INTERNATIONAL BACCALAUREATE

Mathematics: analysis and approaches

MAA

EXERCISES [MAA 1.11-1.12]

COMPLEX NUMBERS (CARTESIAN FORM)

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CARTESIAN FORM

O. Practice questions

1. [Maximum mark: 6] ***[without GDC]***

Let $f(z) = z^2 - 8z + 20$.

(a) Find the discriminant Δ of the quadratic function f. [1]

(b) Find the complex roots of the equation $f(z) = 0$ in the form $z = x \pm y\mathrm{i}$ [3]

(c) Use factorisation to express f in the form $f(z) = (z-h)^2 + k$. [2]

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2. [Maximum mark: 8] ***[with / without GDC]***

The complex roots of the equation $az^2 + bz + c = 0$ are given by $z = \frac{-b \pm \mathrm{i}\sqrt{|\Delta|}}{2a}$.

where $\Delta = b^2 - 4ac$

(a) Find the complex roots of the equation $4z^2 - 8z + 13 = 0$ expressing your answers in the form $z = x \pm y\mathrm{i}$. [4]

(b) Confirm that the sum S and the product P of the roots are given by

(i) $S = -\frac{b}{a}$. (ii) $P = \frac{c}{a}$. [4]

3. [Maximum mark: 18] ***[with / without GDC]***

Let $z_1 = 3 + 4\mathrm{i}$ and $z_2 = 10 + 5\mathrm{i}$

(a) Find the following results in the form $x + y\mathrm{i}$.

(i) $z_1 + z_2$. (ii) $z_2 - z_1$ (iii) $z_1 z_2$ (iv) $\frac{z_2}{z_1}$ [6]

(b) Find the following powers in the form $x + y\mathrm{i}$

(i) ${z_1}^2$. (ii) ${z_1}^3$ [6]

(c) Find

(i) $|z_1|$ (ii) $|z_2|$ (iii) $|z_2 - z_1|$ [6]

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4. [Maximum mark: 6] ***[without GDC]***

(a) Find $(1-i\sqrt{3})^2$ in the form $a+bi$, where $a,b \in Z$. [3]

(b) Find $(1-i\sqrt{3})^3$. [3]

[Confirm the results by your GDC]

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5. [Maximum mark: 4]

(a) Given that $(a-2)+3i = 7+(b-1)i$, find the value of a and of b, where $a,b \in \mathbb{Z}$. [2]

(b) Given that $(c-2)+(d-1)i = 0$, find the value of c and of d, where $c,d \in \mathbb{Z}$. [2]

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6. [Maximum mark: 9] ***[with / without GDC]***

Let $z = x + y\text{i}$. Find the values of x and y if

$$(2+5\text{i})z = 1+17\text{i}$$

(a) by substituting $z = x + y\text{i}$ in the equation and solving the simultaneous equations. [5]

(b) by using division (an equation of the form $az = b$, implies $z = b/a$) [4]

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A. Exam style questions (SHORT)

7. [Maximum mark: 6] ***[without GDC]***

Let the complex number z be given by $z = 1 + \frac{\mathrm{i}}{\mathrm{i} - \sqrt{3}}$.

Express z in the form $a + b\mathrm{i}$, giving the **exact** values of the real constants a, b.

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8. [Maximum mark: 5] ***[without GDC]***

Express $\frac{1}{(1 - i\sqrt{3})^3}$ in the form $\frac{a}{b}$ where $a, b \in Z$.

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9. [Maximum mark: 6] ***[with / without GDC]***

Let $z=\frac{2}{1-\mathrm{i}}+1-4\mathrm{i}$. Express z^2 in the form $x+y\mathrm{i}$ where $x, y \in$ Z.

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10. [Maximum mark: 6] ***[without GDC]***

Consider the equation $2(p+\mathrm{i}q)=q-\mathrm{i}p-2(1-\mathrm{i})$, where p and q are both real numbers. Find p and q.

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11. [Maximum mark: 6] ***[without GDC]***

Given that $(a+\mathrm{i})\,(2-b\mathrm{i})\ =(7-\mathrm{i})$, find the value of a and of b, where $a,b \in \mathbb{Z}$.

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12. [Maximum mark: 4] ***[with / without GDC]***

Find the values of a and b, where a and b are real, given that $(a+b\mathrm{i})(2-\mathrm{i})=5-\mathrm{i}$

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13. [Maximum mark: 4] ***[with without GDC]***

Let $z = x + y\mathrm{i}$. Find the values of x and y if $(1-\mathrm{i})z = 1-3\mathrm{i}$.

14. [Maximum mark: 5] ***[without GDC]***

The complex number z satisfies $\mathrm{i}(z+2) = 1-2z$, where $\mathrm{i} = \sqrt{-1}$. Write z in the form $z = a + b\mathrm{i}$, where a and b are real numbers.

15. [Maximum mark: 6] ***[without GDC]***

The two complex numbers $z_1 = \frac{a}{1+i}$ and $z_2 = \frac{b}{1-2i}$ where $a, b \in R$, are such that $z_1 + z_2 = 3$. Calculate the value of a and of b.

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16. [Maximum mark: 6] ***[without GDC]***

Solve the following equation for z, where z is a complex number.

$$\frac{z}{3+4\mathrm{i}} + \frac{z-1}{5\mathrm{i}} = \frac{5}{3-4\mathrm{i}}$$

Give your answer in the form $a + b\mathrm{i}$, where $a, b \in \mathbb{Z}$

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17. [Maximum mark: 10] ***[without GDC]***

Solve the equations

(a) $(2+5\mathrm{i})z+9=3z+19\mathrm{i}$ [5]

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18. [Maximum mark: 6] ***[without GDC]***

Given that $|z| = 2\sqrt{5}$, find the complex number z that satisfies the equation

$$\frac{25}{z} - \frac{15}{z^*} = 1 - 8\text{i}.$$

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19. [Maximum mark: 6] ***[with / without GDC]***

Let z_1 and z_2 be complex numbers. Solve the simultaneous equations

$$2z_1 + 3z_2 = 7$$
$$z_1 + \text{i}z_2 = 4 + 4\text{i}$$

Give your answers in the form $z = a + b\text{i}$, where $a, b \in \mathbb{Z}$.

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20*. [Maximum mark: 7] ***[without GDC]***

Consider $w = \frac{z}{z^2+1}$ where $z = x + y\mathrm{i}$, $y \neq 0$ and $z^2 + 1 \neq 0$.

Given that $\operatorname{Im} w = 0$, show that $|z| = 1$.

21*. [Maximum mark: 6] ***[without GDC]***

If $z = x + y\mathrm{i}$ is a complex number and $|z+16| = 4|z+1|$, find the value of $|z|$.

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POLYNOMIALS

O. Practice questions

22. [Maximum mark: 7] ***[without GDC]***

Consider the polynomial $f(z)=z^3-3z^2+7z-5$

(a) Confirm that 1 is a root of $f(z)$ [1]

It is given that $f(1+2\mathrm{i})=0$

(b) Write down the two complex roots of $f(z)$. [1]

(c) Write down the three linear factors of $f(z)$ (with complex coefficients). [3]

(d) Express $f(z)$ in the form $(z-a)(z^2+bz+c)$ where $a,b,c\in Z$. [2]

23. [Maximum mark: 8] ***[without GDC]***

Consider the polynomial

$$f(z) = 2(z-1+2\mathrm{i})(z-1-2\mathrm{i})(z-1)(z-2)$$

(a) Write down the four roots of the polynomial. [2]

(b) Express $(z-1+2\mathrm{i})(z-1-2\mathrm{i})$ in the form z^2+Bz+C [2]

(c) Express $f(z)$ in the form $az^4+bz^3+cz^2+dz+e$. [2]

(d) Confirm that the sum and the product of roots are given by $S=-\frac{b}{a}$ and $P=\frac{e}{a}$. [2]

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24. [Maximum mark: 8] ***[with / without GDC]***

Consider the polynomial

$$f(z) = 2z^4 + az^3 + 26z^2 + bz + 20$$

Given that 1 and 2 are roots

(a) find the values of a and b. [4]

(b) find the other two roots of $f(z)$. [4]

25. [Maximum mark: 6] ***[without GDC]***

Consider the polynomial

$$f(z) = 2z^4 - 10z^3 + 26z^2 - 38z + 20$$

Given that $z = 1 - 2\text{i}$ is a root find the other 3 roots of $f(z)$.

26. [Maximum mark: 6] ***[without GDC]***

Find the polynomial $f(z)$ of degree 4, given that

- 3 of the roots are $1-2\mathrm{i}$, 1, 2
- the remainder when $f(z)$ is divided by $z+1$ is 96

A. Exam style questions (SHORT)

27. [Maximum mark: 5] ***[without GDC]***

$(z+2\mathrm{i})$ is a factor of $2z^3-3z^2+8z-12$. Find the other two factors.

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28. [Maximum mark: 6] ***[without GDC]***

Let $P(z)=z^3+az^2+bz+c$, where a,b and $c\in\mathbb{R}$. Two of the roots of $P(z)=0$ are -2 and $(-3+2\mathrm{i})$. Find the value of a, of b and of c.

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29. [Maximum mark: 6] ***[without GDC]***

The polynomial $P(z)=z^3+mz^2+nz-8$ is divisible by $(z+1+\mathrm{i})$, where $z\in C$ and $m,n\in R$. Find the value of m and of n.

METHOD A: Substitute the solution $z=-1-\mathrm{i}$.
[not the ideal way but good for practice!]

METHOD B: Find first the three roots of $P(z)$, hence the factorization and expand [Much quicker!]

B. Exam style questions (LONG)

30. [Maximum mark: 10] ***[without GDC]***

(a) Evaluate $(1+\mathrm{i})^2$, where i = $\sqrt{-1}$. [2]

(b) Prove, by mathematical induction, that $(1+\mathrm{i})^{4n} = (-4)^n$, where $n \in \mathbb{N}^*$. [6]

(c) Hence or otherwise, find $(1+\mathrm{i})^{32}$. [2]

31. [Maximum mark: 13] ***[without GDC]***

Let $z = a + b\mathrm{i}$ and $w = c + d\mathrm{i}$,

(a) Express zw in the form $x + y\mathrm{i}$ [2]

(b) Show that $|zw|^2 = (ac)^2 + (bd)^2 + (ac)^2 + (bd)^2$ [2]

(c) Show that $\overline{z + w} = \overline{z} + \overline{w}$ [2]

(d) Show that $\overline{zw} = \overline{z}\,\overline{w}$ [3]

(e) Show that $|zw| = |z||w|$ [4]

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32. [Maximum mark: 12] ***[without GDC]***

It is given that $\overline{zw} = \overline{z}\,\overline{w}$ and $|zw| = |z||w|$ for any complex numbers z and w.

Show, by using mathematical induction, that for any $n \geq 2$ it holds

(a) $\overline{z^n} = \overline{z}^{\,n}$ [6]

(b) $|z^n| = |z|^n$ [6]

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