equals 25^3. Find another number that is both a perfect square and a perfect cube. How many of these can you find?

Student Response

The smallest examples are 0, 1, 64, 729, and 4 096.

Activity Synthesis

Ask partners to share disagreements in their responses, if any. Then, focus the whole-class discussion on the last two questions. Select a couple of previously identified students to share their interpretations of the fourth question.

Highlight that a cube with a volume of 6^3 cubic units has an edge length of 6 units, because we know there are $6 \times 6 \times 6$ unit cubes in a cube with that edge length.

In other words, we can express the volume of a cube using a number (216), a product of three numbers $(6 \times 6 \times 6)$, or an expression with exponent (6^3). This idea can be extended to all cubes. The volume of a cube with edge length s is: $s \times s \times s = s^3$ Students will have more opportunities to generalise the expressions for the volume of a cube in the next lesson.

Representation: Develop Language and Symbols. Create a display of important terms and vocabulary. Invite students to suggest language or diagrams to include that will support their understanding and memory: squaring, cubing, exponent.
Supports accessibility for: Conceptual processing; Language; Memory Reading, Writing, Speaking: Clarify, Critique, Correct. Use this routine before the whole-class discussion of the last two questions. Display an incorrect response for students to consider. For example, "If the volume of a cube is 6^3 cm³, then the edge length is 216 cm because $6 \times 6 \times 6$ is 216." Ask students to identify the error, critique the reasoning, and write a correct explanation. Listen for students who clarify that the volume of a cube can be represented as an exponent or a value and that neither of these represent the edge length. Invite students to share their critiques and corrected explanations with the class. Listen for and amplify the language students use to describe ways to generalise the relationship between the three representations of volume: a number, a product of three numbers, or an expression with exponent. This helps students evaluate, and improve upon, the written mathematical arguments of others, as they clarify their understanding of exponents.
Design Principle(s): Optimise output (for explanation); Maximise meta-awareness

Lesson Synthesis

Review the language and notation for **squaring** and **cubing** a number. Remind students we use this notation for square and cubic units, too.

- When we multiply two of the same number together like 10×10 we say we are **squaring** the number. We write, for example, $10^2 = 100$ and say, "Ten squared is one hundred."

- When we multiply three of the same number together like $10 \times 10 \times 10$, we say we are **cubing** the number. We write, for example, $10^3 = 1\,000$ and say, "Ten cubed is one thousand."

- Exponents are used to write square and cubic units. The area of a square with side length 7 km is 7^2 km². The volume of a cube with side length 2 millimetres is 2^3 mm³.

17.5 Exponent Expressions

Cool Down: 5 minutes

Anticipated Misconceptions

Students may perform calculations on the second question, which is not necessary since the target is an expression with an exponent.

Student Task Statement

1. Which is larger, 5^2 or 3^3?

2. A cube has an edge length of 21 cm. Use an exponent to express its volume.

Student Response

1. $3^3 = 27$ and $5^2 = 25$, so 3^3 is larger than 5^2.

2. 21^3 cm^3 or 21^3 cubic centimetres

Student Lesson Summary

When we multiply two of the same numbers together, such as 5×5, we say we are **squaring** the number. We can write it like this: 5^2

Because $5 \times 5 = 25$, we write $5^2 = 25$ and we say, "5 squared is 25."

When we multiply three of the same numbers together, such as $4 \times 4 \times 4$, we say we are **cubing** the number. We can write it like this: 4^3

Because $4 \times 4 \times 4 = 64$, we write $4^3 = 64$ and we say, "4 cubed is 64."

We also use this notation for square and cubic units.

- A square with side length 5 inches has area 25 in^2.

- A cube with edge length 4 cm has volume 64 cm^3.

To read 25 in^2, we say "25 square inches," just like before.

The area of a square with side length 7 kilometres is 7^2 km^2. The volume of a cube with edge length 2 millimetres is 2^3 mm^3.

In general, the area of a square with side length s is s^2, and the volume of a cube with edge length s is s^3.

Glossary

- cubed

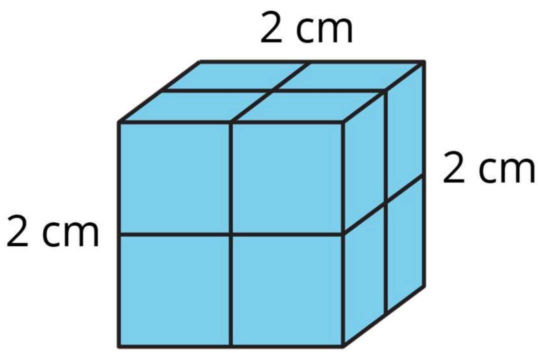
- exponent

- squared

Lesson 17 Practice Problems

1. **Problem 1 Statement**

What is the volume of this cube?



Solution

8 cm^3 ($2 \times 2 \times 2 = 8$)

2. **Problem 2 Statement**

a. Decide if each number on the list is a perfect square.

16

20

25

100

125

144

225

10 000

b. Write a sentence that explains your reasoning.

Solution

a. All of these numbers, except 20 and 125, are perfect squares.

b. Answers vary. Sample response: Perfect squares can be found by multiplying a whole number by itself.

3. **Problem 3 Statement**

a. Decide if each number on the list is a perfect cube.

1

3

8

9

27

64

100

125

b. Explain what a perfect cube is.

Solution

a. All of the numbers except 3, 9, and 100 are perfect cubes.

b. Answers vary. Sample response: Perfect cubes can be found by using a whole number as a factor three times.

4. **Problem 4 Statement**

a. A square has side length 4 cm. What is its area?

b. The area of a square is 49 m^2. What is its side length?

c. A cube has edge length 3 in. What is its volume?

Solution

a. 16 cm^2

b. 7 m

c. 27 in^3

5. **Problem 5 Statement**

Shape A and shape B are cuboids.

– Cuboid A is 3 inches by 2 inches by 1 inch.

– Cuboid B is 1 inch by 1 inch by 6 inches.

Select **all** statements that are true about the two cuboids.

a. They have the same volume.

b. They have the same number of faces.

c. More inch cubes can be packed into cuboid A than into cuboid B.

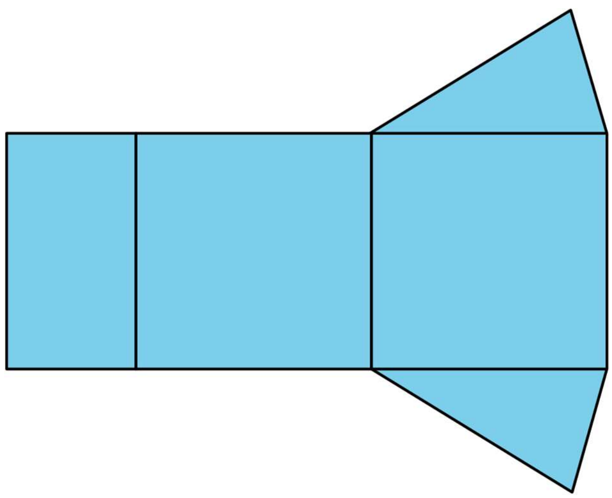
d. The two cuboids have the same surface area.

e. The surface area of cuboid B is greater than that of cuboid A.

Solution ["A", "B", "E"]

6. **Problem 6 Statement**

a. What polyhedron can be assembled from this net?

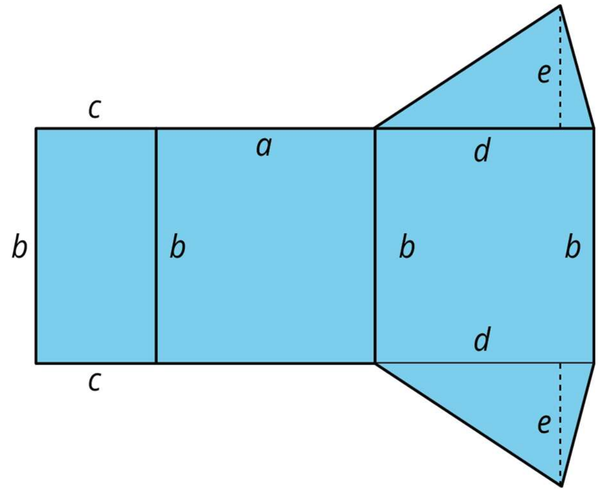


b. What information would you need to find its surface area? Be specific, and label the diagram as needed.

Solution

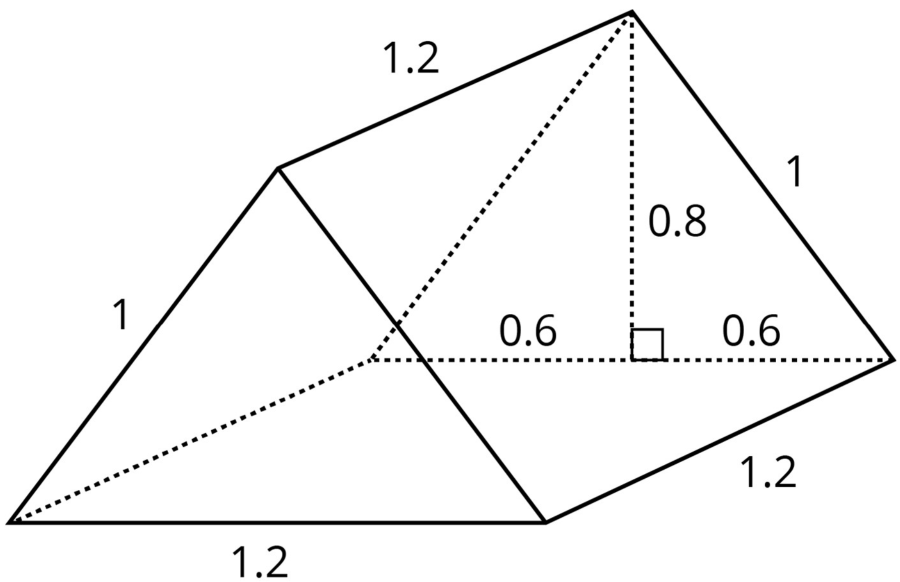
a. Triangular prism

b. Length and width of each rectangular face (as shown in the diagram), as well as the height of the triangular faces



7. **Problem 7 Statement**

Find the surface area of this triangular prism. All measurements are in metres.



Solution

4.8 square metres. Sample reasoning:

- There are two triangular faces with an area of 0.48 square metres each. $\frac{1}{2} \times (1.2) \times (0.8) = 0.48$.
- There are two rectangular faces with area of 1.2 square metres each. $1 \times (1.2) = 1.2$.

- There is one rectangular face with an area of $(1.2) \times (1.2) = 1.44$ square metres.
- $2 \times (0.48) + 2 \times (1.2) + (1.44) = 4.8$, or 4.8 square metres.