Lesson 5: How many groups? (Part 2)

Goals

- Coordinate multiplication and division equations and pattern block diagrams in which the red trapezium represents one whole.
- Create a diagram to represent and solve a problem asking "How many groups?" in which the divisor is a non-unit fraction, and explain (orally) the solution method.
- Identify or generate a multiplication or division equation that represents a given situation involving a fractional divisor.

Learning Targets

- I can find how many groups there are when the number of groups and the amount in each group are not whole numbers.

Lesson Narrative

In this lesson, students continue to work with division situations involving questions like "how many groups?" or "how many of this in that?" Unlike in the previous lesson, they encounter situations where the quotient is not a whole number, and they must attend to the whole when representing the answer as a fraction. They represent the situations with multiplication equations (e.g., "? groups of $\frac{1}{2}$ make 8" can be expressed as $? \times \frac{1}{2} = 8$) and division equations ($8 \div \frac{1}{2} = ?$).

Building On

- Apply and extend previous understandings of division to divide unit fractions by whole numbers and whole numbers by unit fractions. Students able to multiply fractions in general can develop strategies to divide fractions in general, by reasoning about the relationship between multiplication and division.

Addressing

- Interpret and compute quotients of fractions, and solve word problems involving division of fractions by fractions, e.g., by using visual fraction models and equations to represent the problem. For example, create a story context for $\left(\frac{2}{3}\right) \div \left(\frac{3}{4}\right)$ and use a visual fraction model to show the quotient; use the relationship between multiplication and division to explain that $\left(\frac{2}{3}\right) \div \left(\frac{3}{4}\right) = \frac{8}{9}$ because $\frac{3}{4}$ of $\frac{8}{9}$ is $\frac{2}{3}$. In general, $\left(\frac{a}{b}\right) \div \left(\frac{c}{d}\right) = \frac{ad}{bc}$. How much chocolate will each person get if 3 people share $\frac{1}{2}$ lb of chocolate equally? How many $\frac{3}{4}$ cup servings are in $\frac{2}{3}$ of a cup of yogurt? How wide is a rectangular strip of land with length $\frac{3}{4}$ mi and area $\frac{1}{2}$ square mi?

Instructional Routines

- Group Presentations

- Discussion Supports

Required Materials

Geometry toolkits
tracing paper, graph paper, coloured pencils, scissors, and an index card to use as a straightedge or to mark right angles.

Pattern blocks
Tools for creating a visual display
Any way for students to create work that can be easily displayed to the class. Examples: chart paper and markers, whiteboard space and markers, shared online drawing tool, access to a document camera.

Required Preparation

Prepare enough pattern blocks such that each group of 3–4 students has at least 1 hexagon and 4 of each of the other shapes (triangle, rhombus, and trapezium).

Student Learning Goals

Let's use blocks and diagrams to understand more about division with fractions.

5.1 Reasoning with Fraction Strips

Warm Up: 5 minutes

In this warm-up, students continue to think of division in terms of equal-sized groups, using fraction strips as an additional tool for reasoning.

Notice how students transition from concrete questions (the first three) to symbolic ones (the last three). Framing division expressions as "how many of this fraction in that number?" may not yet be intuitive to students. They will further explore that connection in this lesson. For now, support them using whole-number examples (e.g., ask: "how do you interpret 6 ÷ 2?").

The divisors used here involve both unit fractions and non-unit fractions. The last question shows a fractional divisor that is not on the fraction strips. This encourages students to transfer the reasoning used with fraction strips to a new problem, or to use an additional strategy (e.g., by first writing an equivalent fraction).

As students work, notice those who are able to modify their reasoning effectively, even if the approach may not be efficient (e.g., adding a row of $\frac{1}{10}$s to the fraction strips). Ask them to share later.

Launch

Give students 2–3 minutes of quiet work time.

Anticipated Misconceptions

Since the fraction strips do not show tenths, students might think that it is impossible to answer the last question. Ask them if they can think of another fraction that is equivalent to $\frac{2}{10}$.

Student Task Statement

Write a fraction or whole number as an answer for each question. If you get stuck, use the fraction strips. Be prepared to share your reasoning.

1. How many $\frac{1}{2}$s are in 2?
2. How many $\frac{1}{5}$s are in 3?
3. How many $\frac{1}{8}$s are in $1\frac{1}{4}$?
4. $1 \div \frac{2}{6} = ?$
5. $2 \div \frac{2}{9} = ?$
6. $4 \div \frac{2}{10} = ?$

1	1																
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$														
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Student Response

1. 4
2. 15
3. 10
4. 3
5. 9

6. 20

Activity Synthesis

For each of the first five questions, select a student to share their response and ask the class to indicate whether they agree or disagree.

Focus the discussion on two things: how students interpreted expressions such as $1 \div \frac{2}{6}$, and on how they reasoned about $4 \div \frac{2}{10}$. Select a few students to share their reasoning.

For the last question, highlight strategies that are effective and efficient, such as using a unit fraction that is equivalent to $\frac{2}{10}$, finding out how many groups of $\frac{1}{5}$ are in 1 and then multiplying it by 4, etc.

5.2 More Reasoning with Pattern Blocks

25 minutes (there is a digital version of this activity)

This activity serves two purposes: to explicitly bridge "how many of this in that?" questions and division expressions, and to explore division situations in which the quotients are not whole numbers. (Students explored similar questions previously, but the quotients were whole numbers.)

Once again students move from reasoning concretely and visually to reasoning symbolically. They start by thinking about "how many rhombuses are in a trapezium?" and then express that question as multiplication ($? \times \frac{2}{3} = 1$ or $\frac{2}{3} \times ? = 1$) and division ($1 \div \frac{2}{3}$). Students think about how to deal with a remainder in such problems.

As students discuss in groups, listen for their explanations for the question "How many rhombuses are in a trapezium?" Select a couple of students to share later—one person to elaborate on Diego's argument, and another to support Jada's argument.

Instructional Routines

- Discussion Supports

Launch

Arrange students in groups of 3–4. Provide access to pattern blocks and geometry toolkits. Give students 10 minutes of quiet work time for the first three questions and a few minutes to discuss their responses and collaborate on the last question.

Classrooms with no access to pattern blocks or those using the digital materials can use the provided applet. Physical pattern blocks are still preferred, however.

Representation: Develop Language and Symbols. Display or provide charts with symbols and meanings. Emphasise the difference between this activity where students must find what fraction of a trapezium each of the shapes represents, compared to the hexagon in the

previous lesson. Create a display that includes an image of each shape labelled with the name and the fraction it represents of a trapezium. Keep this display visible as students move on to the next problems.
Supports accessibility for: Conceptual processing; Memory

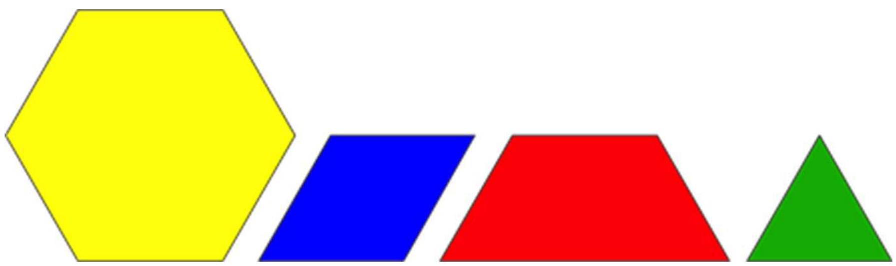
Anticipated Misconceptions

Some students may not notice that in this task, the trapezium—not the hexagon—represents 1 whole. Encourage them to revisit the task statement to check.

Student Task Statement

Your teacher will give you pattern blocks. Use them to answer the questions.

1. If the trapezium represents 1 whole, what do each of the other shapes represent? Be prepared to show or explain your reasoning.



2. Use pattern blocks to represent each multiplication equation. Use the trapezium to represent 1 whole.

   a. $3 \times \frac{1}{3} = 1$

   b. $3 \times \frac{2}{3} = 2$

3. Diego and Jada were asked "How many rhombuses are in a trapezium?"

   – Diego says, "$1\frac{1}{3}$. If I put 1 rhombus on a trapezium, the leftover shape is a triangle, which is $\frac{1}{3}$ of the trapezium."

   – Jada says, "I think it's $1\frac{1}{2}$. Since we want to find out 'how many rhombuses,' we should compare the leftover triangle to a rhombus. A triangle is $\frac{1}{2}$ of a rhombus."

   Do you agree with either of them? Explain or show your reasoning.

4. Select **all** the equations that can be used to answer the question: "How many rhombuses are in a trapezium?"

   – $\frac{2}{3} \div ? = 1$

- $? \times \frac{2}{3} = 1$
- $1 \div \frac{2}{3} = ?$
- $1 \times \frac{2}{3} = ?$
- $? \div \frac{2}{3} = 1$

Student Response

1.

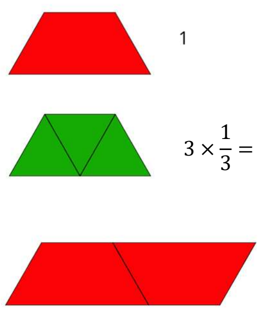
a. $\frac{1}{3}$ because three triangles make a trapezium.

b. $\frac{2}{3}$ because two triangles make a rhombus and each triangle represents $\frac{1}{3}$.

c. 2 because two trapeziums make a hexagon.

2.

a.

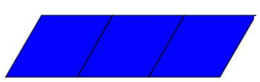


1

$3 \times \frac{1}{3} = 1$

$2 \times 1 = 2$

b.



$3 \times \frac{2}{3} = 2$

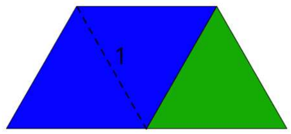
3. Jada's $1\frac{1}{2}$ is the right answer. Sample reasoning: Since the question is “how many rhombuses,” the leftover space should be compared to a rhombus. A triangle is half of a rhombus, so we can fit $1\frac{1}{2}$ rhombuses in a trapezium.

4. $? \times \frac{2}{3} = 1$ and $1 \div \frac{2}{3} = ?$

Activity Synthesis

Focus the whole-class discussion on the last two questions, especially on how the visual representation helps us reason about Jada and Diego's points of view, and on the connections between the verbal and numerical representations of the situation.

Select two previously identified students to explain why Diego or Jada are correct. Display a visual representation of "how many rhombus are in a trapezium?" for all to see (as shown here), or use the applet at https://ggbm.at/VmEqZvke for illustration.



To highlight number of groups and size of one group in the problem, discuss questions such as:

- "This is a 'how many groups of this in that?' question. What makes 1 group, in this case?" (One rhombus.)
- "How do we know whether to compare the remainder to the rhombus or the trapezium?" (Since a rhombus makes 1 group, we need to compare the remainder to the rhombus.)

If students struggle to compare the remainder to a rhombus, ask: "How many triangles are in a trapezium?" and point out that the answer is "3 triangles." Here, the answer to "how many rhombuses are in a trapezium?" would be "(some number of) rhombuses."

The fact that there are two 1 wholes to keep track of may be a source of confusion (the trapezium represents the quantity 1 and the rhombus represents 1 group). Students will have opportunities to make clearer distinctions between the two 1 wholes in upcoming activities.

Speaking, Listening: Discussion Supports. Use this routine to support whole-class discussion. After a student shares their reasoning for whether they agree with Diego or Jada, ask students to restate what they heard using precise mathematical language. Ask the original speaker if their peer was accurately able to restate their thinking. Call students' attention to any words or phrases that helped to clarify the original statement. This provides more students with an opportunity to produce language as they interpret the reasoning of others.
Design Principle(s): Support sense-making

5.3 Drawing Diagrams to Show Equal-sized Groups

Optional: 20 minutes

This activity gives students additional practice in using diagrams and equations to represent division situations involving whole numbers and fractions.

For each problem, many kinds of visual representations are possible, but creating a meaningful representation may be challenging nonetheless. Urge students to use the contexts to generate ideas for useful diagrams, and to start with a draft and modify it as needed. Students may also use the fraction strips in the warm-up as a starting point for drawing diagrams.

As students work and discuss, monitor for effective diagrams or those that can be generalised to different situations (e.g., rectangles, bar models, and number lines). Assign one problem for each group to record on a visual display.

Instructional Routines

- Group Presentations
- Discussion Supports

Launch

Arrange students in group of 3–4. Give students 8–10 minutes of quiet work time and a few minutes to share their responses with their group. Tell each group they will be asked to present their solution to one problem. Provide access to geometry toolkits and tools for creating a visual display.

During group discussion, ask students to exchange feedback on each other's diagrams and to notice any that might be particularly effective, efficient, or easy to understand. Then, they should record the diagram and equations for their assigned problem on a visual display and be prepared to explain them.

Representation: Internalise Comprehension. Demonstrate and encourage students to use colour coding and annotations to illustrate connections between representations. For example, use the same colour to represent the 3 miles in the diagram and the equation, $3 \div \frac{3}{2} = 2$, then label each as "dividend."
Supports accessibility for: Visual-spatial processing

Anticipated Misconceptions

Students may mistake the divisor and the dividend in the problems. Ask students to discuss (in their groups) the number or quantity being divided, and the reasonableness of the different ways of setting up each problem given the context. Representing the situations with objects may also help.

Student Task Statement

For each situation, draw a diagram for the relationship of the quantities to help you answer the question. Then write a multiplication equation or a division equation for the relationship. Be prepared to share your reasoning.

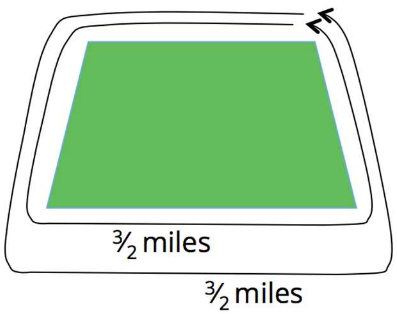
1. The distance around a park is $\frac{3}{2}$ miles. Noah rode his bicycle around the park for a total of 3 miles. How many times around the park did he ride?

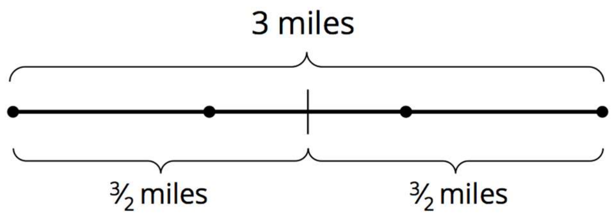
2. You need $\frac{3}{4}$ yard of ribbon for one gift box. You have 3 yards of ribbon. How many gift boxes do you have ribbon for?

3. The water hose fills a bucket at $\frac{1}{3}$ gallon per minute. How many minutes does it take to fill a 2-gallon bucket?

Student Response

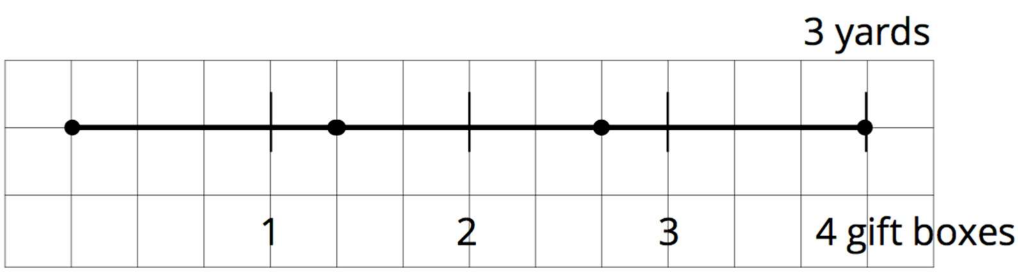
1. 2 (times around the park). Sample diagrams:





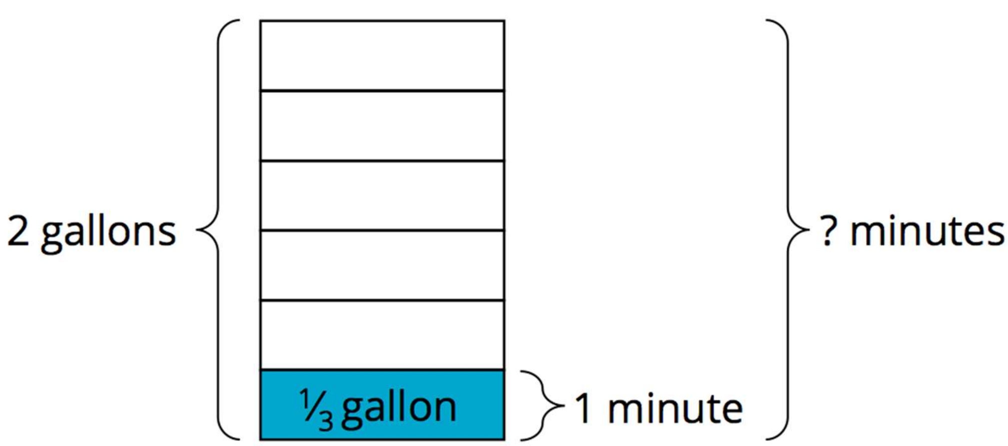
Equation: $2 \times \frac{3}{2} = 3$, or $3 \div \frac{3}{2} = 2$

2. 4 (gift boxes). Sample diagram:



Equation: $4 \times \frac{3}{4} = 3$, or $3 \div \frac{3}{4} = 4$

3. 6 (minutes). Sample diagram:



Equation: $6 \times \frac{1}{3} = 2$, or $2 \div \frac{1}{3} = 6$

Are You Ready for More?

How many heaped teaspoons are in a heaped tablespoon? How would the answer depend on the shape of the spoons?

Student Response

Answers vary. Sample response: There are around 3 heaped teaspoons in a heaped tablespoon, since there are 3 level teaspoons in 1 tablespoon. The size of the heaped spoonfuls depends on the shape of the spoon, since a wide and flat spoon will be able to have more material sit on top of the levelled spoonful.

Activity Synthesis

Invite each group to present the solution on their visual display. Ask the rest of the class to think about two things: whether the equations make sense, and how the presented diagram shows the number of groups, the size of each group, and a total amount. Doing so will help students see the structure of the problems in the equations and diagrams.

Make sure students understand how each situation can be expressed by multiplication and division equations. Students should recognise that a question such as "how many batches are in 4 cups if each batch requires $\frac{2}{3}$ cups?" can be written as both $? \times \frac{2}{3} = 4$ and $4 \div \frac{2}{3} = ?$. With repeated reasoning, they see that a division expression such as $5 \div \frac{3}{8}$ can be interpreted as "how many $\frac{3}{8}$ s are in 5?"

Speaking, Representing: Discussion Supports. Give students additional time to make sure that everyone in their group can explain their visual display and the relationship between the quantities represented. Prompt groups to rehearse what they will say when they share with the whole class. Rehearsing provides students with additional opportunities to speak and clarify their thinking. This will help students improve their explanations of their group's reasoning during the whole-class discussion.
Design Principle(s): Optimise output (for explanation)

Lesson Synthesis

In this lesson, we answered questions such as "how many of this fraction in that number?" or "how many groups of this in that?" We used pattern blocks, fraction strips, diagrams, and equations to help us make sense of those situations. The answers to those questions, we noticed, may not be whole numbers.

- "We can think of the question 'how many $\frac{3}{4}$ are in 2?' in terms of equal-size groups. What do the $\frac{3}{4}$ and 2 represent? What are we looking for?" ($\frac{3}{4}$ is the size of each group. The 2 is the total amount. We are looking for the number of groups.)

- "What multiplication equation can we write for this situation?" ($?\times \frac{3}{4} = 2$)
- "What division equation can we write?" ($2 \div \frac{3}{4} =?$)
- "We can draw a diagram and count how many groups of $\frac{3}{4}$ there are in 2. How many whole groups of $\frac{3}{4}$ are there?" (2 whole groups.)
- "How do we deal with a remainder that is less than one whole group?" (We can compare the size of the remainder with the amount in one group. In $2 \div \frac{3}{4} =?$, each group is $\frac{3}{4}$, and the remainder is $\frac{2}{4}$, which is $\frac{2}{3}$ of one group.)

5.4 Bags of Tangerines

Cool Down: 5 minutes

This cool-down assesses students' ability to represent a division situation with diagrams and equations. The diagrams they create do not need to be of a specific type.

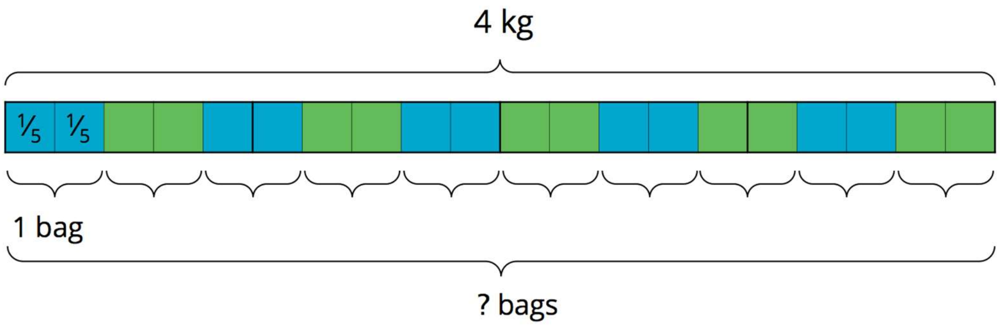
Student Task Statement

A grocery store sells tangerines in $\frac{2}{5}$ kg bags. A customer bought 4 kg of tangerines for a school party. How many bags did he buy?

1. Select **all** equations that represent the situation.
   a. $4 \times \frac{2}{5} =?$
   b. $?\times \frac{2}{5} = 4$
   c. $\frac{2}{5} \div 4 =?$
   d. $4 \div \frac{2}{5} =?$
   e. $?\div \frac{2}{5} = 4$
2. Draw a diagram to represent the situation. Answer the question.

Student Response

1. B ($?\times \frac{2}{5} = 4$) and D ($4 \div \frac{2}{5} =?$)
2. 10 bags. Diagrams vary. Sample diagram:



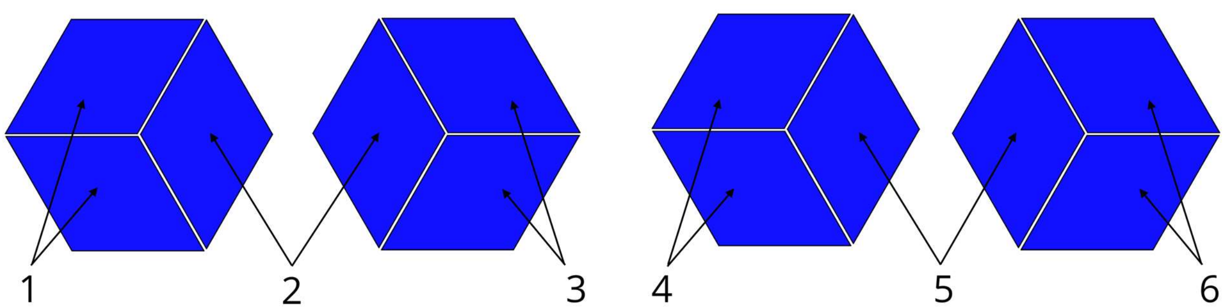
Student Lesson Summary

Suppose one batch of cookies requires $\frac{2}{3}$ cup flour. How many batches can be made with 4 cups of flour?

We can think of the question as being: "How many $\frac{2}{3}$ are in 4?" and represent it using multiplication and division equations.

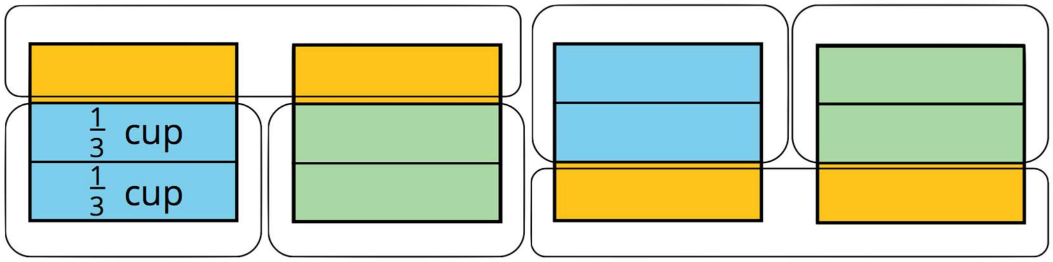
$? \times \frac{2}{3} = 4$ $4 \div \frac{2}{3} = ?$

Let's use pattern blocks to visualise the situation and say that a hexagon is 1 whole.



Since 3 rhombuses make a hexagon, 1 rhombus represents $\frac{1}{3}$ and 2 rhombuses represent $\frac{2}{3}$. We can see that 6 pairs of rhombuses make 4 hexagons, so there are 6 groups of $\frac{2}{3}$ in 4.

Other kinds of diagrams can also help us reason about equal-sized groups involving fractions. This example shows how we might reason about the same question from above: "How many $\frac{2}{3}$-cups are in 4 cups?"



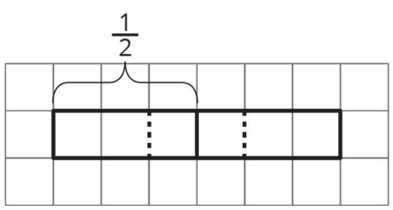
We can see each "cup" partitioned into thirds, and that there are 6 groups of $\frac{2}{3}$ cup in 4 cups. In both diagrams, we see that the unknown value (or the "?" in the equations) is 6. So we can now write:

$6 \times \frac{2}{3} = 4, 4 \div \frac{2}{3} = 6$

Lesson 5 Practice Problems

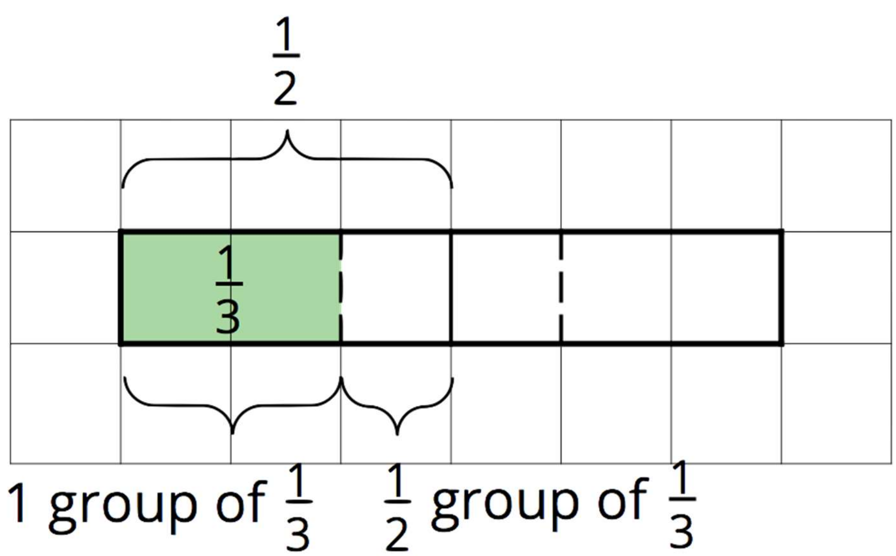
1. **Problem 1 Statement**

   Use the bar model to find the value of $\frac{1}{2} \div \frac{1}{3}$. Show your reasoning.



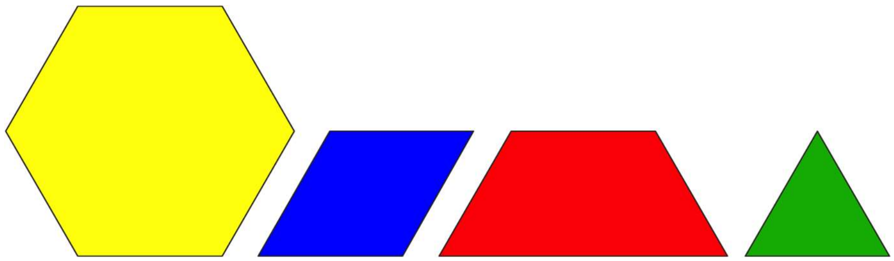
Solution

$1\frac{1}{2}$



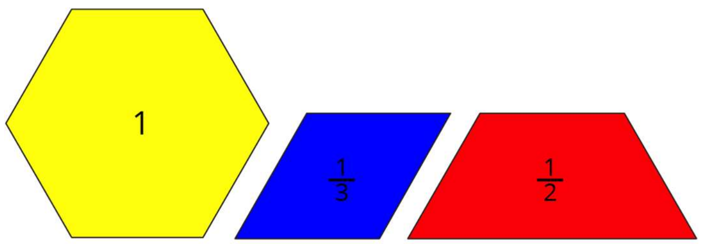
2. **Problem 2 Statement**

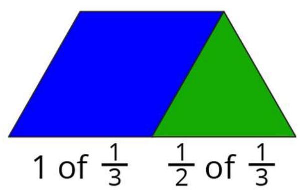
What is the value of $\frac{1}{2} \div \frac{1}{3}$? Use pattern blocks to represent and find this value. The yellow hexagon represents 1 whole. Explain or show your reasoning.



Solution

$1\frac{1}{2}$. Explanations vary. Sample explanations:





One rhombus and $\frac{1}{2}$ of a rhombus compose one trapezium.

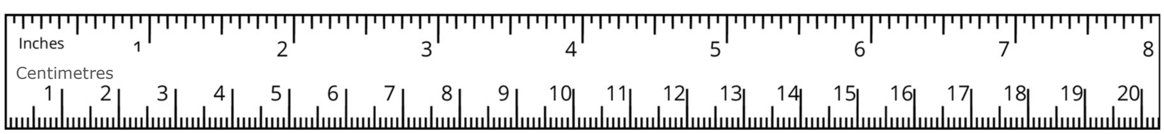
3. **Problem 3 Statement**

Use a standard inch ruler to answer each question. Then, write a multiplication equation and a division equation that answer the question.

a. How many $\frac{1}{2}$ s are in 7?

b. How many $\frac{3}{8}$ s are in 6?

c. How many $\frac{5}{16}$ s are in $1\frac{7}{8}$?



Solution

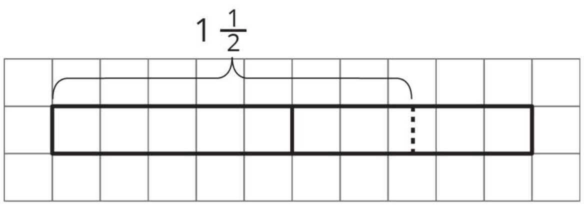
a. Multiplication: $14 \times \frac{1}{2} = 7$ (or equivalent), division: $7 \div \frac{1}{2} = 14$

b. Multiplication: $16 \times \frac{3}{8} = 6$ (or equivalent), division: $6 \div \frac{3}{8} = 16$

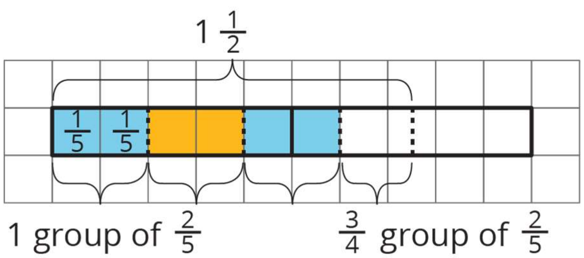
c. Multiplication: $6 \times \frac{5}{16} = 1\frac{7}{8}$ (or equivalent), division: $1\frac{7}{8} \div \frac{5}{16} = 6$

4. **Problem 4 Statement**

Use the bar model to answer the question: How many $\frac{2}{5}$s are in $1\frac{1}{2}$? Show your reasoning.



Solution

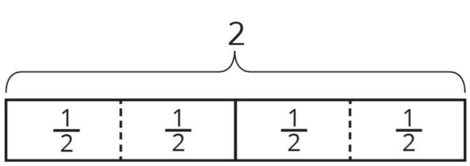


$3\frac{3}{4}$

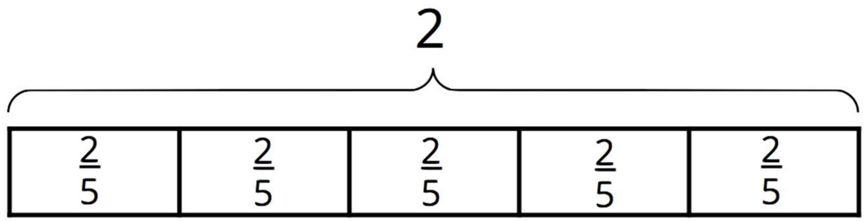
5. **Problem 5 Statement**

Write a multiplication equation and a division equation to represent each sentence or diagram.

a. There are 12 quarters in 3.

b. 

c. How many $\frac{2}{3}$s are in 6?

d. 

Solution

a. $12 \times \frac{1}{4} = 3$ (or equivalent, e.g., $3 \div 12 = \frac{1}{4}$ or $3 \div \frac{1}{4} = 12$)

b. $4 \times \frac{1}{2} = 2$ (or equivalent, e.g., $2 \div 4 = \frac{1}{2}$ or $2 \div \frac{1}{2} = 4$)

c. $? \times \frac{2}{3} = 6$ (or equivalent), $6 \div \frac{2}{3} = ?$ (or equivalent)

d. $5 \times \frac{2}{5} = 2$ (or equivalent), $2 \div 5 = \frac{2}{5}$ (or equivalent)

6. **Problem 6 Statement**

At a farmer's market, two vendors sell fresh milk. One vendor sells 2 litres for £3.80, and another vendor sells 1.5 litres for £2.70. Which is the better deal? Explain your reasoning.

Solution

Answers vary. Sample response:

- 1.5 litres at £2.70 is a better deal. The 1.5-litre-size costs £1.80 per litre since $2.70 \div 1.5 = 1.80$. The 2-litre size costs £1.90 per litre because $3.80 \div 2 = 1.90$. The 1.5-litre bottle is less expensive per litre.

7. **Problem 7 Statement**

A recipe uses 5 cups of flour for every 2 cups of sugar.

a. How much sugar is used for 1 cup of flour?

b. How much flour is used for 1 cup of sugar?

c. How much flour is used with 7 cups of sugar?

d. How much sugar is used with 6 cups of flour?

Solution

a. $\frac{2}{5}$ or 0.4 cups of sugar are used for every cup of flour.

b. $\frac{5}{2}$ or 2.5 cups of flour are used for every cup of sugar.

c. 17.5. $(2.5) \times 7 = 17.5$ so with 7 cups of sugar, there will be 17.5 or $17\frac{1}{2}$ cups of flour.

d. $(0.4) \times 6 = 2.4$ so with 6 cups of flour, there will be 2.4 or $2\frac{2}{5}$ cups of sugar.

flour (cups)	sugar (cups)
5	2
1	$\frac{2}{5}$
$\frac{5}{2}$	1
$\frac{35}{2}$	7
6	$\frac{12}{5}$