Lesson 14: Solving more systems

Goals
- Calculate values that are a solution for a system of equations, and explain (orally) the solution method.
- Generalise (orally) a process for solving systems of equations using substitution.
- Justify (orally and in writing) that a particular system of equations has no solutions using the structure of the equations.

Learning Targets
- I can use the structure of equations to help me figure out how many solutions a system of equations has.

Lesson Narrative

In previous lessons, students have worked with contexts where two quantities are changing at different (or possibly the same) rate, and they must find when they are equal. Such systems are represented by equations of the form $y = mx + c$ and are solved by setting the two expressions for y equal to each other.

In this lesson, students progress to other types of systems with different structures. They learn that examining structure is a good first step since it is sometimes possible to recognise an efficient method for solving the system through observation. They see that if at least one of the equations has a single variable isolated, then that expression can be substituted into the other equation in place of y or x to get a single equation in one variable that can be solved. Finally, students use the structure of a system of equations to reason about its lack of solutions.

Building On
- Understand solving an equation or inequality as a process of answering a question: which values from a specified set, if any, make the equation or inequality true? Use substitution to determine whether a given number in a specified set makes an equation or inequality true.

Addressing
- Analyse and solve pairs of simultaneous linear equations.

Instructional Routines
- Algebra Talk
- Stronger and Clearer Each Time
- Collect and Display
- Discussion Supports

Student Learning Goals

Let's solve systems of equations.

14.1 Algebra Talk: Solving Systems Mentally

Warm Up: 5 minutes

The purpose of this warm-up is to encourage students to use substitution to solve equations mentally and see that sometimes they cannot use this method.

Instructional Routines

- Algebra Talk
- Discussion Supports

Launch

Display each problem one at a time. Give students 30 seconds of quiet think time followed by a whole-class discussion. Leave each problem displayed throughout the talk.

Representation: Internalise Comprehension. To support working memory, provide students with sticky notes or mini whiteboards.
Supports accessibility for: Memory; Organisation

Student Task Statement

Solve these without writing anything down:

$$\begin{cases} x = 5 \\ y = x - 7 \end{cases}$$

$$\begin{cases} y = 4 \\ y = x + 3 \end{cases}$$

$$\begin{cases} x = 8 \\ y = \text{-}11 \end{cases}$$

Student Response

- $x = 5$ and $y = \text{-}2$
- $x = 1$ and $y = 4$
- $x = 8$ and $y = \text{-}11$

Activity Synthesis

After each problem, ask students to share their solutions. Record and display their responses for all to see. As students share their strategies, highlight the term *substitution* as a strategy to solve an equation. For the third question, ask students why they could not use the same strategy as they did in the earlier questions.

Speaking: Discussion Supports.: Display sentence frames to support students when they explain their strategy. For example, "First, I _____ because . . ." or "I noticed _____ so I . . . ." Some students may benefit from the opportunity to rehearse what they will say with a partner before they share with the whole class.
Design Principle(s): Optimise output (for explanation)

14.2 Challenge Yourself

15 minutes

In this activity, students solve systems of linear equations that lend themselves to substitution. There are 4 kinds of systems presented: one kind has both equations given with the y value isolated on one side of the equation, another kind has one of the variables given as a constant, a third kind has one variable given as a multiple of the other, and the last kind has one equation given as a linear combination. This progression of systems nudges students towards the idea of substituting an expression in place of the variable it is equal to.

Notice which kinds of systems students think are least and most difficult to solve.

In future years, students will manipulate equations to isolate one of the variables in a linear system of equations. For now, students do not need to solve a system like $x + 2y = 7$ and $2x - 2y = 2$ using this substitution method.

Instructional Routines

- Collect and Display

Launch

Arrange students in groups of 2. Give students 10 minutes of quiet work time. Encourage students to check in with their partner between questions. Tell students that if there is disagreement, they should work to reach an agreement. Follow with a whole-class discussion.

Anticipated Misconceptions

Some students may have trouble transitioning from systems where both equations are given with one variable isolated to other kinds of systems. Ask these students to look at a system where one of the variables is given as a constant. For example, ask them to look at equation B:

$$\begin{cases} y = 7 \\ x = 3y - 4 \end{cases}$$

Ask, “If y is equal to 7, then what is $3y$ equal to?” If a student continues to struggle, refer them back to this example and then ask, “In this new problem, do we know what expression y (or x) is equal to? Then whenever we see y (or x), what can we replace it with instead?”

Student Task Statement

Here are a lot of systems of equations:

A $\begin{cases} y = 4 \\ x = -5y + 6 \end{cases}$

B $\begin{cases} y = 7 \\ x = 3y - 4 \end{cases}$

C $\begin{cases} y = \frac{3}{2}x + 7 \\ x = -4 \end{cases}$

D $\begin{cases} y = -3x + 10 \\ y = -2x + 6 \end{cases}$

E $\begin{cases} y = -3x - 5 \\ y = 4x + 30 \end{cases}$

F $\begin{cases} y = 3x - 2 \\ y = -2x + 8 \end{cases}$

G $\begin{cases} y = 3x \\ x = -2y + 56 \end{cases}$

H $\begin{cases} x = 2y - 15 \\ y = -2x \end{cases}$

I $\begin{cases} 3x + 4y = 10 \\ x = 2y \end{cases}$

J $\begin{cases} y = 3x + 2 \\ 2x + y = 47 \end{cases}$

K $\begin{cases} y = -2x + 5 \\ 2x + 3y = 31 \end{cases}$

L $\begin{cases} x + y = 10 \\ x = 2y + 1 \end{cases}$

1. Without solving, identify 3 systems that you think would be the least difficult to solve and 3 systems that you think would be the most difficult to solve. Be prepared to explain your reasoning.

2. Choose 4 systems to solve. At least one should be from your "least difficult" list and one should be from your "most difficult" list.

Student Response

1. Answers vary. Sample response: A, B, and C seem easy since one of the variable solutions is already given. J, K, and L seem the most difficult since there are multiple terms and the variables are on the same side of the equation in some of the equations.

2. Four of these answers:

   a. $x = \text{-}14, y = 4$

   b. $x = 17, y = 7$

   c. $x = \text{-}4, y = 1$

   d. $x = 4, y = \text{-}2$

   e. $x = \text{-}5, y = 10$

   f. $x = 2, y = 4$

   g. $x = 8, y = 24$

   h. $x = \text{-}3, y = 6$

   i. $x = 2, y = 1$

   j. $x = 9, y = 29$

   k. $x = \text{-}4, y = 13$

   l. $x = 7, y = 3$

Activity Synthesis

This discussion has two main takeaways. The first is to formalise the idea of substitution in a system of equations. Another is to recognise systems where both equations are written with one variable isolated are actually special cases of substitution.

Invite students to share which systems they thought would be easiest to solve and which would be hardest. To involve more students in the conversation, consider asking:

- "Did you change your mind about any of the systems being more or less difficult after you solved them?"
- "What was similar in these problems? What was different?" (The systems vary slightly in how they are presented, but all of the problems can be solved by replacing a variable with an expression it is equal to.)
- "Will your strategy work for the other systems in this list?" (Yes, substitution works in all the given problems.)

Tell students that the key underlying concept for all of these problems is that it is often helpful to replace a variable with the expression it is equal to, and that this "replacing" is called "substitution." Point out that even setting the expressions for y in the first two problems equal to each other is really substituting y in one equation with the expression it is equal to as given by the other equation. It may be helpful for students to hear language like, "Since y is equal to $\text{-}2x$, that means wherever I see y, I can substitute in $\text{-}2x$."

Representation: Internalise Comprehension. Use colour and annotations to illustrate student thinking. As students describe their strategies and the relationships they noticed, use colour and annotations to scribe their thinking on a display of each problem so that it is visible for all students.
Supports accessibility for: Visual-spatial processing; Conceptual processing Representing, Speaking, Listening: Collect and Display. As students discuss which systems they thought would be easiest to solve and which would be hardest, create a table with the headings "least difficult" and "most difficult" in the two columns. Circulate through the groups and record student language in the appropriate column. Look for phrases such as "different variables on the same side," "variables already isolated," and "various terms." Invite students to share strategies they can use to address the features that make these systems of equations more difficult to solve. This will help students begin to generalise and make sense of the structures of equations for substitution.
Design Principle(s): Support sense-making

14.3 Five Does Not Equal Seven

15 minutes

In this activity, students are asked to make sense of Tyler's justification for the number of solutions to the system of equations. This activity continues the thread of reasoning about the structure of an equation and the focus should be on what, specifically, in the equations students think Tyler sees that makes him believe the system has no solutions.

Instructional Routines

- Stronger and Clearer Each Time

Launch

Give students 2–3 minutes of quiet think time to read the problem and decide if they agree or disagree with Tyler. Use the remaining time for a whole-class discussion.

Student Task Statement

Tyler was looking at this system of equations:

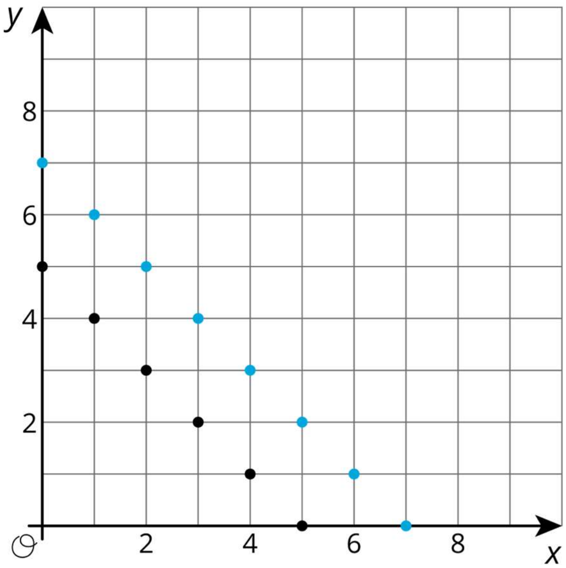
$$\begin{cases} x + y = 5 \\ x + y = 7 \end{cases}$$

He said, "Just looking at the system, I can see it has no solution. If you add two numbers, that sum can't be equal to two different numbers."

Do you agree with Tyler?

Student Response

Yes. Explanations vary. Sample explanation: The sum of two numbers cannot be equal to both 5 and 7. Students may choose to make a graph to show that the lines will never cross.



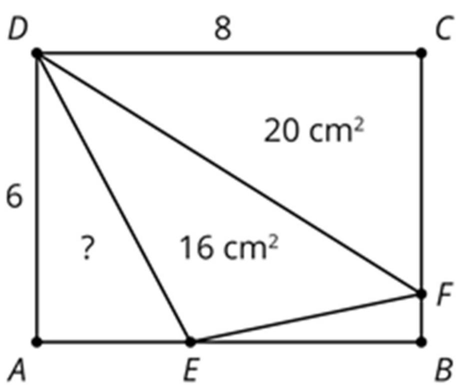
Are You Ready for More?

In rectangle $ABCD$, side AB is 8 centimetres and side BC is 6 centimetres. F is a point on BC and E is a point on AB. The area of triangle DFC is 20 square centimetres, and the area of triangle DEF is 16 square centimetres. What is the area of triangle AED?

Student Response

$\frac{48}{5}$ square centimetres or equivalent. Since the area of triangle DFC is 20 and $DC = 8$, $CF = 5$ and hence $FB = 1$. The area of rectangle $ABCD$ is 48 square centimetres. Summing the areas of the four triangles, we get $48 = 20 + 16 + \frac{1}{2} \times 6 \times AE + \frac{1}{2} \times BE$. We also have $AE + EB = 8$. This is a system of equations where one solution is $AE = \frac{16}{5}$ leading to the area of triangle AED is $\frac{48}{5}$.

It may simplify the work to use a variable to represent the lengths of AE and EB.



Activity Synthesis

The goal of this discussion is to look at one way to reason about the structure of a system of equations in order to determine the solution and then have students make their own reasoning about a different, but similar, system of equations.

Poll the class to see how many students agree with Tyler and how many students disagree with Tyler. If possible, invite students from each side to explain their reasoning. As students explain, it should come out that Tyler is correct and, if no student brings up the idea, make sure to point out that we can also visualise this by graphing the equations in the system and noting that the lines look parallel and will never cross.

In the previous activity, students noticed that if they knew what one variable was equal to, they could substitute that value or expression into another equation in the same problem. Point out that, in this problem, the expression $(x + y)$ is equal to 5 in the first equation. If, in the second equation, we replace $(x + y)$ with 5, the resulting equation is $5 = 7$ which cannot be true regardless of the choice of x and y.

Display the following system and ask students how many solutions they think it has and to give a signal when they think they know:

$$\begin{cases} 4x + 2y = 8 \\ 2x + y = 5 \end{cases}$$

Once the majority of the class signals they have an answer, invite several students to explain their thinking. There are multiple ways students might use to reason about the number of solutions this system has. During the discussion, encourage students to use the terms *coefficient* and *constant term* in their reasoning. Introduce these terms if needed to help students recall their meanings. Bring up these possibilities if no students do so in their explanations:

- “Re-write the second equation to isolate the y variable and substitute the new expression into the first equation in order to find that the system of equations has no solutions.”
- “Notice that both equations are lines with the same gradient but different y-intercepts, which means the system of equations has no solutions.”
- “Notice that $4x + 2y$ is double $2x + y$, but 8 is not 5 doubled, so the system of equations must have no solution.”

Writing: Stronger and Clearer Each Time. If time allows, use this routine to give students an opportunity to summarise the whole-class discussion. Display the prompt, “How can you use the structure of a system of equations to determine when there are no solutions?” Give students 3–4 minutes to write a response, then invite students to meet with 2–3 partners, to share and get feedback on their writing. Encourage listeners to ask their partner clarifying questions such as, “What do you notice about the coefficients and constant terms in the equations?”; “What do the equations tell us about the rates of change and initial values?; “Can you use the example of Tyler’s problem to explain that more?” Students can

borrow ideas and language from each partner to strengthen their final product. This will help students solidify their understanding of how to use the structure of the system of equations to determine the number of solutions.
Design Principle(s): Optimise output; Cultivate conversation

Lesson Synthesis

To emphasise the concepts from this lesson, consider displaying the three systems and asking these discussion questions:

$$\begin{cases} x = 2 \\ y = 3x - 1 \end{cases}$$

$$\begin{cases} x = 2y + 4 \\ x = 9 - 3y \end{cases}$$

$$\begin{cases} x = 2y + 3 \\ y = 2x - 9 \end{cases}$$

- "What is the first step you would take to solve the first system?" (Since we already know the x value of the solution, we only need to find the y value. Substituting 2 in for x in the other equation should help us solve for the y value that makes both equations true when x is 2.)
- "What steps would you take to solve the second system?" (Since we know two expressions that are equal to x, we can set those expressions equal to one another. Therefore, we know that $2y + 4 = 9 - 3y$ which can be solved using the techniques to solve equations with a single variable. Once we know the value for y, we can find the value for x from either of the original equations from the system.)
- "For the third system, a student begins the substitution method by writing $y = 2 \times 2y + 3 - 9$ then $y = 4y - 6$. What has this student done wrong?" (When substituting for x, the student did not multiply the *entire* expression by 2.)

14.4 Solve It

Cool Down: 5 minutes

This cool-down asks students to solve a system of equations presented in an algebraic form. Although no method is specified, the main ideas from this lesson as well as a lack of a coordinate plane may lead students to use a substitution method which is both efficient and effective on this system.

Student Task Statement

Solve this system of equations: $\begin{cases} y = 2x \\ x = -y + 6 \end{cases}$

Student Response

(2,4). Sample response: Students may use the substitution method to rewrite the system as the one variable equation $x = \text{-}(2x) + 6$, then solve.

Student Lesson Summary

When we have a system of linear equations where one of the equations is of the form $y =$ [stuff] or $x =$ [stuff], we can solve it algebraically by using a technique called *substitution*. The basic idea is to replace a variable with an expression it is equal to (so the expression is like a substitute for the variable). For example, let's start with the system:

$$\begin{cases} y = 5x \\ 2x - y = 9 \end{cases}$$

Since we know that $y = 5x$, we can substitute $5x$ for y in the equation $2x - y = 9$,

$2x - (5x) = 9$,

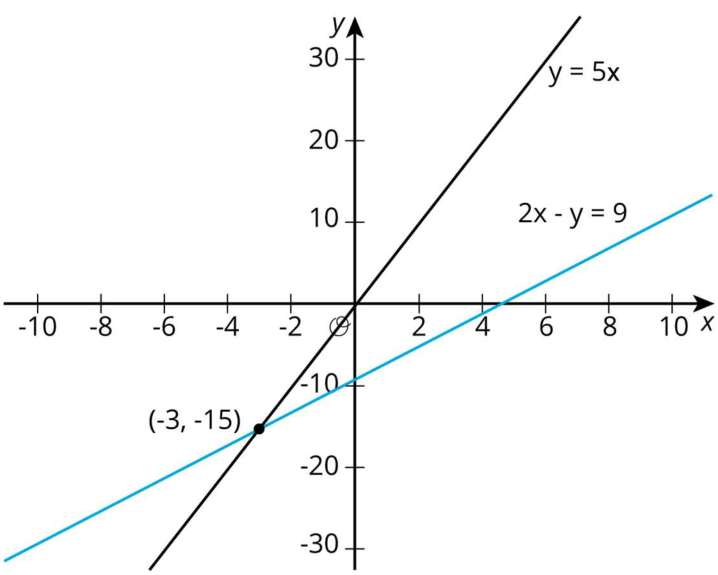
and then solve the equation for x,

$x = \text{-}3$.

We can find y using either equation. Using the first one: $y = 5 \times \text{-}3$. So

(-3,-15)

is the solution to this system. We can verify this by looking at the graphs of the equations in the system:



Sure enough! They intersect at (-3,-15).

We didn't know it at the time, but we were actually using substitution in the last lesson as well. In that lesson, we looked at the system

$$\begin{cases} y = 2x + 6 \\ y = \text{-}3x - 4 \end{cases}$$

and we substituted $2x + 6$ for y into the second equation to get $2x + 6 = \text{-}3x - 4$. Go back and check for yourself!

Lesson 14 Practice Problems

1. **Problem 1 Statement**

   Solve: $\begin{cases} y = 6x \\ 4x + y = 7 \end{cases}$

   Solution

   $\left(\frac{7}{10}, \frac{21}{5}\right)$

2. **Problem 2 Statement**

   Solve: $\begin{cases} y = 3x \\ x = \text{-}2y + 70 \end{cases}$

   Solution

   (10,30)

3. **Problem 3 Statement**

   Which equation, together with $y = \text{-}1.5x + 3$, makes a system with one solution?

   a. $y = \text{-}1.5x + 6$

   b. $y = \text{-}1.5x$

   c. $2y = \text{-}3x + 6$

   d. $2y + 3x = 6$

   e. $y = \text{-}2x + 3$

   Solution E

4. **Problem 4 Statement**

   The system $x - 6y = 4$, $3x - 18y = 4$ has no solution.

   a. Change one constant or coefficient to make a new system with one solution.

   b. Change one constant or coefficient to make a new system with an infinite number of solutions.

Solution

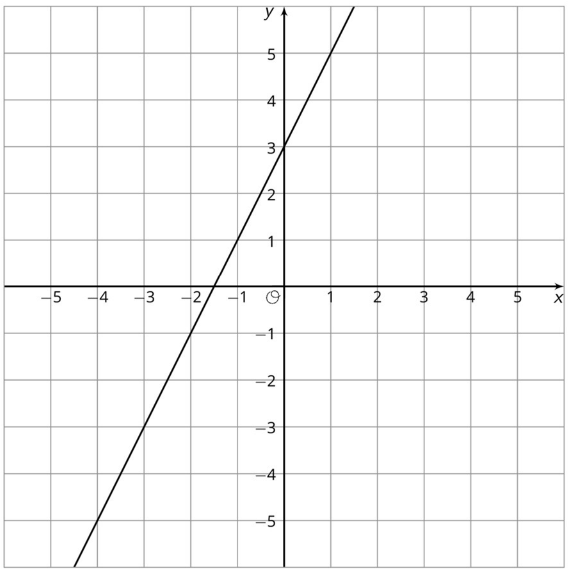
a. Answers vary. Sample response: $2x - 6y = 4$

b. Answers vary. Sample response: $3x - 18y = 12$

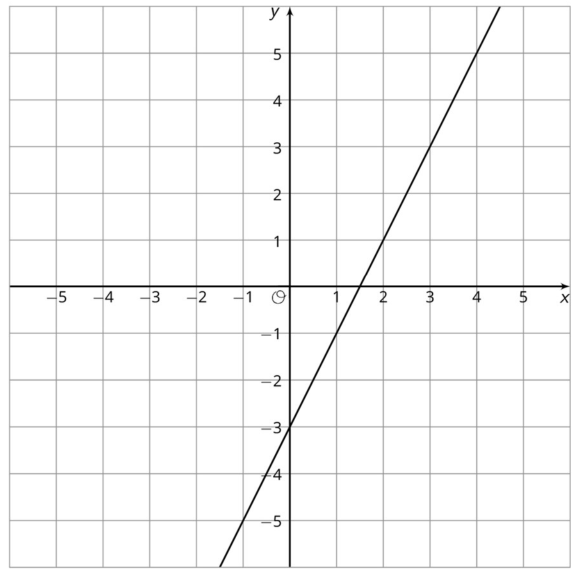
5. **Problem 5 Statement**

Match each graph to its equation.

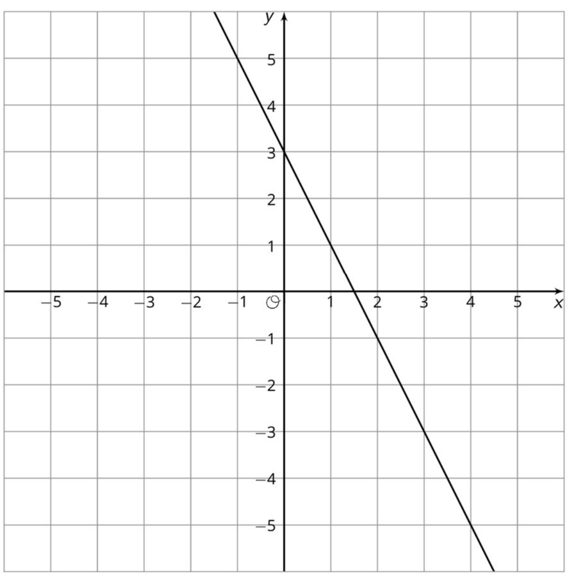
A



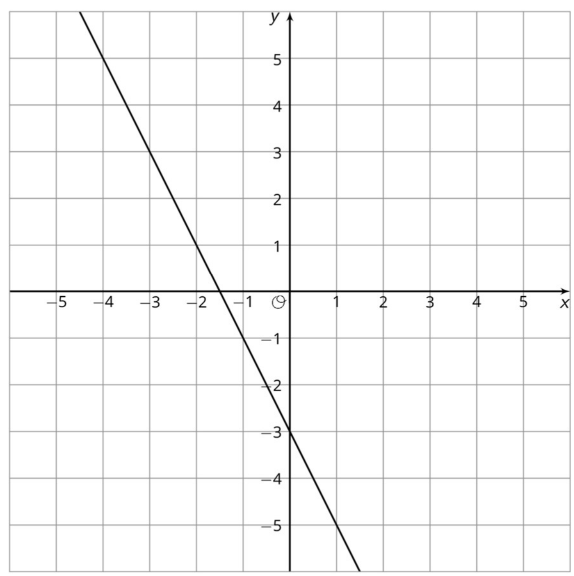
B



C



D



a. $y = 2x + 3$

b. $y = -2x + 3$

c. $y = 2x - 3$

d. $y = -2x - 3$

Solution

a. A

b. C

c. B

d. D

6. **Problem 6 Statement**

Here are two points: (-3,4), (1,7). What is the gradient of the line between them?

a. $\frac{4}{3}$

b. $\frac{3}{4}$

c. $\frac{1}{6}$

d. $\frac{2}{3}$

Solution B