Lesson 7: Exploring the area of a circle

Goals

- Create a table and a graph that represent the relationship between the diameter and area of circles of various sizes, and justify (using words and other representations) that this relationship is not proportional.

- Estimate the area of a circle on a grid by decomposing and approximating it with polygons.

Learning Targets

- If I know a circle's radius or diameter, I can find an approximation for its area.

- I know whether or not the relationship between the diameter and area of a circle is proportional and can explain how I know.

Lesson Narrative

This lesson is the first of two lessons that develop the formula for the **area of a circle**. Students start by estimating the area inside different circles, deepening their understanding of the concept of area as the number of unit squares that cover a region, and discovering that area (unlike circumference) is not proportional to diameter.

Next, they investigate how the area of a circle compares to the area of a square that has side lengths equal to the circle's radius. Students may choose tools strategically from their geometry toolkits to help them make these comparisons. Students find an approximate formula: the area of a circle is a little bigger than $3r^2$, and they check their earlier estimates with this formula. At this point, it is a reasonable guess that the exact formula is $A = \pi r^2$, but the next lesson will focus on using informal dissection arguments to establish this formula.

When we say "area of a circle" we technically mean "area of the region enclosed by a circle." However, "area of a circle" is the phrase most commonly used.

Addressing

- Draw, construct, and describe geometrical shapes and describe the relationships between them.

- Solve real-life and mathematical problems involving angle measure, area, surface area, and volume.

- Know the formulae for the area and circumference of a circle and use them to solve problems; give an informal derivation of the relationship between the circumference and area of a circle.

- Decide whether two quantities are in a proportional relationship, e.g., by testing for equivalent ratios in a table or graphing on a coordinate grid and observing whether the graph is a straight line through the origin.

Building Towards

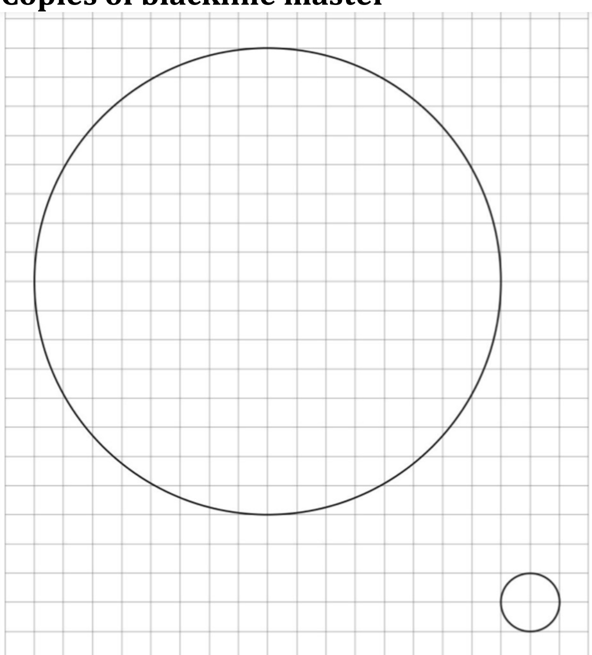
- Know the formulas for the area and circumference of a circle and use them to solve problems; give an informal derivation of the relationship between the circumference and area of a circle.

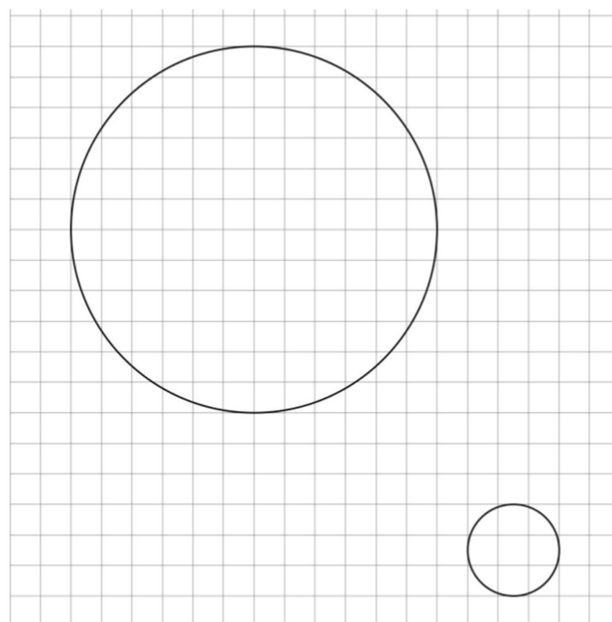
Instructional Routines

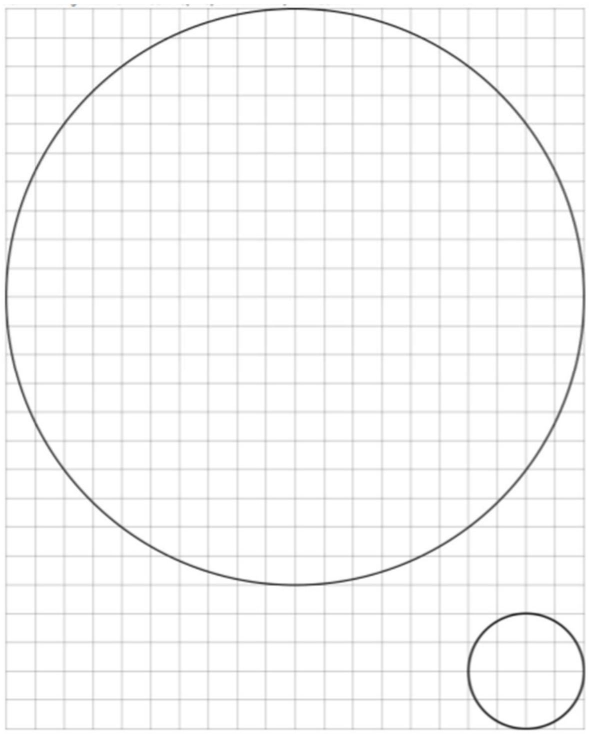
- Group Presentations
- Collect and Display
- Compare and Connect

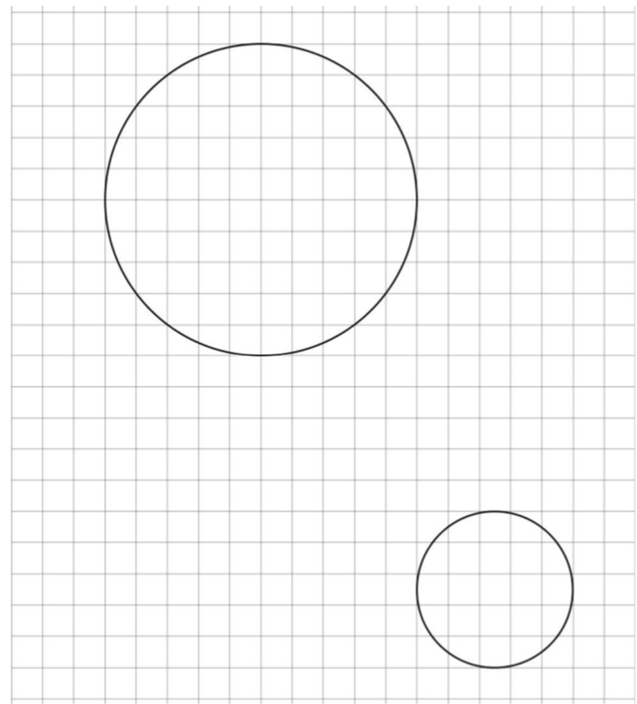
Required Materials

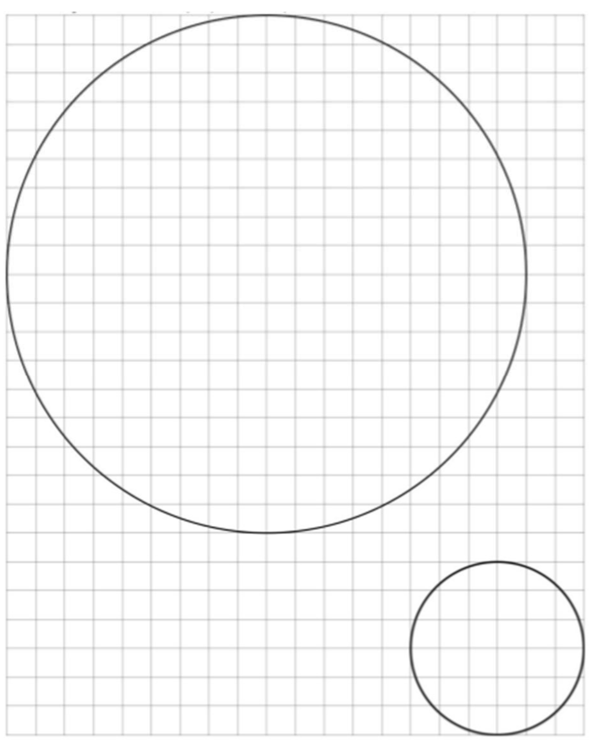
Copies of blackline master

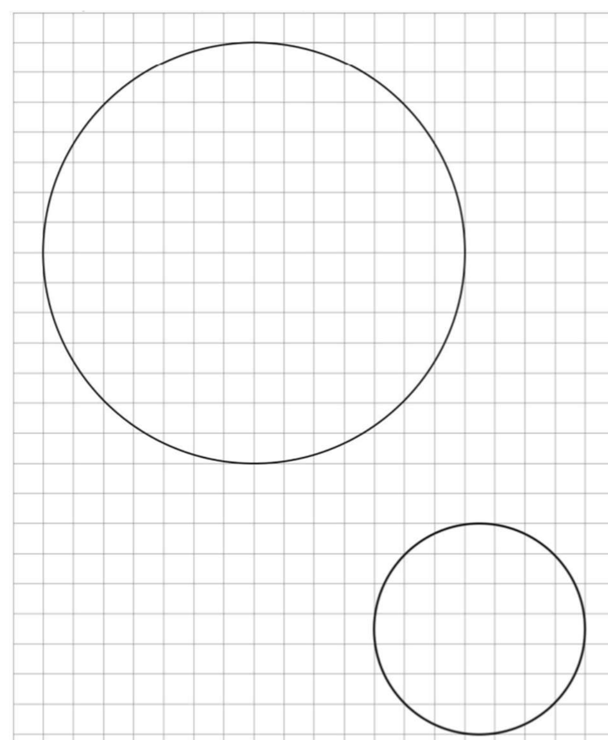












Geometry toolkits

tracing paper, graph paper, coloured pencils, scissors, and an index card to use as a straightedge or to mark right angles, plus a ruler and protractor. Clear protractors with no holes and with radial lines printed on them are recommended.

Required Preparation

For the first activity, you will need the Estimating Areas of Circles blackline master. Prepare 1 copy for every 12 students. (Each group of 2 students gets one of the pages.)

For the second activity, make sure students have access to their geometry toolkits, especially tracing paper and scissors, if they so choose (but try not to influence students' choices about what tools to use).

Student Learning Goals

Let's investigate the areas of circles.

7.1 Estimating Areas

Warm Up: 5 minutes

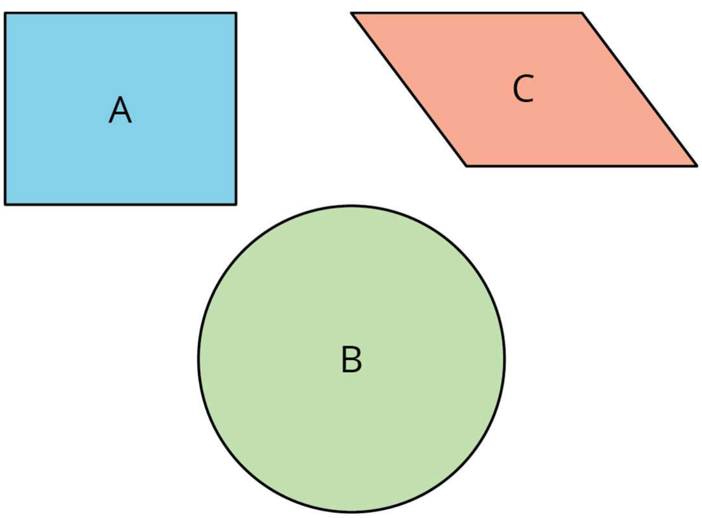
The purpose of this warm-up is for students to estimate the area of a circle using what they know about the area of polygons. The first picture with no grid prompts students to visualise decomposing and rearranging pieces of the shapes in order to compare their areas. Using the grid, students are able to estimate the areas and discuss their strategies.

Instructional Routines

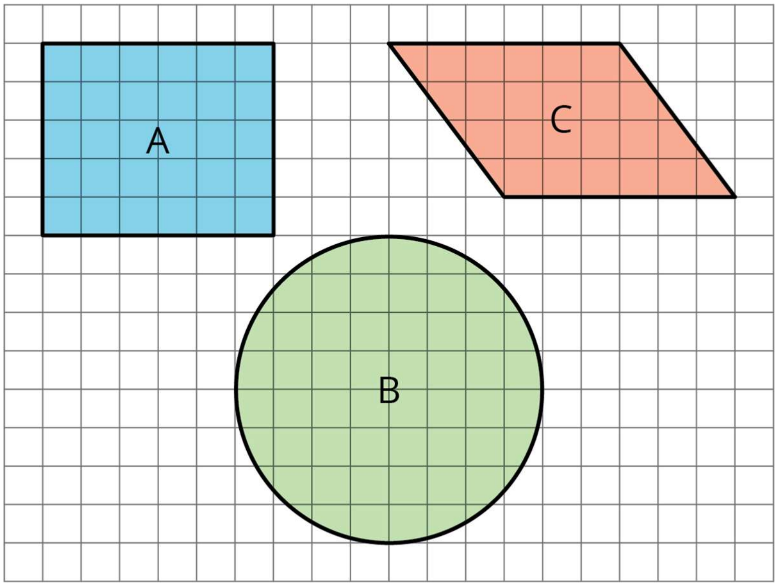
- Compare and Connect

Launch

Arrange students in groups of 3. Display the first image with no grid for all to see.



Ask students to give a signal when they have an idea which shape has the largest area. Give students 30 seconds of quiet think time followed by 1 minute to discuss their reasoning with a partner. Next display the image on a grid.



Ask students to discuss with their group how they would find or estimate the area of each of the shapes. Tell them to share their ideas with their group.

Student Task Statement

Your teacher will show you some shapes. Decide which shape has the largest area. Be prepared to explain your reasoning.

Student Response

Shape B appears to be the largest.

To find the area of shape A, multiply the length and width. To find the area of shape C, multiply the length of the base and its corresponding height. To estimate the area of shape B, find the area of the rectangle in the middle, and then count partial squares. Or, find the enclosing rectangle and subtract the empty partial squares.

Activity Synthesis

Invite selected students to share their strategies and any information in the image that would inform their responses. After each explanation, solicit questions from the class that could help the student clarify his or her reasoning. Ask the whole class to discuss their strategies for finding or estimating the area. Ask them if they think it is possible to calculate the area of the circle *exactly*. Tell them that trying to find the area of a circle will be the main topic for this lesson.

Refer to *Compare and Connect*. Prompt students with questions like: What information was useful for solving the problem? What formulas or prior knowledge did you use to approach the problem? What did you do that was similar to another student? How did you estimate when there was not a complete grid block?

7.2 Estimating Areas of Circles

20 minutes (there is a digital version of this activity)

In a previous lesson, students measured various circular objects and graphed the measurements to see that there appears to be a proportional relationship between the diameter and circumference of a circle. In this activity, students use a similar process to see that the relationship between the diameter and area of a circle is *not* proportional. This echoes the earlier exploration comparing the length of a diagonal of a square to the area of the square, which was also not proportional.

Each group estimates the area of one smaller circle and one larger circle. After estimating the area of their circles, students graph the class's data on a coordinate grid to notice that the data points curve upward instead of making a straight line through the origin. Watch for students who use different methods for estimating the area of the circles (counting full and partial grid squares inside the circle, surrounding the circle with a square and removing full and partial grid squares) and invite them to share in the discussion.

For classes using the digital version, students can record the class data in the spreadsheet and graph points directly on the grid using the Point tool. Note: you have to click on the graph side of the applet for the point tool to appear.

Instructional Routines

- Group Presentations
- Compare and Connect

Launch

Arrange students in groups of 2.

For classes using the print version, distribute the grids with the circles already drawn—one set of circles to each group of students from the Estimating Areas of Circles blackline master.

For classes using the digital version of the activity, assign each group of students a pair of diameters from this set:

1. 2 cm and 16 cm
2. 5 cm and 10 cm
3. 3 cm and 12 cm
4. 6 cm and 18 cm
5. 4 cm and 20 cm
6. 7 cm and 14 cm

Encourage students to look for strategies that will help them efficiently count the area of their assigned circles. Give students 4–5 minutes of group work time.

After students have finished estimating the areas of their circles, display the blank coordinate grid from the activity statement and have students plots points for their measurements. Give students 3–4 minutes of quiet work time, followed by whole-class discussion.

Anticipated Misconceptions

Some students might be unsure on how to count the squares around the border of the circle that are only partially included. Let them come up with their own idea, but if they need additional support, suggest that they round up to a whole square when it looks like half or more of the square is within the circle and round down to no square when it looks like less than half the square is within the circle.

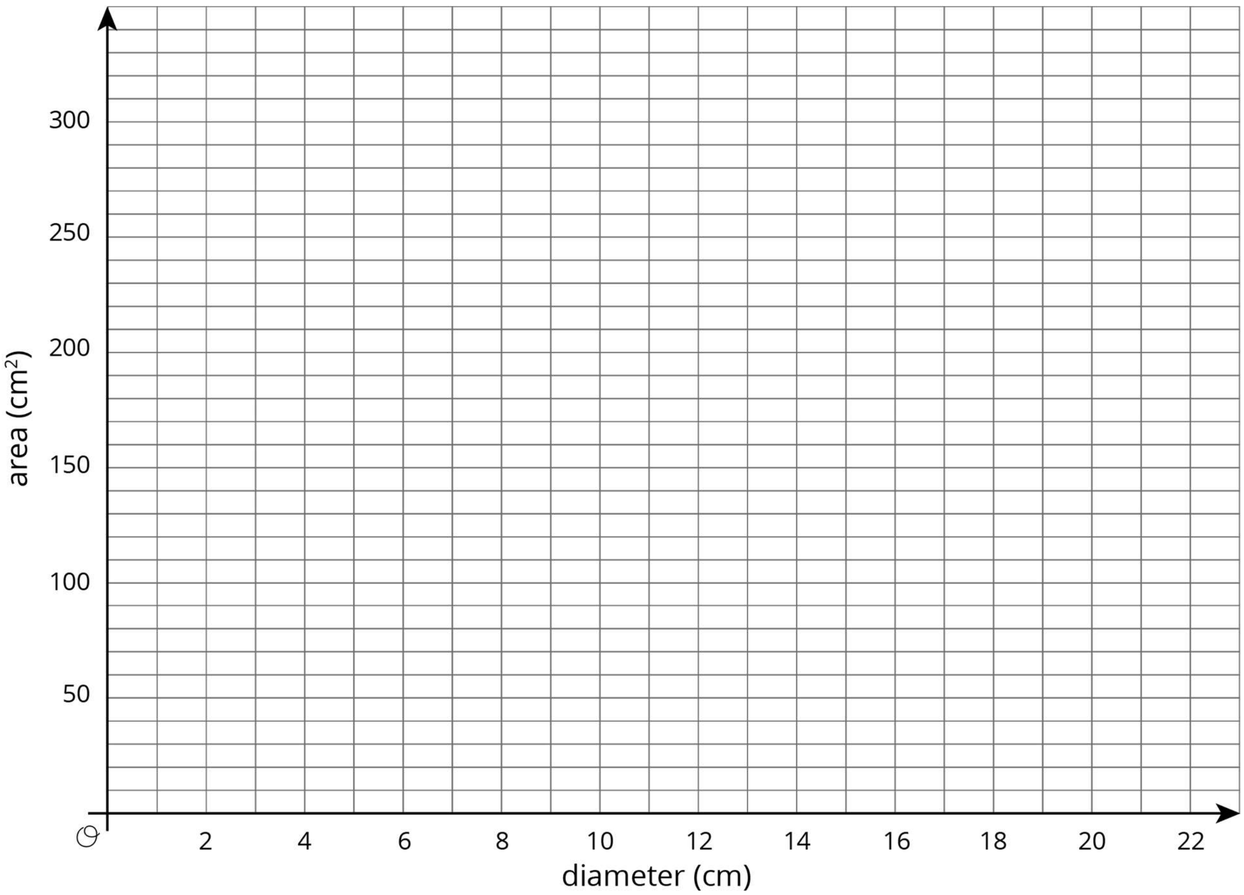
Student Task Statement

Your teacher will give your group two circles of different sizes.

1. For each circle, use the squares on the graph paper to measure the diameter and estimate the **area of the circle**. Record your measurements in the table.

diameter (cm)	estimated area (cm^2)

2. Plot the values from the table on the class coordinate grid. Then plot the class's data points on your coordinate grid.



3. In a previous lesson, you graphed the relationship between the diameter and circumference of a circle. How is this graph the same? How is it different?

Student Response

1. Each group has 2 of the rows from this table (and the values in the right column are approximate):

diameter (cm)	estimated area (cm^2)
2	3
3	7
4	12
5	19
6	27
7	38
10	78
12	108
14	147
16	200

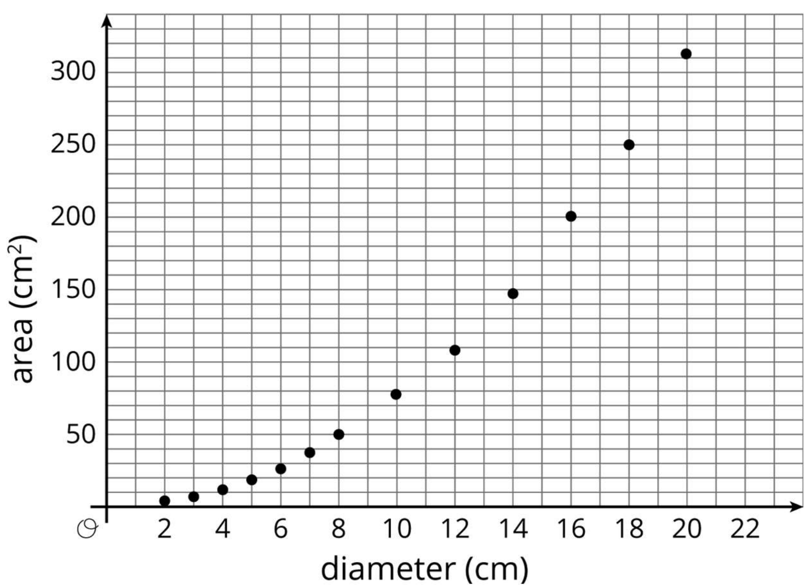
18	250
20	312

Possible solutions for a circle of radius 6:

- Count the number of squares that fit completely inside the circle to get an underestimate: 88 cm^2. Next, count the number of squares that cover any part of the circle to get an overestimate: 128 cm^2. Then, average the two: about 108 cm^2.

- Count the number of squares that fit completely inside the circle: 88 cm^2. Next, estimate the partial squares that make up the gaps: about 24 cm^2. Then, add the two: about 112 cm^2 total.

2.



3. Both the graphs of circumference and area go up as the diameter increases. However, the area graph goes up faster the bigger the diameter, while the circumference goes up by about the same amount every time. That means the area graph curves upward instead of being a straight line through the origin, so it does not represent a proportional relationship.

Are You Ready for More?

How many circles of radius 1 unit can you fit inside each of the following so that they do not overlap?

1. a circle of radius 2 units?
2. a circle of radius 3 units?
3. a circle of radius 4 units?

If you get stuck, consider using coins or other circular objects.

Student Response

1. 2
2. 7
3. 11

Activity Synthesis

Invite selected students to share their strategies for estimating the area of their circle.

Next, ask "Is the relationship between the diameter and the area of a circle a proportional relationship?" (No.) Invite students to explain their reasoning. (The points do not lie on a straight line through (0,0).)

To help students see and express that the relationship is not proportional, consider adding a column to the table of measurements to record the quotient of the area divided by the diameter. Here is a table of sample values.

diameter (cm)	**estimated area (cm^2)**	area ÷ diameter
2	3	1.5
3	7	2.3
4	12	3.0
5	19	3.8
6	27	4.5
7	38	5.4
10	78	7.8
12	108	9.0
14	147	10.5
16	200	12.5
18	250	13.9
20	312	15.6

Remind students that there is a proportional relationship between diameter and circumference, even though there is not between diameter and area. Recall that students saw the same phenomenon when they examined the relationship between the diagonal of a square and its perimeter (proportional) and the diagonal of a square and its area (not proportional).

Speaking, Listening: Compare and Connect. Ask students to prepare a visual display that shows how they estimated the area of their circle. As they work, look for students with different strategies that overestimate or underestimate the area. As students investigate each other's work, ask them to share what worked well in a particular approach. Listen for and amplify any comments that make the estimation of the area more precise. Then encourage students to make connections between the expressions and diagrams used to estimate the area of a circle. Listen for and amplify language students use to make sense of the area of a circle as the number of unit squares enclosed by the circle. This will foster students' meta-awareness and support constructive conversations as they compare strategies for estimating the area of a circle and make connections between expressions and diagrams used to estimate the area of a circle. *Design Principles(s): Cultivate conversation; Maximise meta-awareness*

7.3 Covering a Circle

Optional: 20 minutes

In this activity students compare the area of a circle of radius r with the area of a square of side length r through trying to cover the circle with different amounts of squares. The task is open-ended so the students can look for a very rough estimate or can look for a more precise estimate. In either case, they find that the circle has area greater than 2 times the square, less than 4 times the square, and that 3 times the square looks like a good estimate.

In the discussion, students will generalise their estimates for different values of r. An optional video shows how to cut up 3 squares and place them inside a circle. Since there is a little white space still showing around the cut pieces, that means that the area of a circle with radius r is close to, but a little bit more than, $3r^2$.

Watch for how students use the square and circle provided in the problem.

Instructional Routines

- Collect and Display

Launch

Keep students in the same groups. Provide access to geometry toolkits.

Provide access to tools and assistive technologies such as physical cut-outs of the square or a digital version that students can manipulate.
Supports accessibility for: Visual-spatial processing; Conceptual processing; Organisation
Conversing, Reading: Collect and Display. As students work in groups to make sense of the problem, circulate and listen to groups as they discuss the number of squares it would take to cover the circle exactly. Write down the words and phrases students use to justify why it definitely takes more than 2 squares and less than 4 squares to cover the circle exactly. As groups cut and reposition the squares in the circle, include a diagram or picture to show this in the visual display. As students review the language and diagrams collected in the visual display, encourage students to clarify the meaning of a word or phrase. This routine will provide feedback to students in a way that supports sense-making while simultaneously increasing meta-awareness of language.
Design Principle(s): Support sense-making; Maximise meta-awareness

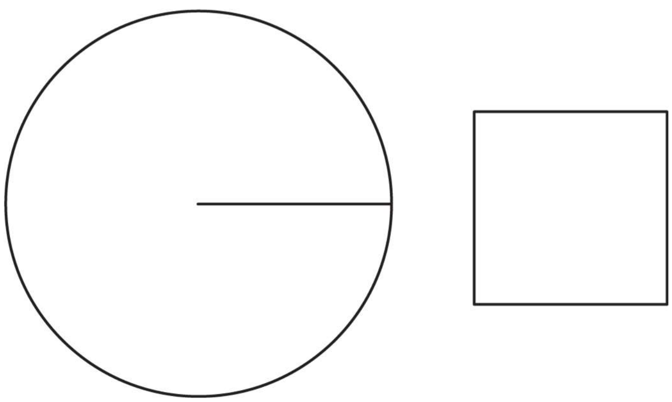
Anticipated Misconceptions

Students may focus solely on the radius of the circle and side length of the square, not relating their work to area. As these students work, ask them what they find as they try to cover the circle each time. Reinforce the idea that as they cover the circle, they are comparing the area of the circle and squares.

If students arrive at the idea that 4 squares suffice to completely cover the circle, ask them if there is any excess. Could they cover the square with $3\frac{1}{2}$ squares, for example?

Student Task Statement

Here is a square whose side length is the same as the radius of the circle.



How many of these squares do you think it would take to cover the circle exactly?

Student Response

It definitely takes more than 2 squares and less than 4 squares. It would probably take somewhere close to 3 squares. Sample reasoning: I traced the square onto tracing paper 4 times and cut them out. I saw that only 3 of these would fit completely on the circle.

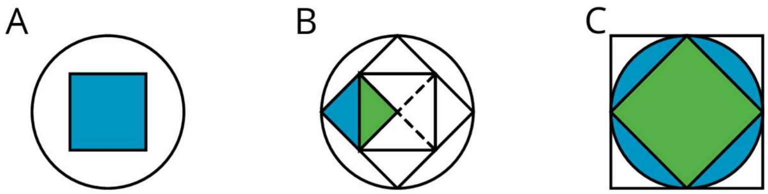
Activity Synthesis

The goal of this discussion is for students to recognise that the area of the circle with radius r is a little more than $3r^2$, for any size circle.

Ask the class:

- "Can two squares completely cover the circle?" (No.)
- "Can four squares completely cover the circle?" (Yes.)
- "Can three squares completely cover the circle?" (It's hard to tell for sure.)

Invite students to explain their reasoning. Consider displaying this image for students to refer to during their explanations.



- Shape A shows that the area of the circle is larger than the area of the square, because the square can be placed inside the circle and more white space remains.
- Shape B shows that the area of the circle is larger than twice the area of the square, because the squares can be cut and repositioned to fit within the circle and some white space still remains.
- Shape C shows that the area of the circle is smaller than four times the area of the square, because the squares completely cover the circle and the corners go outside the circle.
- Shape C also shows that it is reasonable to conclude that the area of the circle is approximately equal to three times the area of the square, because it looks like the blue shaded regions (inside the circle) are close in area to the white shaded regions (outside the circle but inside the square).

Consider showing this video which makes it more apparent that three of these squares can be cut and repositioned to fit entirely within the circle.

Video 'Area of a Circle' available here: https://player.vimeo.com/video/304137873.

Since there is a little white space remaining around the cut pieces, that means it would take a little bit more than three squares to cover the circle. The area of the circle is a little bit more than three times the area of one of those squares. At this point, some students may suggest that it takes exactly π squares to cover the circle. This will be investigated in more detail in the next lesson. If not mentioned by students, it does *not* need to be brought up in this discussion.

Next, guide students towards the expression $3r^2$ by asking questions like these:

- "Does the size of the circle affect how many radius squares it takes to cover the circle?" (No, the entire picture can be scaled.)
- "If the radius of the circle were 4 units, what would be the area of the square? What would be the area of the circle?" (16 units^2 and a little more than 3×16, or 48 units^2)
- "If the radius of the circle were 11 units, what would be the area of the square? What would be the area of the circle?" (121 units^2 and a little more than 3×121, or 363 units^2)
- "If the circle has radius r, what would be the area of the square? What would be the area of the circle?" (r^2 units^2 and a little more than $3r^2$ units^2)

Lesson Synthesis

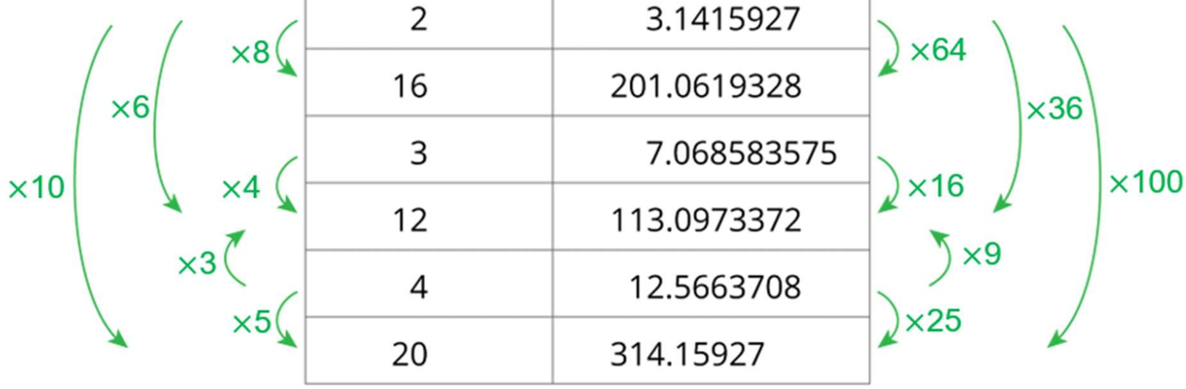
Pose the following question: "If you have a square with side lengths equal to the radius of a circle, how many of these squares does it take to cover the circle?"

Tell students what approximation to use for this value. Have students use this approximation along with the area of such a square to calculate the *area* of each circle they were assigned at the beginning of class. Record their answers in a table displayed for all to see and discuss:

- "How many times larger is the diameter?"
- "Does the area increases by the same factor?"
- "Is the relationship between the diameter and area of a circle a proportional relationship? How do you know?"

Draw arrows with the scale factors to the left side of the table to illustrate the relationship between the diameters. Draw arrows on the right side of the table and label them with the factor the area is increasing by, such as "× 64" or just write "not × 8".

diameter (cm)	area of circle (cm^2)
2	3.1415927
16	201.0619328
3	7.068583575
12	113.0973372
4	12.5663708
20	314.15927



7.4 Areas of Two Circles

Cool Down: 5 minutes

Anticipated Misconceptions

If students think that the diameter and area of a circle are proportional, they will likely choose C because $20 \times 3 = 60$ and $300 \times 3 = 900$.

Student Task Statement

- Circle A has a diameter of approximately 20 inches and an area of 300 in^2.
- Circle B has a diameter of approximately 60 inches.

Which of these could be the area of circle B? Explain your reasoning.

1. About 100 in^2
2. About 300 in^2
3. About 900 in^2
4. About 2 700 in^2

Student Response

About 2 700 in^2. The diameter of circle B is 3 times bigger than the diameter of circle A, so the area of circle B is larger than the area of circle A. The pattern shows that the area grew quickly, so 900 is probably not large enough. The radius of circle B is 30 inches, so the area is about 3×30^2 in^2 (and is definitely more than 30^2 because a square of side 30 inches fits inside the circle with a lot of space left).

Student Lesson Summary

The circumference C of a circle is proportional to the diameter d, and we can write this relationship as $C = \pi d$. The circumference is also proportional to the radius of the circle, and the constant of proportionality is $2 \times \pi$ because the diameter is twice as long as the radius. However, the **area of a circle** is *not* proportional to the diameter (or the radius).

The area of a circle with radius r is a little more than 3 times the area of a square with side r so the area of a circle of radius r is approximately $3r^2$. We saw earlier that the circumference of a circle of radius r is $2\pi r$. If we write C for the circumference of a circle, this proportional relationship can be written $C = 2\pi r$.

The area A of a circle with radius r is approximately $3r^2$. Unlike the circumference, the area is not proportional to the radius because $3r^2$ cannot be written in the form kr for a number k. We will investigate and refine the relationship between the area and the radius of a circle in future lessons.

Glossary

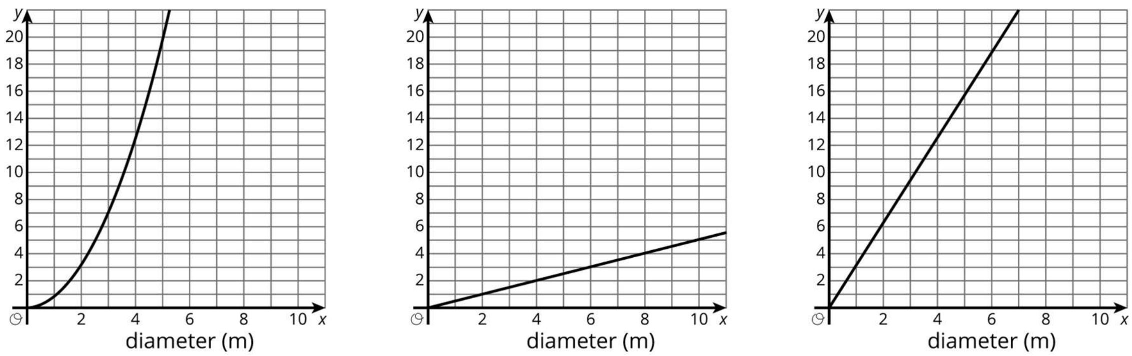
- area of a circle

Mathematics Education Innovation

Lesson 7 Practice Problems

1. **Problem 1 Statement**

The x-axis of each graph has the diameter of a circle in metres. Label the y-axis on each graph with the appropriate measurement of a circle:
radius (m), circumference (m), or area (m^2).



Solution

The first graph shows the relationship between the diameter and area of a circle, because it is not a proportional relationship. The second graph shows the relationship between the diameter and the radius, because it is proportional and the constant of proportionality is $\frac{1}{2}$. The third graph shows the relationship between the diameter and the circumference, because is it proportional and the constant of proportionality is π.

2. **Problem 2 Statement**

Circle A has area 500 in^2. The diameter of circle B is three times the diameter of circle A. Estimate the area of circle B.

Solution

About 4 500 in^2. If the diameter is 3 times greater, the area must be 3^2, or 9 times greater.

3. **Problem 3 Statement**

Lin's bike travels 100 metres when her wheels rotate 55 times. What is the circumference of her wheels?

Solution

About 1.82 metres because $100 \div 55 \approx 1.82$

4. **Problem 4 Statement**

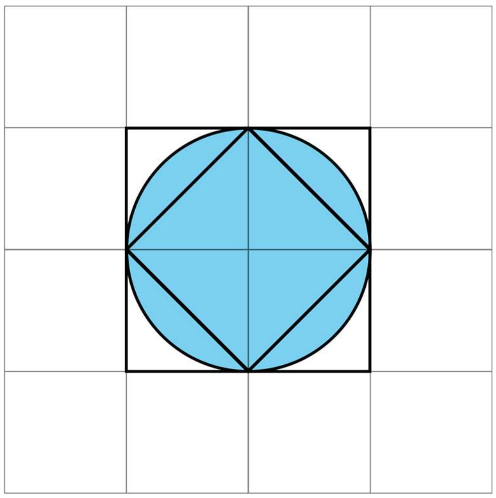
Priya drew a circle whose circumference is 25 cm. Clare drew a circle whose diameter is 3 times the diameter of Priya's circle. What is the circumference of Clare's circle?

Solution

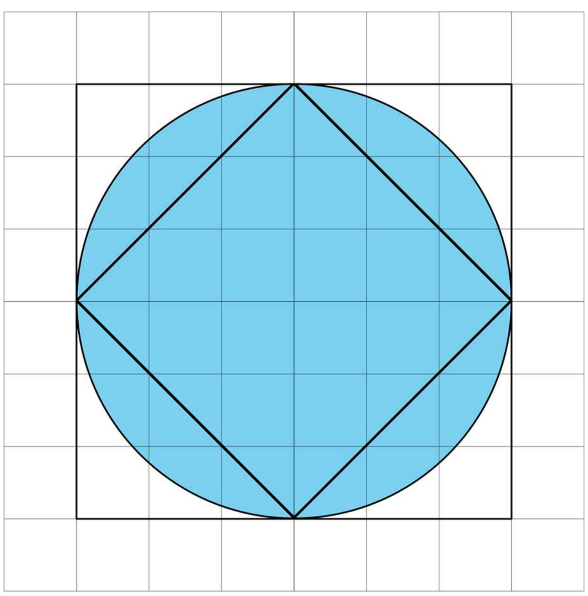
75 cm

5. **Problem 5 Statement**

   a. Here is a picture of two squares and a circle. Use the picture to explain why the area of this circle is more than 2 square units but less than 4 square units.



   b. Here is another picture of two squares and a circle. Use the picture to explain why the area of this circle is more than 18 square units and less than 36 square units.



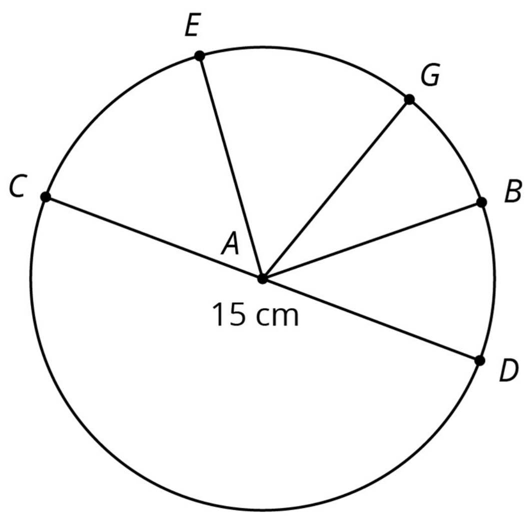
Solution

a. The square inside the circle has an area of 2 square units because it is made of 4 triangles each with area $\frac{1}{2}$ square unit, and $\frac{4}{2} = 2$. The square outside the circle has an area of 4 square units, because $2^2 = 4$.

b. The square inside the circle has an area of 18 square units because $12 + \frac{12}{2} = 18$ (the square inside the circle contains 12 full grid squares and 12 half grid squares). The square outside the circle has an area of 36 square units because $6^2 = 36$.

6. **Problem 6 Statement**

Point A is the centre of the circle, and the length of CD is 15 centimetres. Find the circumference of this circle.



Solution

About 47 cm because $15 \times \pi \approx 47$