Mathematics Education Innovation

Lesson 9: Formula for the area of a triangle

Goals

- Compare, contrast, and critique (orally) different strategies for determining the area of a triangle.
- Generalise a process for finding the area of a triangle, and justify (orally and in writing) why this can be abstracted as $\frac{1}{2} \times b \times h$.
- Recognise that any side of a triangle can be considered its base, choose a side to use as the base when calculating the area of a triangle, and identify the corresponding height.

Learning Targets

- I can use the area formula to find the area of any triangle.
- I can write and explain the formula for the area of a triangle.
- I know what the terms “base” and “height” refer to in a triangle.

Lesson Narrative

In this lesson students begin to reason about area of triangles more methodically: by generalising their observations up to this point and expressing the area of a triangle in terms of its **base** and **height**.

Students first learn about bases and heights in a triangle by studying examples and non-examples. They then identify base-height measurements of triangles, use them to determine area, and look for a pattern in their reasoning to help them write a general formula for finding area. Students also have a chance to build an informal argument about why the formula works for any triangle.

Addressing

- Write expressions that record operations with numbers and with letters standing for numbers. For example, express the calculation “Subtract y from 5” as $5 - y$.
- Evaluate expressions at specific values of their variables. Include expressions that arise from formulas used in real-world problems. Perform arithmetic operations, including those involving whole-number exponents, in the conventional order when there are no brackets to specify a particular order (Order of Operations). For example, use the formulas $V = s^3$ and $A = 6s^2$ to find the volume and surface area of a cube with sides of length $s = \frac{1}{2}$.
- Find the area of right-angled triangles, other triangles, special quadrilaterals, and polygons by composing into rectangles or decomposing into triangles and other shapes; apply these techniques in the context of solving real-world and mathematical problems.

Building Towards

- Find the area of right-angled triangles, other triangles, special quadrilaterals, and polygons by composing into rectangles or decomposing into triangles and other shapes; apply these techniques in the context of solving real-world and mathematical problems.

Instructional Routines

- Collect and Display
- Discussion Supports
- Notice and Wonder

Required Materials

Geometry toolkits

tracing paper, graph paper, coloured pencils, scissors, and an index card to use as a straightedge or to mark right angles.

Student Learning Goals

Let's write and use a formula to find the area of a triangle.

9.1 Bases and Heights of a Triangle

Warm Up: 10 minutes

In this activity, students think about the meaning of **base** and **height** in a triangle by studying examples and non-examples. The goal is for them to see that in a triangle:

- Any side can be a base.
- A segment that represents a height must be drawn at a right angle to the base, but can be drawn in more than one place. The length of this perpendicular segment is the distance between the base and the vertex opposite it.
- A triangle can have three possible bases and three corresponding heights.

Students may draw on their experience with bases and heights in a parallelogram and observe similarities. Encourage this, as it would help them conceptualise base-height pairs in triangles.

As students discuss with their partners, listen for how they justify their decisions or how they know which statements are true.

Instructional Routines

- Notice and Wonder

Launch

Display the examples and non-examples of bases and heights for all to see. Give students a minute to observe them. Ask them to be ready to share at least one thing they notice and one thing they wonder. Give the class a minute to share some of their observations and questions.

Tell students they will now use the examples and non-examples to determine what is true about bases and heights in a triangle. Arrange students in groups of 2. Give them 2–3 minutes of quiet think time and then a minute to share their response with a partner.

Anticipated Misconceptions

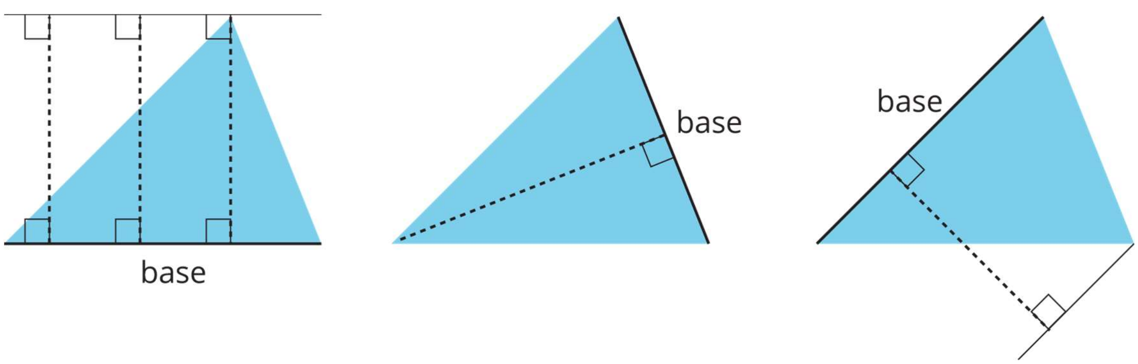
Some students may struggle to interpret the diagrams. Ask them to point out parts of the diagrams that might be unclear and clarify as needed.

Students may not remember from their experience with parallelograms that a height needs to be perpendicular to a base. Consider posting a diagram of a parallelogram—with its base and height labelled—in a visible place in the room so that it can serve as a reference.

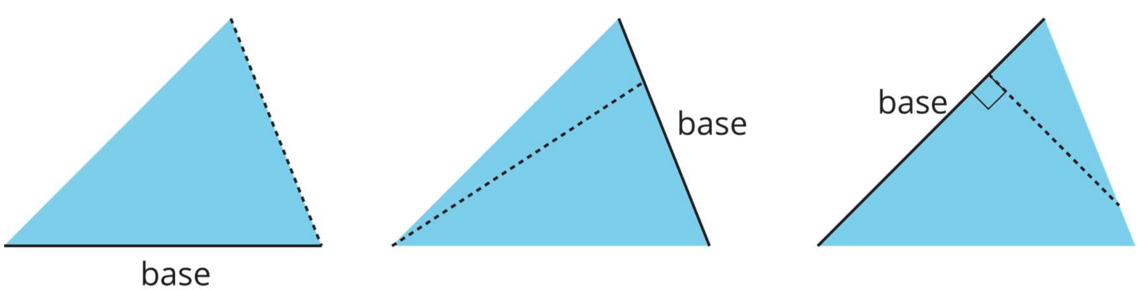
Student Task Statement

Study the examples and non-examples of **bases** and **heights** in a triangle.

- Examples: These dashed segments represent heights of the triangle.



- Non-examples: These dashed segments do *not* represent heights of the triangle.



Select **all** the statements that are true about bases and heights in a triangle.

1. Any side of a triangle can be a base.

2. There is only one possible height.
3. A height is always one of the sides of a triangle.
4. A height that corresponds to a base must be drawn at an acute angle to the base.
5. A height that corresponds to a base must be drawn at a right angle to the base.
6. Once we choose a base, there is only one segment that represents the corresponding height.
7. A segment representing a height must go through a vertex.

Student Response

Only statements 1 and 5 are true.

Activity Synthesis

For each statement, ask students to indicate whether they think it is true. For each "true" vote, ask one or two students to explain how they know. Do the same for each "false" vote. Encourage students to use the examples and counterexamples to support their argument (e.g., "The last statement is not true because the examples show dashed segments or heights that do not go through a vertex.") which means more than one height." Agree on the truth value of each statement before moving on. Record and display the true statements for all to see.

Students should see that only statements 1 and 5 are true—that any side of a triangle can be a base, and a segment for the corresponding height must be drawn at a right angle to the base. What is missing—an important gap to fill during discussion—is the length of any segment representing a height.

Ask students, "How long should a segment that shows a height be? If we draw a perpendicular line from the base, where do we stop?" Solicit some ideas from students.

Explain that the length of each perpendicular segment is the distance between the base and the vertex opposite of it. The **opposite vertex** is the vertex that is not an endpoint of the base. Point out the opposite vertex for each base. Clarify that the segment does not have to be drawn through the vertex (although that would be a natural place to draw it), as long as it maintains that distance between the base and the opposite vertex.

It is helpful to connect this idea to that of heights in a parallelogram. Consider duplicating the triangle and use the original and the copy to compose a parallelogram. The height for a chosen base in the triangle is also the height of the parallelogram with the same base.

Students will have many opportunities to make sense of bases and heights in this lesson and an upcoming one, so they do not need to know how to draw a height correctly at this point.

9.2 Finding a Formula for Area of a Triangle

20 minutes

This task culminates in writing a formula for the area of triangles. By now students are likely to have developed the intuition that the area of a triangle is half of that of a parallelogram with the same base and height. This activity encapsulates that work in an algebraic expression.

Students first find the areas of several triangles given base and height measurements. They then generalise the numerical work to arrive at an expression for finding the area of any triangle.

If needed, remind students how they reasoned about the area of triangles in the previous lesson (i.e. by composing a parallelogram, enclosing with one or more rectangles, etc.). Encourage them to refer to their previous work and use tracing paper as needed. Students might write $b \times h \div 2$ or $b \times h \times \frac{1}{2}$ as the expression for the area of any triangle. Any equivalent expression should be celebrated.

At the end of the activity, consider giving students a chance to reason more abstractly and deductively, i.e., to think about why the expression $b \times h \div 2$ would hold true for all triangles. See Activity Synthesis for prompts and diagrams that support such reasoning.

Instructional Routines

- Discussion Supports

Launch

Arrange students in groups of 2–3. Explain that they will now find the area of some triangles using what they know about base-height pairs in triangles and the relationships between triangles and parallelograms.

Give students 5–6 minutes to complete the activity and access to geometry toolkits, especially tracing paper. Ask them to find the area at least a couple of triangles independently before discussing with their partner(s).

Anticipated Misconceptions

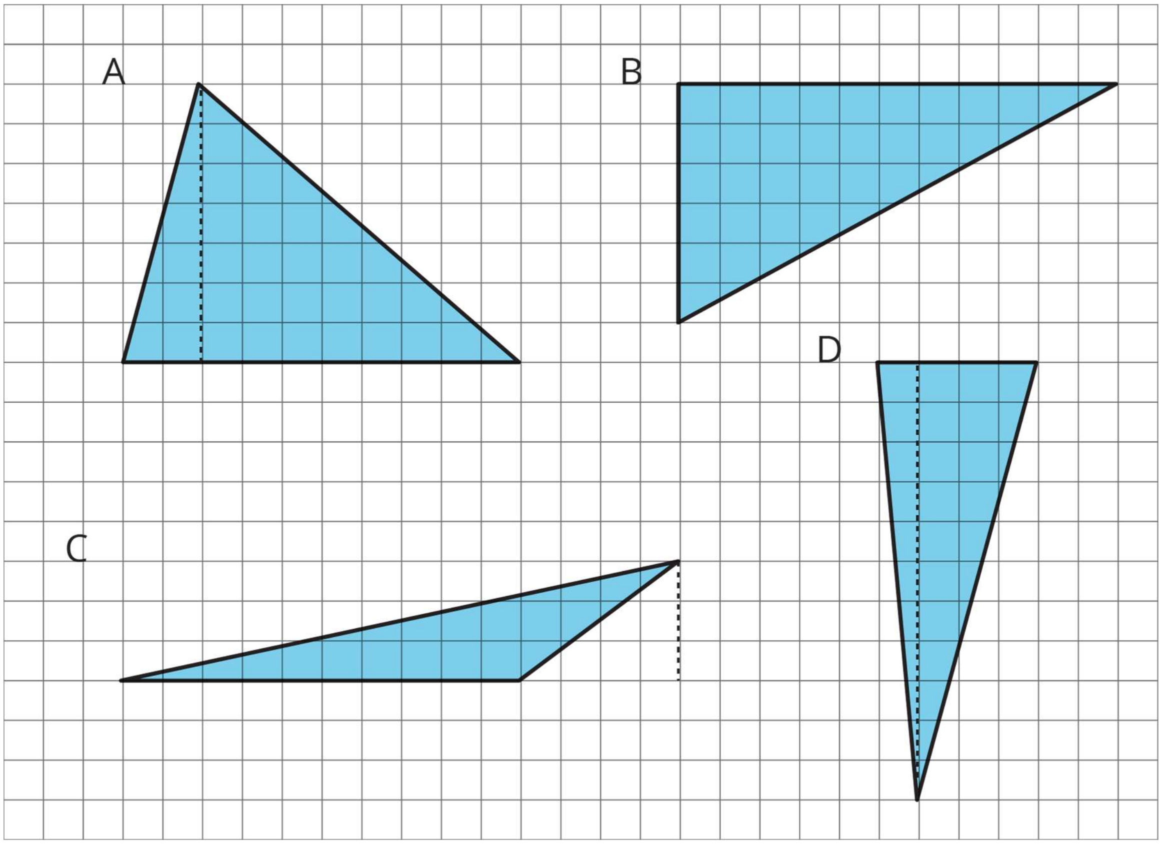
Students may not be inclined to write an expression using the variables b and h and instead replace the variables with numbers of their choice. Ask them to reflect on what they did with the numbers for the first four triangles. Then, encourage them to write the same operations but using the letters b and h rather than numbers.

Student Task Statement

For each triangle:

- Identify a base and a corresponding height, and record their lengths in the table.

- Find the area of the triangle and record it in the last column of the table.



triangle	base (units)	height (units)	area (square units)
A			
B			
C			
D			
any triangle	b	h	

In the last row, write an expression for the area of any triangle, using b and h.

Student Response

triangle	base (units)	height (units)	area (square units)
A	10	7	35
B	11 (or 6)	6 (or 11)	33
C	10	3	15
D	4	11	22
any triangle	b	h	$b \times h \div 2$ (or equivalent)

1. Explanations vary. Sample responses:

- We can make a parallelogram from any triangle using the same base and height. The triangle will be half of the parallelogram. The area of a parallelogram is the length of the base times the length of the height, so the area of the triangle will be $b \times h \div 2$.

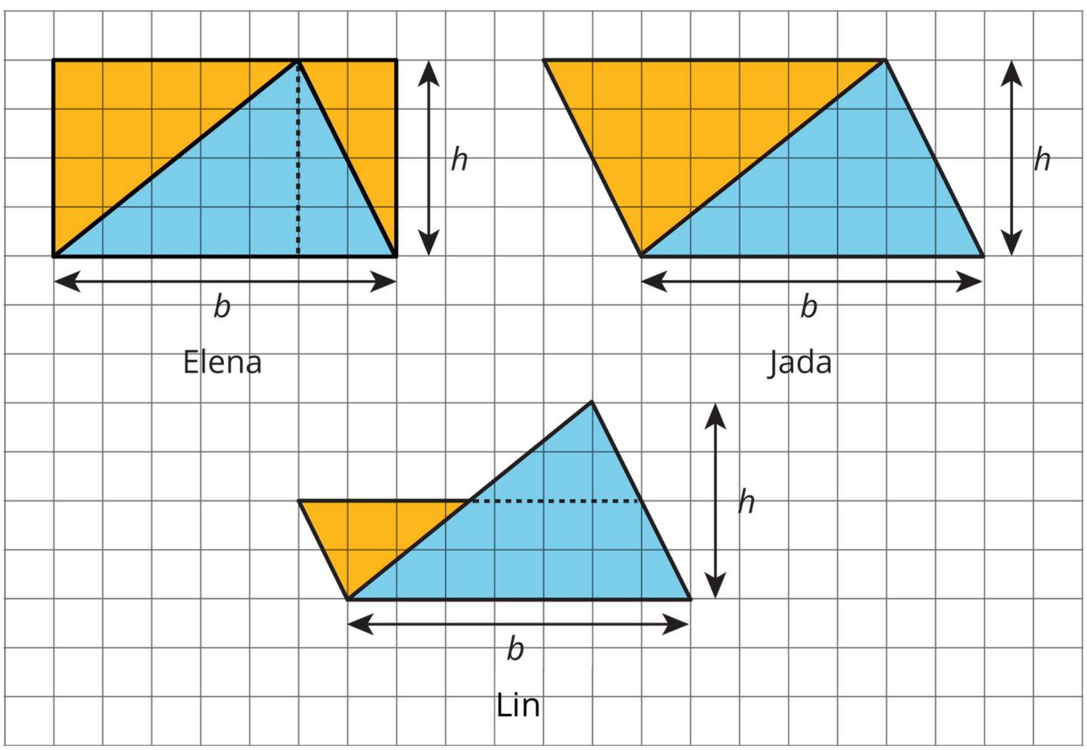
- I can cut off the top half of a triangle and rotate it to make a parallelogram. That parallelogram has a base of b and a height that is half of the original triangle, which is $\frac{1}{2} \times h$, so its area is $b \times \frac{1}{2} \times h$. Since the parallelogram is just the triangle rearranged, the area of the triangle is also $\frac{1}{2} b \times h$.

Activity Synthesis

Select a few students to share their expression for finding the area of any triangle. Record each expression for all to see.

To give students a chance to reason logically and deductively about their expression, ask, "Can you explain why this expression is true for *any* triangle?"

Display the following diagrams for all to see. Give students a minute to observe the diagrams. Ask them to choose one that makes sense to them and use that diagram to explain or show in writing that the expression $b \times h \div 2$ works for finding the area of any triangle. (Consider giving each student an index card or a sheet of paper on which to write their reasoning so that their responses could be collected, if desired.)



When dealing only with the variables b and h and no numbers, students are likely to find Jada's and Lin's diagrams more intuitive to explain. Those choosing to use Elena's diagram are likely to suggest moving one of the extra triangles and joining it with the other to form a non-rectangular parallelogram with an area of $b \times h$. Expect students be less comfortable reasoning in abstract terms than in concrete terms. Prepare to support them in piecing together a logical argument using only variables.

If time permits, select students who used different diagrams to share their explanation, starting with the most commonly used diagram (most likely Jada's). Ask other students to support, refine, or disagree with their arguments. It time is limited, consider collecting students' written responses now and discussing them in an upcoming lesson.

Representation: Develop Language and Symbols. Create a display of important terms and concepts. Invite students to suggest language or diagrams to include that will support their understanding. Include the following terms and maintain the display for reference throughout the unit: opposite vertex, base (of a triangle), height (of a triangle), and formula for finding area.
Supports accessibility for: Memory; Language Speaking: Discussion Supports. Use this routine to support whole-class discussion when students share their expressions for finding the area of any triangle. Provide students with 1–2 minutes of quiet think time to begin to consider why their expression is true for any triangle, before they continue to work with a partner to complete their response. Select 1 or 2 pairs of students to share with the class, then call on students to restate their peers' reasoning. This will give more students the chance to use language to interpret and describe expressions for the area of triangles.
Design Principle(s): Support sense-making; Maximise meta-awareness

9.3 Applying the Formula for Area of Triangles

10 minutes

In this activity, students apply the expression they previously generated to find the areas of various triangles. Each diagram is labelled with two or three measurements. Before calculating, students think about which lengths can be used to find the area of each triangle.

As students work, notice students who choose different bases for triangles B and D. Invite them to contribute to the discussion about finding the areas of right-angled triangles later.

Instructional Routines

- Collect and Display

Launch

Explain to students that they will now practise using their expression to find the area of triangles without a grid. For each triangle, ask students to be prepared to explain which measurement they choose for the base and which one for the corresponding height and why.

Keep students in groups of 2–4. Give students 5 minutes of quiet think time, followed by 1–2 minutes for discussing their responses in their group.

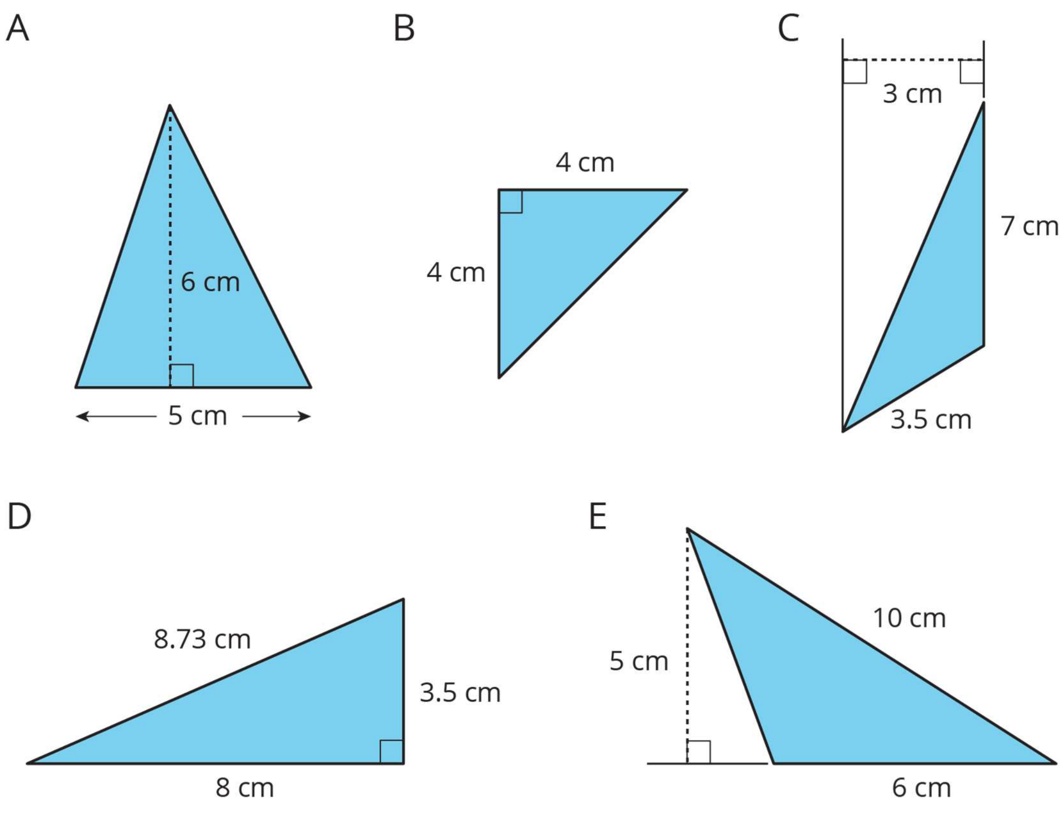
Representing: Collect and Display. While pairs are working, circulate and listen to student talk about identifying the bases and corresponding heights for each of triangles. Write down common or important phrases you hear students say about each triangle, specifically focusing on how students make sense of the base and height of each triangle. Record the words students use to refer to each triangle and display them for all to see during the whole-class discussion.
Design Principle(s): Support sense-making; Maximise meta-awareness

Anticipated Misconceptions

The extra measurement in triangles C, D, and E may confuse some students. If they are unsure how to decide the measurement to use, ask what they learned must be true about a base and a corresponding height in a triangle. Urge them to review the work from the warm-up activity.

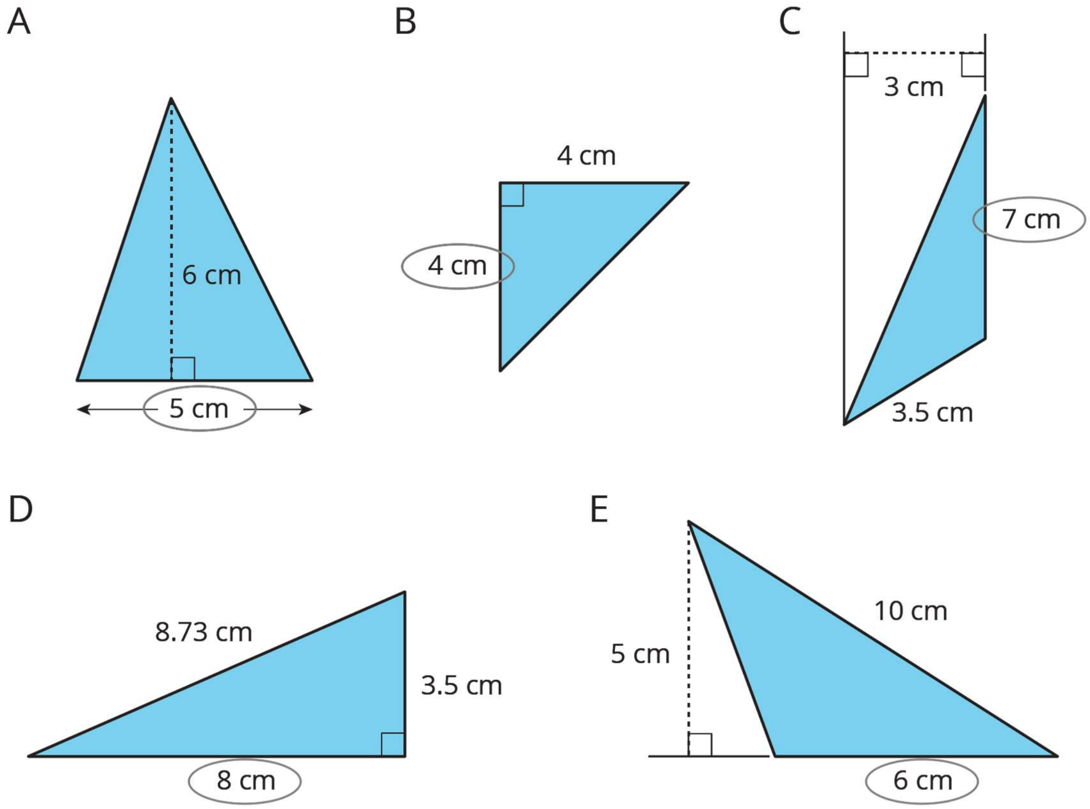
Student Task Statement

For each triangle, circle a base measurement that you can use to find the area of the triangle. Then, find the area of any *three* triangles. Show your reasoning.



Student Response

In B and D either of the given pair of measurements can be the base.



Triangle A: 15 square cm, $b = 5$, $h = 6$, $A = 5 \times 6 \div 2 = 15$

Triangle B: 8 square cm, $b = 4$, $h = 4$, $A = 4 \times 4 \div 2 = 8$

Triangle C: 10.5 square cm, $b = 7$, $h = 3$, $A = 7 \times 3 \div 2 = 10.5$

Triangle D: 14 square cm, $b = 8$, $h = 3.5$, $A = 8 \times (3.5) \div 2 = 14$

Triangle E: 15 square cm, $b = 6$, $h = 5$, $A = 6 \times 5 \div 2 = 15$

Activity Synthesis

The aim of this whole-class discussion is to deepen students' awareness of the base and height of triangles. Discuss questions such as:

- For triangle A, can we say that the 6-cm segment is the base and the 5-cm segment is the height? Why or why not? (No, the base of a triangle is one of its sides.)
- What about for triangle C: Can the 3-cm segment serve as the base? Why or why not? (No, that segment is not a side of the triangle.)
- Can the 3.5-cm side in triangle D serve as the base? Why or why not? (Yes, it is a side of the triangle, but because we don't have the height that corresponds to it, it is not helpful for finding the area here.)

- More than two measurements are given for triangles C, D, and E. Which ones are helpful for finding area? (We need a base and a corresponding height, which means the length of one side of the triangle and the length of a perpendicular segment between that side and the opposite vertex.)
- When it comes to finding area, how are right-angled triangles—like B and D—unique? (Either of the two sides that form the right angle could be the base or the height. In non-right-angled triangles—like A, C, and E—the height segment is not a side of the triangle; a different line segment has to be drawn.)

Lesson Synthesis

The area of a parallelogram can be determined using base and height measurements. In this lesson you learned that we can do the same with triangles.

- "How do we locate the **base** of a triangle? How many possible bases are there?" (Any side of a triangle can be a base. There are 3 possible bases.)
- "How do we locate the **height** once we know the base?" (Find the length of a perpendicular segment that connects the base and its opposite vertex.)

We can use the base-height pair of measurements to find the area of a triangle quite simply.

- "What expression works for finding the area of a triangle?" ($\frac{1}{2} \times b \times h$ or $\frac{b \times h}{2}$)
- "Can you explain briefly why this expression or formula works?" (The area of a triangle is always half of the area of a related parallelogram that shares the same base and height.)

You learned that any side of the triangle can be the base, but not all sides can be the height.

- "Are there cases in which both the base and the height are sides of the triangle? When does that happen?" (Yes. In a right-angled triangle, both the base and height can be the sides of the triangle.)

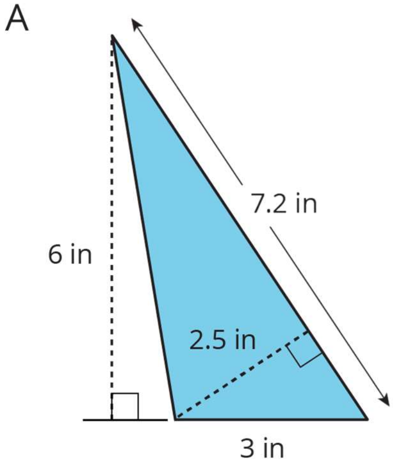
9.4 Two More Triangles

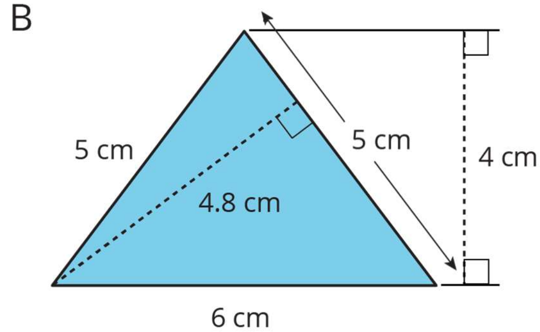
Cool Down: 5 minutes

Students apply what they learned about the area formula and about the base and height of a triangle in this cool-down. Multiple measurements are given, so students need to be attentive in choosing the right pair of measurements that would allow them to calculate the area.

Student Task Statement

For each triangle, identify a base and a corresponding height. Use them to find the area. Show your reasoning.





Student Response

Answers vary. Possible responses:

Triangle A:

- $b = 3, h = 6$, area: 9 sq in, $\frac{1}{2} \times 3 \times 6 = 9$
- $b = 7.2, h = 2.5$, area: 9 sq in, $\frac{1}{2} \times (7.2) \times (2.5) = 9$

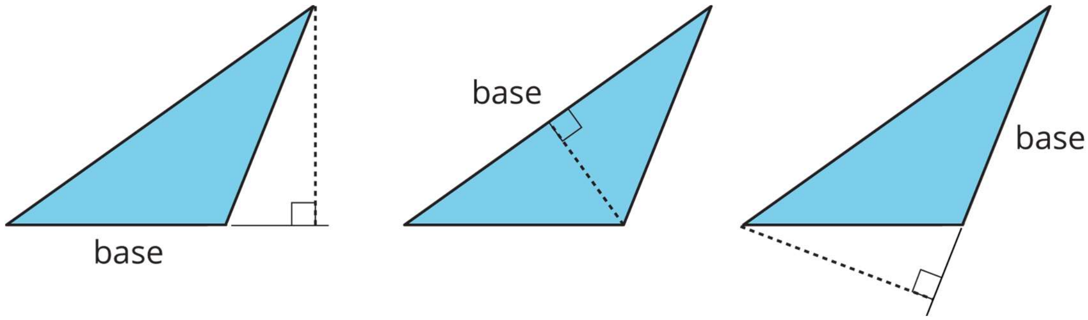
Triangle B:

- $b = 6, h = 4$, area: 12 sq in, $\frac{1}{2} \times 6 \times 4 = 12$
- $b = 5, h = 4.8$, area: 12 sq in, $\frac{1}{2} \times 5 \times (4.8) = 12$

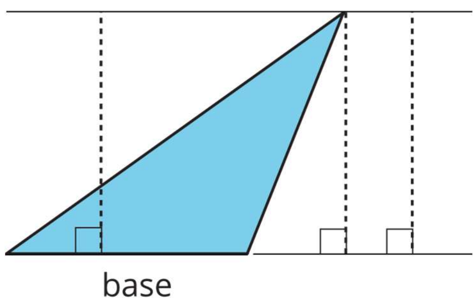
Student Lesson Summary

- We can choose any of the three sides of a triangle to call the **base**. The term “base” refers to both the side and its length (the measurement).
- The corresponding **height** is the length of a perpendicular segment from the base to the vertex opposite of it. The **opposite vertex** is the vertex that is *not* an endpoint of the base.

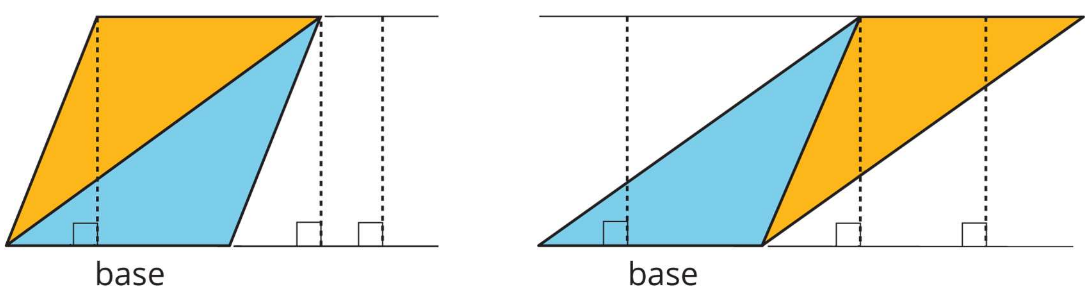
Here are three pairs of bases and heights for the same triangle. The dashed segments in the diagrams represent heights.



A segment showing a height must be drawn at a right angle to the base, but it can be drawn in more than one place. It does not have to go through the opposite vertex, as long as it connects the base and a line that is parallel to the base and goes through the opposite vertex, as shown here.



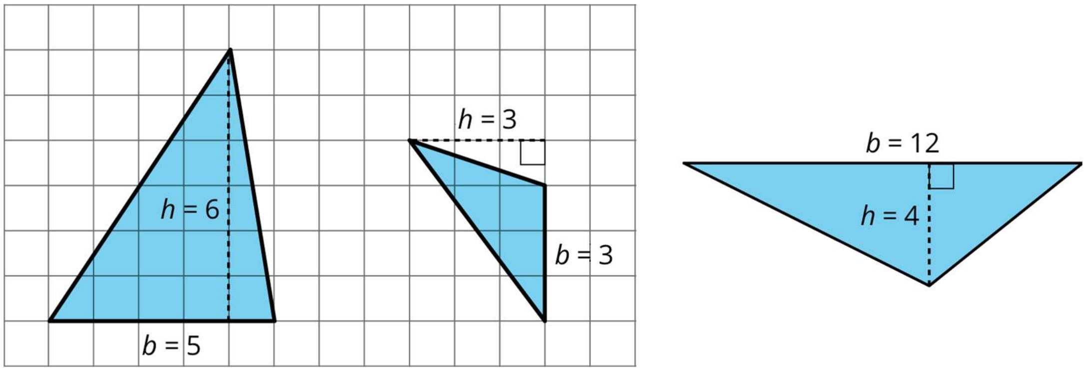
The base-height pairs in a triangle are closely related to those in a parallelogram. Recall that two copies of a triangle can be composed into one or more parallelograms. Each parallelogram shares at least one base with the triangle.



For any base that they share, the corresponding height is also shared, as shown by the dashed segments.

We can use the base-height measurements and our knowledge of parallelograms to find the area of any triangle.

- The formula for the area of a parallelogram with base b and height h is $b \times h$.
- A triangle takes up half of the area of a parallelogram with the same base and height. We can therefore express the area A of a triangle as: $A = \frac{1}{2} \times b \times h$

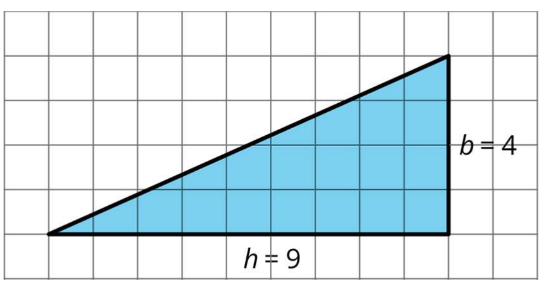


- The area of triangle A is 15 square units because $\frac{1}{2} \times 5 \times 6 = 15$.
- The area of triangle B is 4.5 square units because $\frac{1}{2} \times 3 \times 3 = 4.5$.
- The area of triangle C is 24 square units because $\frac{1}{2} \times 12 \times 4 = 24$.

In each case, one side of the triangle is the base but neither of the other sides is the height. This is because the angle between them is not a right angle.

In right-angled triangles, however, the two sides that are perpendicular can be a base and a height.

The area of this triangle is 18 square units whether we use 4 units or 9 units for the base.



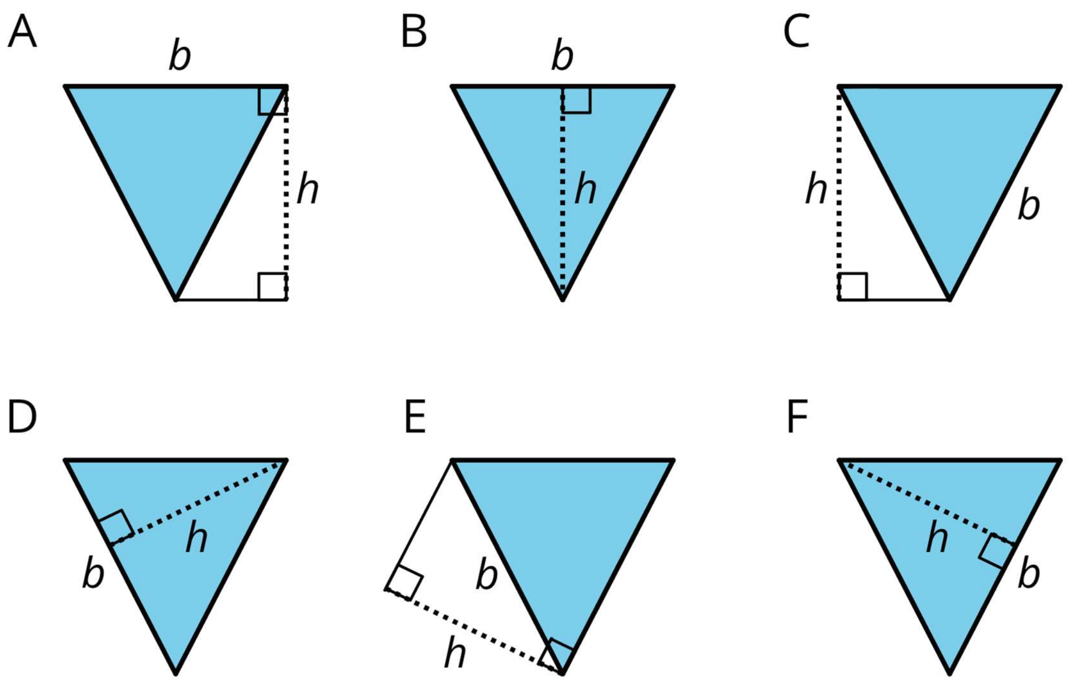
Glossary

- opposite vertex

Lesson 9 Practice Problems

1. **Problem 1 Statement**

   Select **all** drawings in which a corresponding height h for a given base b is correctly identified.



   a. A

   b. B

   c. C

   d. D

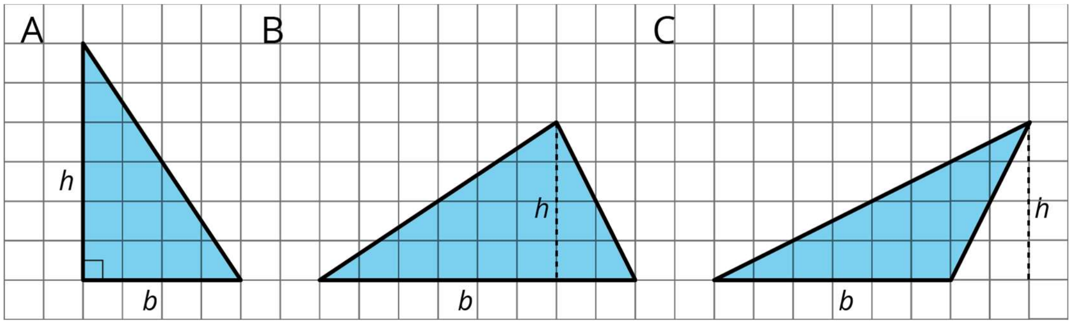
   e. E

   f. F

   Solution ["A", "B", "D", "F"]

2. **Problem 2 Statement**

   For each triangle, a base and its corresponding height are labelled.

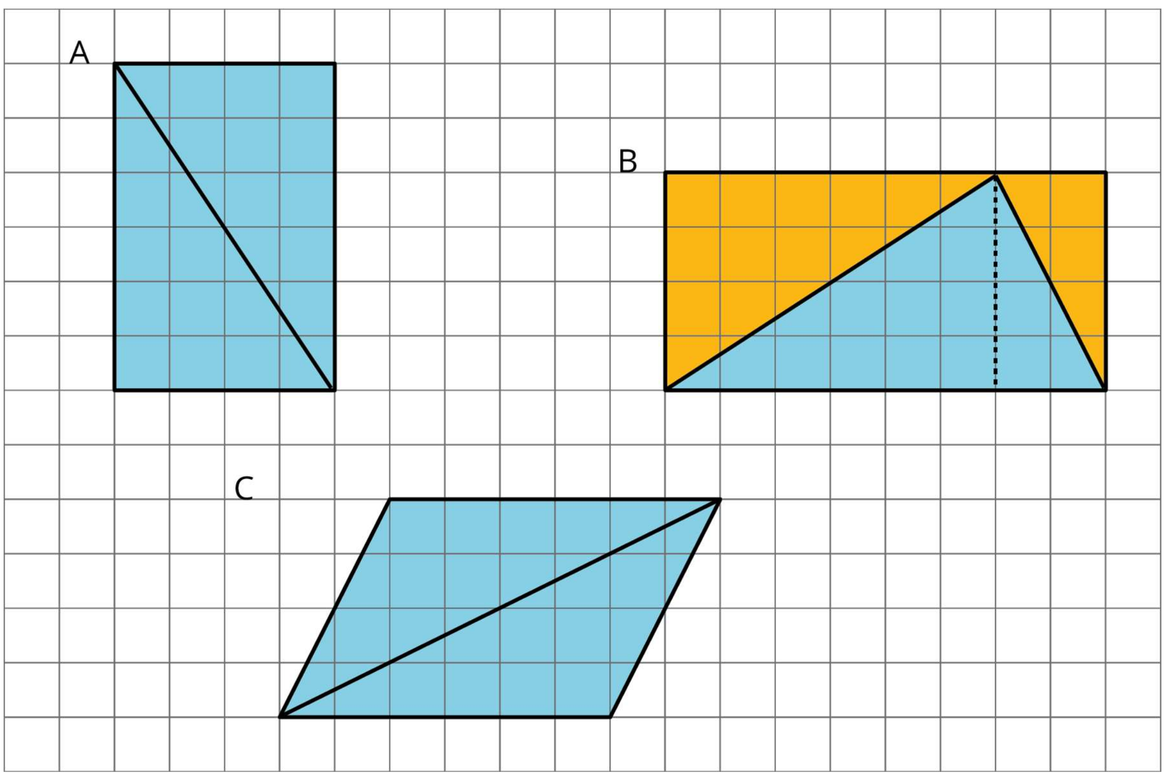


a. Find the area of each triangle.

b. How is the area related to the base and its corresponding height?

Solution

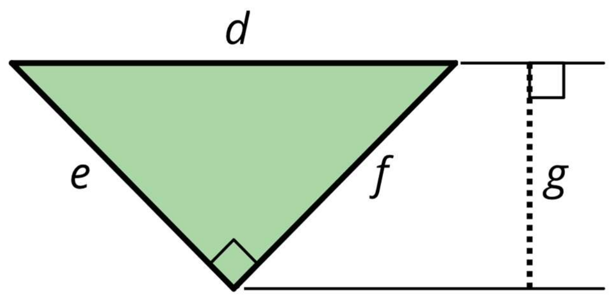
a. Triangle A: 12 square units, triangle B: 16 square units, triangle C: 12 square units



b. In each case, the area of the triangle, in square units, is half of the base times its corresponding height, $\frac{b \times h}{2}$.

3. **Problem 3 Statement**

Here is a right-angled triangle. Name a corresponding height for each base.



a. Side d

b. Side e

c. Side f

Solution

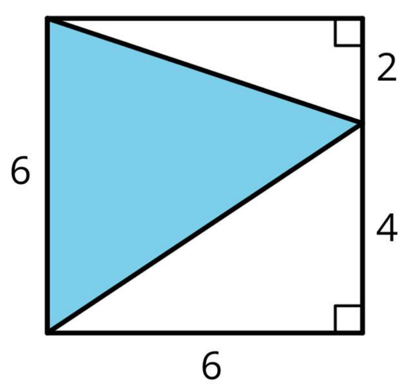
a. Segment g

b. Side f

c. Side e

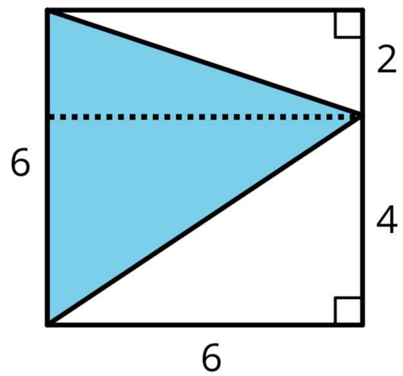
4. **Problem 4 Statement**

Find the area of the shaded triangle. Show your reasoning.



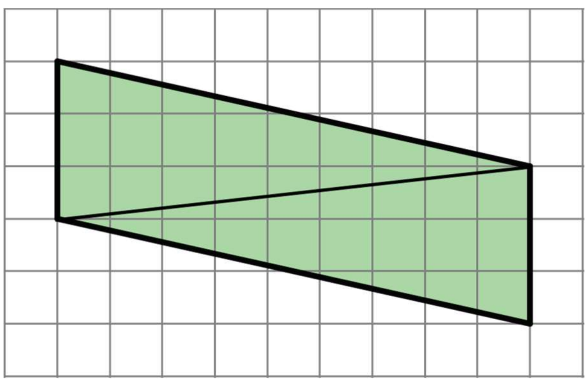
Solution

18 square units. Reasoning varies. One likely approach is by decomposing the triangle with a horizontal line to form two rectangles and to split the triangle into two smaller triangles. The top triangle is half of the top rectangle, so its area is $\frac{1}{2} \times 6 \times 2 = 6$. The bottom triangle is half of the bottom rectangle, so its area is $\frac{1}{2} \times 6 \times 4 = 12$. The area of the original triangle is $6 + 12$ or 18 square units.



5. **Problem 5 Statement**

   Andre drew a line connecting two opposite corners of a parallelogram. Select **all** true statements about the triangles created by the line Andre drew.

   a. Each triangle has two sides that are 3 units long.

   b. Each triangle has a side that is the same length as the diagonal line.

   c. Each triangle has one side that is 3 units long.

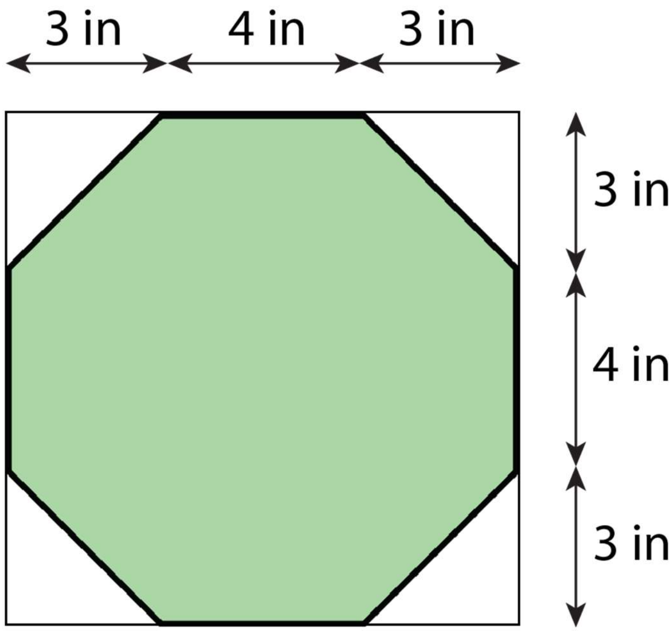
   d. When one triangle is placed on top of the other and their sides are aligned, we will see that one triangle is larger than the other.

   e. The two triangles have the same area as each other.

   Solution ["B", "C", "E"]

6. **Problem 6 Statement**

   Here is an octagon. (Note: The diagonal sides of the octagon are *not* 4 inches long.)



a. While estimating the area of the octagon, Lin reasoned that it must be less than 100 square inches. Do you agree? Explain your reasoning.

b. Find the exact area of the octagon. Show your reasoning.

Solution

a. Yes. Explanations vary. Sample explanation: The octagon fits in a square that is 10 inches by 10 inches, but with four corners of the square removed. The square has an area of 100 square inches, so the area of the octagon must be less than that.

b. 82 square inches. Reasoning varies. Sample reasoning: A 10-inch-by-10-inch square that encloses the octagon has an area of 100 square inches. Two corner triangles compose a 3 inch-by-3 inch square, so their combined area is 9 square inches. $100 - 2(3 \times 3) = 100 - 18 = 82$.