Grades 9-12 (AS)

Duration: 30 min

Tools: one Logifaces Set / 2-3 pairs or 4-6 students

Individual / Pair work

Keywords: Cosine, Formula, Proof, Perpendicular projection, Angle of planes

539 - Angle of Planes

MATHS / TRIGONOMETRY

LOGIFACES METHODOLOGY

Erasmus+

TEACHER

Logifaces

2019-1-HU01-KA201-0612722019-1

DESCRIPTION

In the 9 pcs or 16 pcs Set students choose those blocks whose vertical edges have different lengths (these are blocks 123 and 132) and calculate the angle between the planes of the base and top triangles.
Take a triangle ABC on one of the two intersecting planes and let $A_1B_1C_1$ be the perpendicular projection of the triangle ABC to the other plane. If the angle of inclination of the two planes is α, then $cos(\alpha) = \frac{A(A_1B_1C_1)}{A(ABC)}$, where $A(A_1B_1C_1)$ and $A(ABC)$ denote the areas of the triangles.

LEVEL 1 Using the given formula students calculate the angle between the planes of the base and top triangles in block 123 or 132.

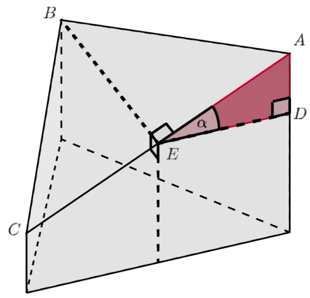
LEVEL 2 Students calculate the angle between the planes of the base and top triangles in block 123 or 132 without knowing the formula, using their knowledge about angles between planes and the symmetry of the block.

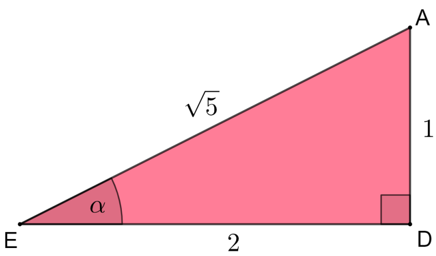
LEVEL 3 Students prove the formula $cos(\alpha) = \frac{A(A_1B_1C_1)}{A(ABC)}$.

SOLUTIONS / EXAMPLES

LEVEL 1 Observe that the perpendicular projection of the top triangle to the plane of the base triangle is the base triangle.
However, the other direction does not hold: the perpendicular projection of the base triangle to the plane of the top triangle is not the top triangle.
The area of the base triangle in standard units is
$4\sqrt{3}$, and the area of the top triangle is $2\sqrt{15}$. Using the formula we get $\alpha \approx 27°$.

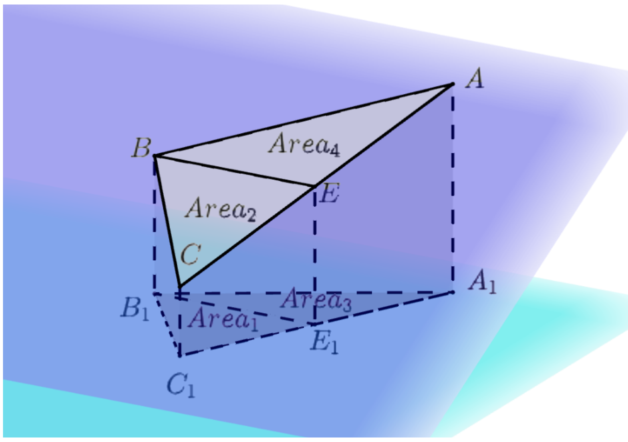
LEVEL 2 In the top triangle of block 123 (or 132) denote by A, B and C the vertices corresponding to the vertical edges of lengths 3, 2 and 1, respectively. Let E be the midpoint of the edge AC. Note that BE is parallel to the plane of the base triangle. Furthermore, the vertical face through the vertices A and C is perpendicular to both the base triangle and the line BE. Therefore the angle between the planes of the triangles is the same as the angle between the non-parallel sides of the trapezium vertical face.
Using this fact, we can calculate the angle as follows. Let D be a point on the vertical edge through A such that AD is perpendicular to DE. Thus $cos(\alpha) = \frac{|DE|}{|AE|} = \frac{2}{\sqrt{5}}$, hence $\alpha \approx 27°$.

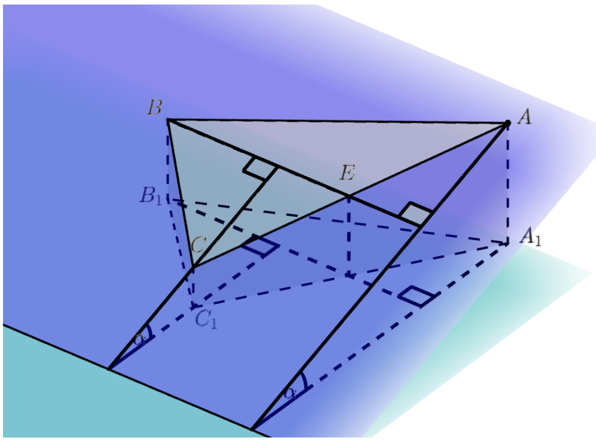




LEVEL 3 If two of the segments AA_1, BB_1 and CC_1 have the same length, then the truncated prism $ABCA_1B_1C_1$ satisfies the conditions of the Level 2 exercise in 538 - Ratio of Areas, hence the proof of the statement can be found in the solution of that exercise.

The other case is when ABC and $A_1B_1C_1$ are two triangles such that their planes intersect, $A_1B_1C_1$ is the perpendicular projection of ABC and $|AA_1| > |BB_1| > |CC_1|$ (this is the case of the blocks 123 and 132).
Let E be a point on the edge AC such that BE is parallel to the plane of the triangle $A_1B_1C_1$ (hence also parallel to the common line of the planes). Such point E exists, because the assumption $|AA_1| > |BB_1| > |CC_1|$ implies that there must be a point E on the segment AC with distance $|BB_1|$ from the plane of the triangle $A_1B_1C_1$.
Cut the truncated prism $ABCA_1B_1C_1$ into two parts with a plane through BE which is perpendicular to the plane of triangle $A_1B_1C_1$. Then we get two truncated prisms with two edges (BE and B_1E_1) parallel to the common line of the planes, thus the statement holds for both of them (see the proof of Level 2 in 538 - Ratio of Areas): $cos(\alpha) = \frac{T_1}{T_2}$ and $cos(\alpha) = \frac{T_3}{T_4}$. It follows that $T_1 = T_2 cos(\alpha)$ and $T_3 = T_4 cos(\alpha)$, thus $\frac{A_b}{A_t} = \frac{T_1+T_3}{T_2+T_4} = \frac{T_2 cos(\alpha)+T_4 cos(\alpha)}{T_2+T_4} = \frac{(T_2+T_4)cos(\alpha)}{T_2+T_4} = cos(\alpha)$.





PRIOR KNOWLEDGE

Angle between two planes, Area of triangle, Trigonometric ratios

RECOMMENDATIONS / COMMENTS

The exercise is suitable for differentiation, as proving the formula is more difficult than applying it.
Exercise 538 - Ratio of Areas is recommended before this exercise. In particular, Level 2 of that exercise is necessary before Level 3 of this exercise.

The calculations can be verified using GeoGebra, see exercise 528 - Read the Results in GeoGebra.