Lesson 3: Reasoning about equations with bar models

Goals

- Coordinate bar models and equations of the form $px + q = r$ or $p(x + q) = r$.
- Create a bar model to represent an equation of the form $px + q = r$ or $p(x + q) = r$, and use it to solve the equation.
- Identify equivalent equations, and justify (using words and other representations) that they are equivalent.

Learning Targets

- I can match equations and bar models that represent the same situation.
- If I have an equation, I can draw a bar model that shows the same relationship.

Lesson Narrative

The purpose of this lesson is to make connections between a bar model and an equation of the form $px + q = r$ or $p(x + q) = r$. Students match bar models to corresponding equations and sort them into categories, and then they draw bar models to represent equations. They use the bar model and the equation to reason about a solution, but it is expected that students reason using any method that makes sense to them. It's not yet time to teach particular methods for solving particular types of equations.

Building On

- Identify when two expressions are equivalent (i.e., when the two expressions name the same number regardless of which value is substituted into them). For example, the expressions $y + y + y$ and $3y$ are equivalent because they name the same number regardless of which number y stands for.

Addressing

- Solve multi-step real-life and mathematical problems posed with positive and negative rational numbers in any form (whole numbers, fractions, and decimals), using tools strategically. Apply properties of operations to calculate with numbers in any form; convert between forms as appropriate; and assess the reasonableness of answers using mental computation and estimation strategies.

Building Towards

- Use variables to represent quantities in a real-world or mathematical problem, and construct simple equations and inequalities to solve problems by reasoning about the quantities.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect

- Stronger and Clearer Each Time
- Collect and Display
- Take Turns
- Think Pair Share

Student Learning Goals

Let's see how equations can describe bar models.

3.1 Find Equivalent Expressions

Warm Up: 10 minutes

Students learned all about the distributive property earlier in KS3, including how to use the distributive property to rewrite expressions like $6(2x - 3)$ and $12a + 8a$. This is an opportunity for students to remember what the distributive property is all about before they will be expected to use it in the process of solving equations of the form $p(x + q) = r$ later in this unit. If this activity indicates that students remember little of the distributive property from earlier in KS3, heavier interventions may be needed.

Look for students who:

- Rule out expressions by testing values
- Use the term *distributive property*

Instructional Routines

- Think Pair Share

Launch

Ask students to think of anything they know about *equivalent expressions*. Ask if they can:

- Explain why $2x$ and $2 + x$ are not equivalent. (These expressions are equal when x is 2, but not equal for other values of x. Multiplying 2 by a number usually gives a different result than adding that number to 2.)
- Explain why $3 + x$ and $x + 3$ are equivalent. (These expressions are equal no matter the value of x. Also, addition is commutative.)
- Think of another example of two equivalent expressions. (Examples: $2x$ and $x \times 2$, $a + a + a$, and $3a$.)
- Explain what this term means. (Equivalent expressions are equal no matter the value assigned to the variable.)

- Describe ways to decide whether expressions are equivalent.
  (Test some values, draw diagrams for different values, analyse them for properties of the operations involved.)

Arrange students in groups of 2. Give 3 minutes of quiet work time and then invite students to share their responses with their partner, followed by a whole-class discussion.

Student Task Statement

Select **all** the expressions that are **equivalent** to $7(2-3n)$. Explain how you know each expression you select is equivalent.

1. $9-10n$
2. $14-3n$
3. $14-21n$
4. $(2-3n)\times 7$
5. $7\times 2\times (\text{-}3n)$

Student Response

$14-21n$ is equivalent because of the distributive property. $(2-3n)\times 7$ is equivalent because multiplication is commutative.

Activity Synthesis

Select a student who tested values to explain how they know two expressions are not equivalent. For example, $9-10n$ is not equivalent to $7(2-3n)$, because if we use 0 in place of n, $9-10\times 0$ is 9 but $7(2-3\times 0)$ is 14. If no one brings this up, demonstrate an example.

Select a student who used the term *distributive property* to explain why $7(2-3n)$ is equivalent to $14-21n$ to explain what they mean by that term. In general, an expression of the form $a(b+c)$ is equivalent to $ab+ac$.

3.2 Matching Equations to Bar Models

15 minutes

In this activity, the bar models and equations use the same numbers, so students must attend to the meaning of the operations in the equations and to the structure of the bar models.

Look for students who have sensible ways to distinguish diagram A from diagram C, and also diagram D from diagram E. These correspond to using categories to sort the equations like “multiply by 2” vs. “multiply by 5.” Also look for students who categorise equations by “brackets” vs. “no brackets.” These categories correspond to distinguishing A, B, and C from D and E.

Instructional Routines

- Collect and Display
- Take Turns

Launch

Keep students in the same groups. Tell them that, in this activity, they will match some diagrams, like the ones they saw in previous lessons, to corresponding equations. Then, they sort the list of equations into categories of their choosing. When they sort the equations, they should work with their partner to come up with categories, and then take turns sorting each equation into one of their categories, explaining why they are doing so. If necessary, demonstrate this protocol before students start working.

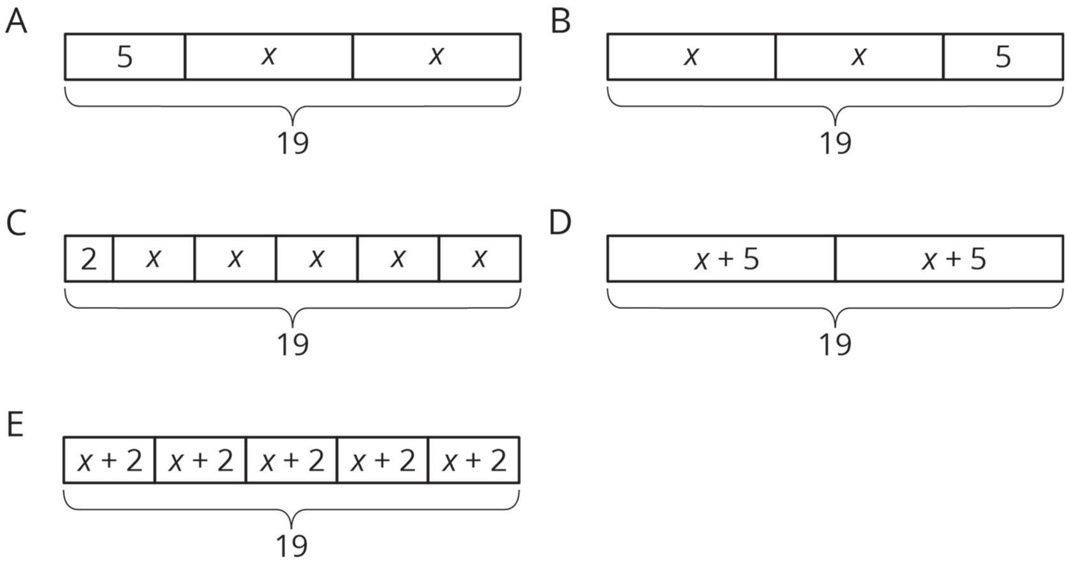
Give students 5 minutes to work with their partner followed by a whole-class discussion.

Representation: Internalise Comprehension. Demonstrate and encourage students to use colour coding and annotations to highlight connections between representations in a problem. For example, use the same colour to highlight the variable in the bar model to the same variable in the matching equation. Give students time to compare their colour coding and annotations during their partner discussion.
Supports accessibility for: Visual-spatial processing Speaking, Representing: Collect and Display. As students explain how the equation matches the diagram, listen for and collect students’ descriptions of the equation (e.g., “two groups of $x + 5$ equal 19”). Display collected language next to the corresponding bar model and equation for all to see. Invite students to borrow language from the displayed examples while sorting into categories, after the matching is complete. This will help students make connection between language, diagrams, and equations.
Design Principle(s): Support sense-making; Maximise meta-awareness

Anticipated Misconceptions

If students don’t know where to begin, encourage them to describe the diagrams and equations in words. For example, diagram E could be described “two groups of $x + 5$ equal 19,” and so could the equation $2(x + 5) = 19$.

Student Task Statement



1. Match each equation to one of the bar models. Be prepared to explain how the equation matches the diagram.

2. Sort the equations into categories of your choosing. Explain the criteria for each category.

- $2x + 5 = 19$
- $2 + 5x = 19$
- $2(x + 5) = 19$
- $5(x + 2) = 19$
- $19 = 5 + 2x$
- $(x + 5) \times 2 = 19$
- $19 = (x + 2) \times 5$
- $19 \div 2 = x + 5$
- $19 - 2 = 5x$

Student Response

- $2x + 5 = 19$ A or B
- $2 + 5x = 19$ C

- $2(x+5)=19$ D
- $5(x+2)=19$ E
- $19=5+2x$ A or B
- $(x+5)\times 2=19$ D
- $19=(x+2)\times 5$ E
- $19\div 2=x+5$ D
- $19-2=5x$ C

For the second question, answers vary. Likely categories include:

- Equation contains brackets vs. no brackets
- 19 on the left side vs. the right side of the =
- Multiply by 2 vs. multiply by 5

Activity Synthesis

Select a student who used the categories "multiply by 2" vs. "multiply by 5" to share their reasoning. Ask them to explain how we can see these categories in the corresponding diagrams. How did they categorise $19\div 2=x+5$ which contains no multiplication?

Select a student to share their reasoning who used the categories "brackets" vs. "no brackets." Did they have any misgivings about $19\div 2=x+5$? It contains no brackets, but the corresponding diagram D also matches $2(x+5)=19$, which has brackets.

If students express uncertainty about $2x+5=19$, spend some time here. Some students are likely to match it to exactly one diagram and some students match it to both A and B. The point isn't that one of these is right; it is to have the conversation about the idea of expressions or equations being identical vs. equivalent. Equivalent expressions or equivalent equations can have different literal interpretations, but when matching equations to bar models, all that matters for the purposes of solving is that the equations are equivalent.

3.3 Drawing Bar Models to Represent Equations

10 minutes (there is a digital version of this activity)

This activity is parallel to one in the previous lesson, except that students are creating a bar model after interpreting an equation rather than interpreting a story. The intention is for students to reason in any way that makes sense to them about the equations and diagrams to figure out the solution to each equation. Do not demonstrate any equation solving procedures yet.

For each equation, monitor for a student who used their diagram to reason about a solution and a student who used the structure of the equation to reason about a solution.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Stronger and Clearer Each Time

Launch

Give students 5 minutes of quiet work time followed by a whole-class discussion.

Representation: Develop Language and Symbols. Display or provide charts with symbols and meanings. For example, display a blank template of a bar model labelling the different parts with generalisations for what content will go inside. In addition, consider using a previous example situation or equation to make connections to the blank template.
Supports accessibility for: Conceptual processing; Memory Speaking, Representing: Stronger and Clearer Each Time. Use this routine to give students a structured opportunity to refine their bar models. Ask students to meet with 2–3 partners to get feedback on their diagram of one or both equations. Listeners should press for details from the equations (e.g., “Where is the ___ from the equation in your diagram?”). This will help students use language to describe connections between the diagrams and the equations.
Design Principle(s): Support sense-making

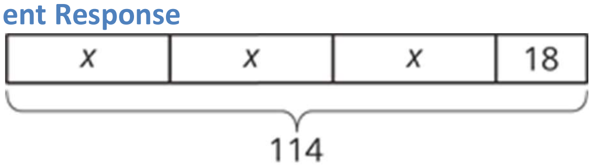
Student Task Statement

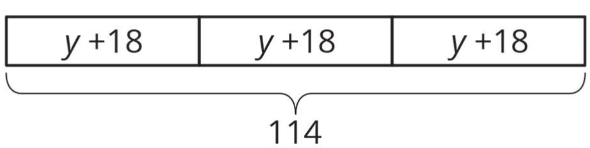
- $114 = 3x + 18$
- $114 = 3(y + 18)$

1. Draw a bar model to match each equation.
2. Use any method to find values for x and y that make the equations true.

Student Response

1.



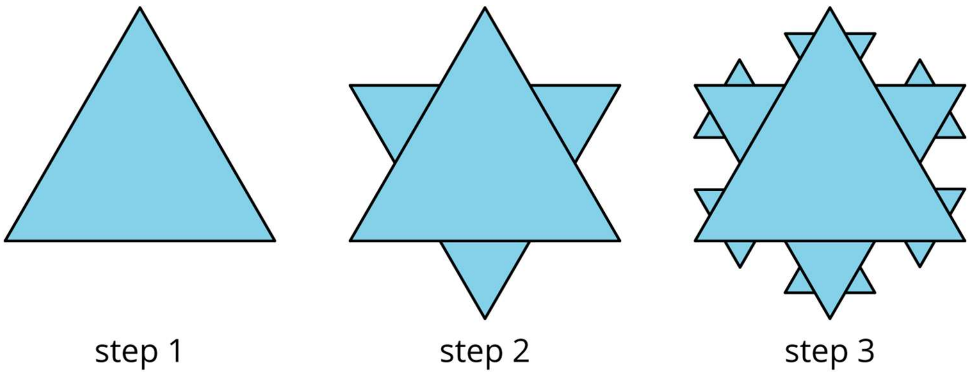


2. $x = 32$ and $y = 20$ Strategies vary.

Are You Ready for More?

To make a Koch snowflake:

- Start with an equilateral triangle that has side lengths of 1. This is step 1.
- Replace the middle third of each line segment with a small equilateral triangle with the middle third of the segment forming the base. This is step 2.
- Do the same to each of the line segments. This is step 3.
- Keep repeating this process.



1. What is the perimeter after step 2? Step 3?
2. What happens to the perimeter, or the length of line traced along the outside of the figure, as the process continues?

Student Response

1. 4, $5\frac{1}{3}$
2. The perimeter increases as the process continues.

Activity Synthesis

For each equation, select a student who used their diagram to reason about a solution and a student who used the structure of the equation to reason about a solution. Ask these students to explain how they arrived at a solution. Display the diagram and the equation side by side as students are explaining, and draw connections between the two representations.
Do not demonstrate any equation-solving procedures yet.

Lesson Synthesis

Display one or more bar models students encountered or created during the lesson, along with their corresponding equations. Ask, “What are some ways that bar models represent equations?” Responses to highlight:

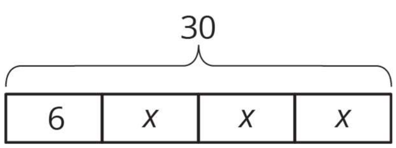
- Multiplication in the equation is represented with multiple copies of the same piece in the diagram.
- The total amount is shown in both the equation and the diagram.
- An unknown amount is represented with a variable.
- Either the equation or the diagram can be used to reason about a solution to the equation.

3.4 Three of These Equations Belong Together

Cool Down: 5 minutes

Student Task Statement

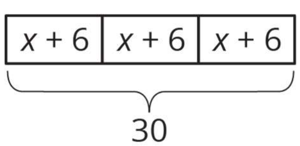
Here is a diagram.



1. Circle the equation that the diagram does *not* match.
   - $6 + 3x = 30$
   - $3(x + 6) = 30$
   - $3x = 30 - 6$
   - $30 = 3x + 6$
2. Draw a diagram that matches the equation you circled.

Student Response

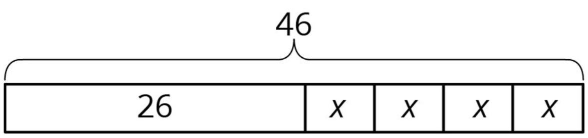
1. $3(x + 6) = 30$ does not match.
2. Sample diagram:



Student Lesson Summary

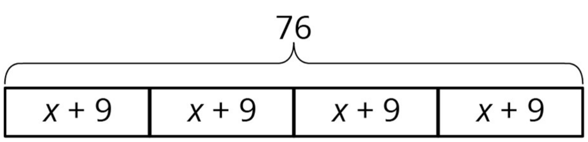
We have seen how bar models represent relationships between quantities. Because of the meaning and properties of addition and multiplication, more than one equation can often be used to represent a single bar model.

Let's take a look at two bar models.



We can describe this diagram with several different equations. Here are some of them:

- $26 + 4x = 46$, because the parts add up to the whole.
- $4x + 26 = 46$, because addition is commutative.
- $46 = 4x + 26$, because if two quantities are equal, it doesn't matter how we arrange them around the equal sign.
- $4x = 46 - 26$, because one part (the part made up of 4 x's) is the difference between the whole and the other part.



For this diagram:

- $4(x + 9) = 76$, because multiplication means having multiple groups of the same size.
- $(x + 9) \times 4 = 76$, because multiplication is commutative.
- $76 \div 4 = x + 9$, because division tells us the size of each equal part.

Glossary

- equivalent expressions

Lesson 3 Practice Problems

1. **Problem 1 Statement**

   Solve each equation mentally.

   a. $2x = 10$

   b. $-3x = 21$

c. $\frac{1}{3}x = 6$

d. $-\frac{1}{2}x = -7$

Solution

a. 5

b. -7

c. 18

d. 14

2. **Problem 2 Statement**

Complete the magic squares so that the sum of each row, each column, and each diagonal in a grid are all equal.

0	7	2
	3	

1		
	3	-2
		5

4	2	0
-1		

Solution

0	7	2
5	3	1
4	-1	6

1	2	6
8	3	-2
0	4	5

3	-2	5
4	2	0
-1	6	1

3. **Problem 3 Statement**

Draw a bar model to match each equation.

a. $5(x + 1) = 20$

b. $5x + 1 = 20$

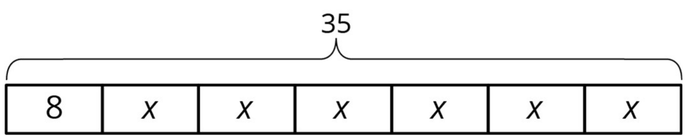
Solution

a. A diagram showing 5 equal parts of $x + 1$ for a total of 20

b. A diagram showing 5 equal parts of x and one part of 1 for a total of 20

4. **Problem 4 Statement**

Select **all** the equations that match the bar model.



a. $35 = 8 + x + x + x + x + x + x$

b. $35 = 8 + 6x$

c. $6 + 8x = 35$

d. $6x + 8 = 35$

e. $6x + 8x = 35x$

f. $35 - 8 = 6x$

Solution ["A", "B", "D", "F"]

5. **Problem 5 Statement**

Each car is travelling at a constant speed. Find the number of miles each car travels in 1 hour at the given rate.

a. 135 miles in 3 hours

b. 22 miles in $\frac{1}{2}$ hour

c. 7.5 miles in $\frac{1}{4}$ hour

d. $\frac{100}{3}$ miles in $\frac{2}{3}$ hour

e. $97\frac{1}{2}$ miles in $\frac{3}{2}$ hour

Solution

a. 45 miles

b. 44 miles

c. 30 miles

d. 50 miles

e. 65 miles