Lesson 11: Scales without units

Goals

- Explain (orally and in writing) how to use scales without units to determine scaled or actual distances.
- Interpret scales expressed without units, e.g., “1 to 50,” (in spoken and written language).

Learning Targets

- I can explain the meaning of scales expressed without units.
- I can use scales without units to find scaled distances or actual distances.

Lesson Narrative

In previous lessons, students worked with scales that associated two distinct measurements—one for the distance on a drawing and one for actual distance. The units used in the two measurements are often different (centimetre and metre, inch and foot, etc.). In this lesson, students see that a scale can be expressed without units. For example, consider the scale 1 to 60. This means that every unit of length on the scale drawing represents an actual length that is 60 times its size, whatever the unit may be (inches, centimetres, etc.).

Expressing the scale as 1 to 60 highlights the scale factor relating the scale drawing to the actual object. Each measurement on the scale drawing is multiplied by 60 to find the corresponding measurement on the actual object. This relates closely to the scaled copies that were examined earlier in the unit in which each copy was related to the original by a scale factor. Students gain a better understanding of both scaled copies and scale drawings as they understanding the common underlying structure.

Addressing

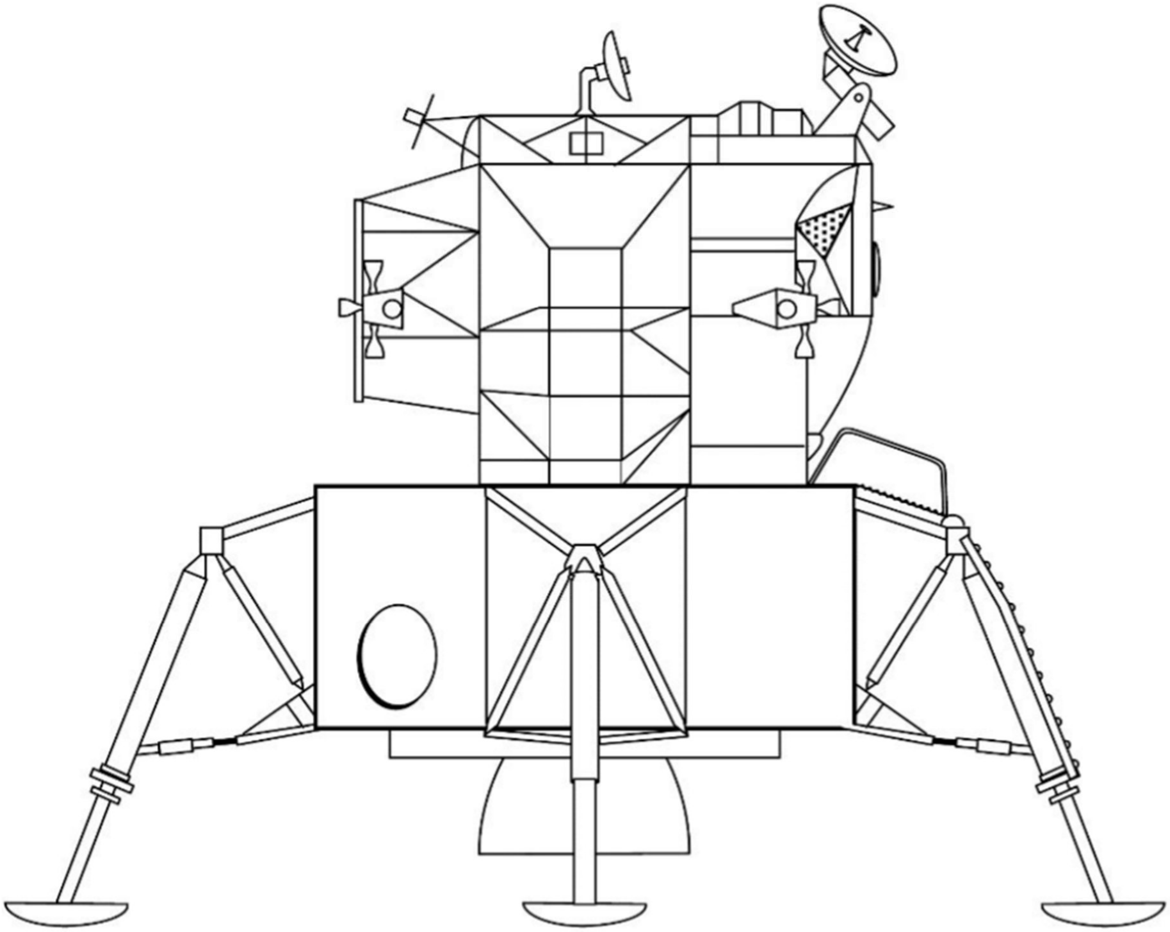
- Solve problems involving scale drawings of geometric shapes, including computing actual lengths and areas from a scale drawing and reproducing a scale drawing at a different scale.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Discussion Supports
- Think Pair Share

Required Materials

Copies of blackline master



Rulers

Required Preparation

You will need the Apollo Lunar Module blackline master for this lesson. Prepare one copy per student.

Ensure students have access to geometry toolkits, especially rulers and graph paper.

Student Learning Goals

Let's explore a different way to express scales.

11.1 One to One Hundred

Warm Up: 5 minutes

This warm-up introduces students to a scale without units and invites them to interpret it using what they have learned about scales so far.

As students work and discuss, notice those who interpret the unitless scale as numbers having the same units, as well as those who see “1 to 100” as comparable to using a scale factor of 100. Invite them to share their thinking later.

Instructional Routines

- Think Pair Share

Launch

Remind students that, until now, we have worked with scales that each specify two units—one for the drawing and one for the object it represents. Tell students that sometimes scales are given without units.

Arrange students in groups of 2. Give students 2 minutes of quiet think time and another minute to discuss their thinking with a partner.

Anticipated Misconceptions

Students might think that when no units are given, we can choose our own units, using different units for the 1 and the 100. This is a natural interpretation given students' work so far. Make note of this misconception, but address it only if it persists beyond the lesson.

Student Task Statement

A map of a park says its scale is 1 to 100.

1. What do you think that means?

2. Give an example of how this scale could tell us about measurements in the park.

Student Response

1. Answers vary. Sample responses:

   - Distances in the park are 100 times bigger than corresponding distances in the map.

   - One unit on the map represents 100 units of distance in the park.

2. Answers vary. Sample responses:

   - If a path is 6 inches long on the map, then we could tell that the actual path is 600 inches long.

   - We could use the scale to tell the size of the park. For example, if the park is 20 inches wide on the map, we can tell the actual park is 2 000 inches wide.

Activity Synthesis

Solicit students' ideas about what the scale means and ask for a few examples of how it could tell us about measurements in the park. If not already mentioned by students, point out that a scale written without units simply tells us how many times larger or smaller an actual measurement is compared to what is on the drawing. In this example, a distance in the park would be 100 times the corresponding distance on the map, so a distance of 12 cm on the map would mean 1 200 cm or 12 m in the park.

Explain that the distances could be in any unit, but because one is expressed as a number times the other, the unit is the same for both.

Tell students that we will explore this kind of scale in this lesson.

11.2 Apollo Lunar Module

15 minutes

In this activity, students use a scale drawing and a scale expressed without units to calculate actual lengths. Students will need to make a choice about which units to use, and some choices make the work easier than others.

Monitor for several paths students may take to determine actual heights of the objects in the drawing. Their choice of units could influence the number of conversions needed and the efficiency of their paths (as shown in the sample student responses). Select students with the following approaches, sequenced in this order, to share during the discussion.

- Measure in cm, find cm for actual spacecraft, then convert to m
- Measure in cm, convert to m for scale drawing, then find spacecraft measurement in m

One other approach students may use is to measure the scale drawing using an inch ruler. This leads to an extra conversion from inches to centimetres or metres. Ask them to consider the unit of interest. Discuss and highlight strategic choices of units during whole-class debriefing.

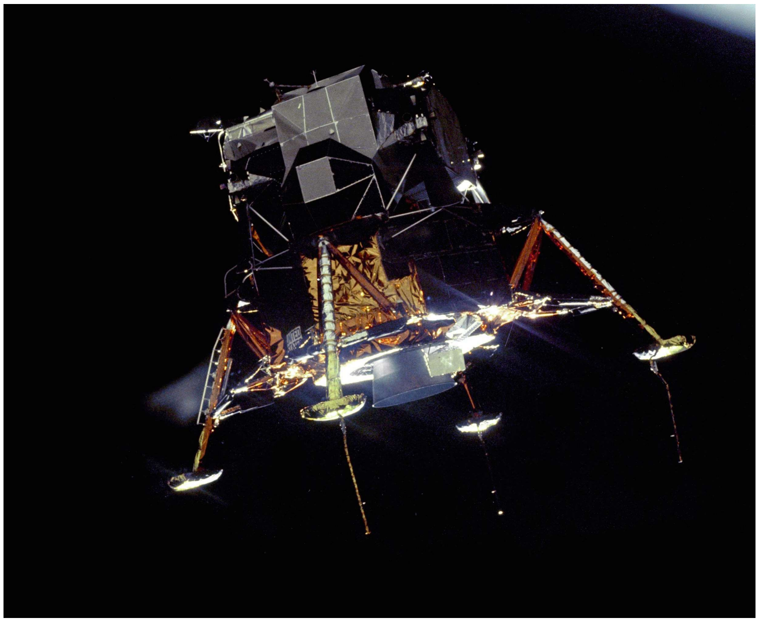
You will need the Apollo Lunar Module blackline master for this activity.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Discussion Supports

Launch

Tell students that Neil Armstrong and Buzz Aldrin were the first people to walk on the surface of the Moon. The Apollo Lunar Module was the spacecraft used by the astronauts when they landed on the Moon in 1969. Consider displaying a picture of the landing module such as this one. Tell students that the landing module was one part of a larger spacecraft that was launched from Earth.



Solicit some guesses about the size of the spacecraft and about how the height of a person might compare to it. Explain to students that they will use a scale drawing of the Apollo Lunar Module to find out.

Arrange students in groups of 2. Give each student a scale drawing of the Apollo Lunar Module (from the blackline master). Provide access to centimetre and inch rulers. Give students 3–4 minutes to complete the first two questions. Ask them to pause briefly and discuss their responses with their partner before completing the rest of the questions.

Students are asked to find heights of people if they are drawn "to scale." Explain that the phrase means "at the same scale" or "at the specified scale."

Action and Expression: Internalise Executive Functions. Chunk this task into more manageable parts for students who benefit from support with organisational skills in problem solving. Check in with students within the first 2-3 minutes of work time to ensure that they have a place to start. If students are unsure how to begin finding the actual length of the landing gear or actual height of the spacecraft, suggest that they first find out the length on the drawing.
Supports accessibility for: Organisation; Attention

Anticipated Misconceptions

If students are unsure how to begin finding the actual length of the landing gear or actual height of the spacecraft, suggest that they first find out the length on the drawing.

Students may measure the height of the spacecraft in centimetres and then simply convert it to metres without using the scale. Ask students to consider the reasonableness of their answer (which is likely around 0.14 m) and remind them to take the scale into account.

Student Task Statement

Your teacher will give you a drawing of the Apollo Lunar Module. It is drawn at a scale of 1 to 50.

1. The "legs" of the spacecraft are its landing gear. Use the drawing to estimate the actual length of each leg on the sides. Write your answer to the nearest 10 centimetres. Explain or show your reasoning.
2. Use the drawing to estimate the actual height of the Apollo Lunar Module to the nearest 10 centimetres. Explain or show your reasoning.
3. Neil Armstrong was 71 inches tall when he went to the surface of the Moon in the Apollo Lunar Module. How tall would he be in the drawing if he were drawn with his height to scale? Show your reasoning.
4. Sketch a stick figure to represent yourself standing next to the Apollo Lunar Module. Make sure the height of your stick figure is to scale. Show how you determined your height on the drawing.

Student Response

1. The leg of the spacecraft is about 350 cm if you just include one straight segment. Sample reasoning:
   - The leg is about 7 cm on the drawing, so the actual length is 7×50 or 350 cm.
   - The leg is about 2.75 inches on the drawing, so the actual length is 137.5 inches. $(2.75) \times 50 = 137.5$. Multiplying 137.5 by 2.54 gives the length in centimetres. $(137.5) \times (2.54) = 349.5$; this is 350 cm rounded to the nearest 10 cm.
2. The Lunar Module was about 7 metres tall. Sample explanations:
   - The spacecraft is about 14 cm tall on the drawing. The actual height is 50 times 14 cm, which is 700 cm. 700 cm is 7 m.
   - 14 cm is 0.14 m, because $14 \div 100 = 0.14$, and $(0.14) \times 50 = 7$, so the spacecraft is about 7 m tall.
   - The spacecraft is about 5.5 inches on the drawing. $(5.5) \times 50 = 275$. The actual height is about 275 inches, which is 698.5 cm. $275 \times (2.54) = 698.5$. 698.5 cm is 6.985 m, or about 7 m. (Do not highlight this solution in class discussion.)
3. Neil Armstrong would be about 1.4 inches tall in the scale drawing. Sample reasoning: $71 \div 50 \approx 1.4$.
4. Drawings vary depending on a student's height. Sample reasoning:
   - My height is 5 feet and 2 inches, which equals 62 inches. $(5 \times 12) + 2 = 62$. My height on the drawing is about $1\frac{1}{4}$ inches, since $62 \div 50 \approx 1.24$.

– I am 155 cm tall. $155 \div 50 = 3.1$. My height is 3.1 cm on the drawing.

Are You Ready for More?

The table shows the distance between the Sun and 8 planets in our solar system.

1. If you wanted to create a scale model of the solar system that could fit somewhere in your school, what scale would you use?

2. The diameter of Earth is approximately 8 000 miles. What would the diameter of Earth be in your scale model?

planet	average distance (millions of miles)
Mercury	35
Venus	67
Earth	93
Mars	142
Jupiter	484
Saturn	887
Uranus	1 784
Neptune	2 795

Student Response

Answers vary. Sample response:

1. The sports hall has a space of about 100 feet by 100 feet. The largest distance we need to represent is between the Sun and Neptune and this is about 2 800 million (or 3 billion) miles. So if 1 foot represents about 30 million miles, the solar system will fit.

2. 8 000 miles is one thousandth of 8 million miles and one millionth of 8 000 million miles. So the diameter of Earth will be about 3 millionths of the distance from Neptune to the Sun. This distance is represented by about 100 feet on the scale model, so the diameter of Earth will be about 3 millionths of 100 feet. This is about $\frac{1}{2\,500}$ of an inch. This is smaller than a fine grain of sand!

Activity Synthesis

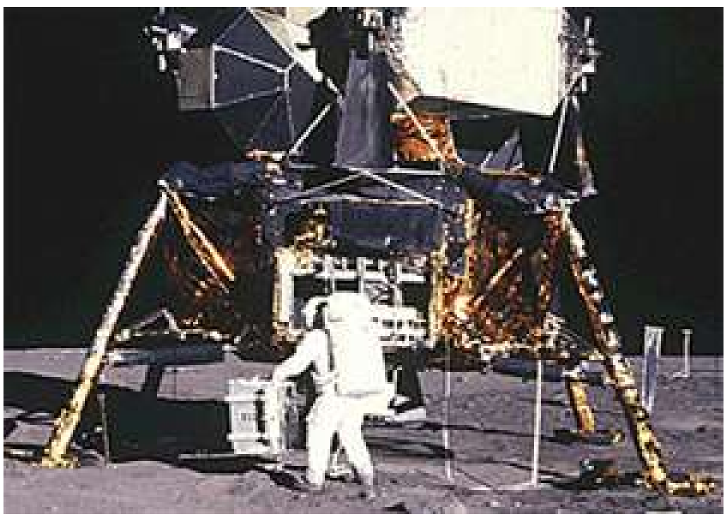
Invite selected students who measured using a centimetre ruler to share their strategies and solutions for the first two questions. Consider recording their reasoning for all to see. Highlight the multiplication of scaled measurements by 50 to find actual measurements. For example, the height of each leg is about 350 cm because $50 \times 7 = 350$.

Discuss whether or how units matter in problems involving unitless scales:

- Does it matter what unit we use to measure the drawing? Why or why not?
- Which unit is more efficient for measuring the height of the lunar module on the drawing—inches or centimetres? (Since the question asks for a height in metres, centimetres would be more efficient since it means fewer conversions. If the question asks for actual height in feet, inches would be a more strategic unit to use.)

Ask a few other students to share their responses to the last two questions. Select those who gave their heights in different units to share their solutions to the last problem. Highlight that, regardless of the starting unit, finding the length on the scale drawing involves dividing the actual measurement by 50. In other words, actual measurements can be translated to scaled measurements with a scale factor of $\frac{1}{50}$.

If time permits, consider displaying a photograph of one of the astronauts next to the Lunar Module, such as shown here, as a way to visually check the reasonableness of students' solutions.



Speaking: Discussion Supports. Use this routine to support whole-class discussion. For each response or observation that is shared, ask students to restate and/or revoice what they heard using precise mathematical language. Consider providing students time to restate what they hear to a partner, before selecting one or two students to share with the class. Ask the original speaker if their peer was accurately able to restate their thinking. Call students' attention to any words or phrases that helped to clarify the original statement. This will provide more students with an opportunity to produce language as they interpret the reasoning of others.
Design Principle(s): Support sense-making

11.3 Same Drawing, Different Scales

15 minutes

In this activity, students explore the connection between a scale with units and one without units. Students are given two equivalent scales (one with units and the other without) and are asked to make sense of how the two could yield the same scaled measurements of an actual object. They also learn to rewrite a scale with units as a scale without units.

Students will need to attend to precision as they work simultaneously with scales with units and without units. A scale of 1 inch to 16 feet is very different than a scale of 1 to 16, and students have multiple opportunities to address this subtlety in the activity.

As students work, identify groups that are able to reason clearly about why the two scales produce the same scale drawing. Two different types of reasoning to expect are:

- Using the two scales and the given dimensions of the car park to calculate and verify the student calculations.
- Thinking about the meaning of the scales, that is, in each case, the actual measurements are 180 times the measurements on the scale drawing.

Instructional Routines

- Discussion Supports

Launch

Ask students: "Is it possible to express the 1 to 50 scale of the Lunar Module as a scale with units? If so, what units would we use?" Solicit some ideas. Students are likely to say "1 inch to 50 inches," and "1 cm to 50 cm." Other units might also come up. Without resolving the questions, explain to students that their next task is to explore how a scale without units and one with units could express the same relationship between scaled lengths and actual lengths.

Keep students in the same groups. Provide access to rulers. Give partners 3–4 minutes to complete the first question and another 3–4 minutes of quiet work time for the last two questions.

Representation: Internalise Comprehension. Represent the same information through different modalities by using diagrams. If students are unsure where to begin, suggest that they draw a diagram to help organise the information provided.
Supports accessibility for: Conceptual processing; Visual-spatial processing

Anticipated Misconceptions

Some students may have trouble getting started. Suggest that they begin by treating each scale separately and find out, for instance, how a scale of 1 inch to 15 feet produces a drawing that is 8 inches by 5 inches.

Student Task Statement

A rectangular car park is 120 feet long and 75 feet wide.

- Lin made a scale drawing of the car park at a scale of 1 inch to 15 feet. The drawing she produced is 8 inches by 5 inches.
- Diego made another scale drawing of the car park at a scale of 1 to 180. The drawing he produced is also 8 inches by 5 inches.

1. Explain or show how each scale would produce an 8 inch by 5 inch drawing.
2. Make another scale drawing of the same car park at a scale of 1 inch to 20 feet. Be prepared to explain your reasoning.
3. Express the scale of 1 inch to 20 feet as a scale without units. Explain your reasoning.

Student Response

1. Answers vary. Sample explanations:
   - In Lin's case, 1 in represents 15 ft, so 120 ft is 8 in ($120 \div 15 = 8$) and 75 ft is 5 in ($75 \div 15 = 5$). In Diego's case, 1 unit on the drawing represents 180 of the same unit in the actual distance, so 1 in represents 180 in. 180 in is equal to 15 ft ($180 \div 12 = 15$). Since the scale here is also 1 in to 15 ft, the drawing will also be 8 in by 5 in.
   - 120 ft is 1 440 in ($120 \times 12 = 1,440$) and 75 ft is 900 in ($75 \times 12 = 900$). If the scale is 1 to 180, the sides of the car park will be $1,440 \div 180$ and $900 \div 180$, or 8 in and 5 in, respectively.
2. Drawing should show a 6 inch by $3\frac{3}{4}$ inch rectangle. Sample reasoning: $120 \div 20 = 6$. $75 \div 20 = 3\frac{3}{4}$.
3. 1 to 240. Sample explanation: 20 ft is 240 in, so 1 in on the drawing represents 240 in of actual distance.

Activity Synthesis

Select a couple of previously identified groups to share their responses to the first question and a couple of other groups for the other questions.

Highlight how scaled lengths and actual lengths are related by a factor of 180 in both scales, and that this factor is shown explicitly in one scale but not in the other.

- In the case of 1 to 180, we know that actual lengths are 180 times as long as scaled lengths (or scaled lengths are $\frac{1}{180}$ of actual lengths).
  If the scaled lengths are given in inches, we can use scaled lengths to find actual lengths in inches and, if desired, convert to feet afterward, and vice versa.
- In the case of 1 in to 15 ft, though we know that actual measurements are *not* 15 times longer than their corresponding measurements on a drawing (because 15 feet is not 15 times larger than 1 inch), it is not immediately apparent what factor relates the two

measurements. Converting the units helps us see the scale factor. Since 1 foot equals 12 inches and $15 \times 12 = 180$, the scale of 1 in to 15 feet is equivalent to the scale of 1 in to 180 in, or 1 to 180.

Speaking, Representing: Discussion Supports. Give students additional time to make sure that everyone in their group can explain their responses to the task statement questions. Invite groups to rehearse what they will say when they share with the whole class. Rehearsing provides students with additional opportunities to speak and clarify their thinking, and will improve the quality of explanations shared during the whole-class discussion. Vary who is selected to represent the work of the group, so that students get accustomed to preparing each other to fill that role.
Design Principle(s): Support sense-making; Cultivate conversation

Lesson Synthesis

- What does it mean when the scale on a scale drawing does not indicate any units?
- How is a scale without units the same as or different from a scale with units?
- How can a scale without units be used to calculate scaled or actual distances?

When a scale does not show units, the same unit is used for both the scaled distance and the actual distance. For instance, a scale of 1 to 500 means that 1 inch on the drawing represents 500 inches in actual distance, and 10 mm on a drawing represents 5,000 mm in actual distance. In other words, the actual distance is 500 times the distance on the drawing, and the scaled distance is $\frac{1}{500}$ of the actual distance. To calculate actual distances, we can multiply all distances on the drawing by the factor 500, regardless of the unit we choose or are given. Likewise, to find scaled distances, we multiply actual distances by $\frac{1}{500}$, regardless of the unit used. 500 and $\frac{1}{500}$ are scale factors that relate the two measurements (actual and scaled).

11.4 Scaled Courtyard Drawings

Cool Down: 5 minutes

Student Task Statement

Andre drew a plan of a courtyard at a scale of 1 to 60. On his drawing, one side of the courtyard is 2.75 inches.

1. What is the actual measurement of that side of the courtyard? Express your answer in inches and then in feet.
2. If Andre made another courtyard scale drawing at a scale of 1 to 12, would this drawing be smaller or larger than the first drawing? Explain your reasoning.

Student Response

1. 165 in, which is 13.75 ft. Sample reasoning: $2.75 \times 60 = 165$. $165 \div 12 = 13.75$.

2. It would be larger. Sample explanation: A scale of 1 to 12 means the length on paper is $\frac{1}{12}$ of the original length (or 10 inches by 13.75 inches), so the drawing would be larger than one drawn at $\frac{1}{60}$ the original length.

Student Lesson Summary

In some scale drawings, the scale specifies one unit for the distances on the drawing and a different unit for the actual distances represented. For example, a drawing could have a scale of 1 cm to 10 km.

In other scale drawings, the scale does not specify any units at all. For example, a map may simply say the scale is 1 to 1 000. In this case, the units for the scaled measurements and actual measurements can be any unit, so long as the same unit is being used for both. So if a map of a park has a scale 1 to 1 000, then 1 inch on the map represents 1 000 inches in the park, and 12 centimetres on the map represent 12 000 centimetres in the park. In other words, 1 000 is the scale factor that relates distances on the drawing to actual distances, and $\frac{1}{1\,000}$ is the scale factor that relates an actual distance to its corresponding distance on the drawing.

A scale with units can be expressed as a scale without units by converting one measurement in the scale into the same unit as the other (usually the unit used in the drawing). For example, these scales are equivalent:

- 1 inch to 200 feet
- 1 inch to 2 400 inches (because there are 12 inches in 1 foot, and $200 \times 12 = 2\,400$)
- 1 to 2 400

This scale tells us that all actual distances are 2 400 times their corresponding distances on the drawing, and distances on the drawing are $\frac{1}{2\,400}$ times the actual distances they represent.

Lesson 11 Practice Problems

Problem 1 Statement

A scale drawing of a car is presented in the following three scales. Order the scale drawings from smallest to largest. Explain your reasoning. (There are about 1.1 yards in a metre, and 2.54 cm in an inch.)

a. 1 in to 1 ft

b. 1 in to 1 m

c. 1 in to 1 yd

Solution

b, c, a. Explanations vary. Sample responses:

- Of the three units, 1 ft is the smallest unit, and 1 m is the largest. Therefore, a drawing with scale 1 in to 1 ft will require the most number units (the largest), and a drawing with scale 1 in to 1 m will require the least (the smallest).
- Each scale was converted into a scale without units. 1 in to 1 ft is equivalent to 1 to 12. 1 in to 1 m is equivalent to 2.54 cm to 100 cm, which is roughly 1 to 39. And 1 in to 1 yd is equivalent to 1 to 36.

Problem 2 Statement

Which scales are equivalent to 1 inch to 1 foot? Select **all** that apply.

a. 1 to 12

b. $\frac{1}{12}$ to 1

c. 100 to 0.12

d. 5 to 60

e. 36 to 3

f. 9 to 108

Solution ["A", "B", "D", "F"]

Problem 3 Statement

A model airplane is built at a scale of 1 to 72. If the model plane is 8 inches long, how many feet long is the actual airplane?

Solution

48 feet. The actual airplane is 72 times the length of the model. $8 \times 72 = 576$. 576 inches is 48 feet, as $576 \div 12 = 48$.

Problem 4 Statement

Quadrilateral A has side lengths 3, 6, 6, and 9. Quadrilateral B is a scaled copy of A with a shortest side length equal to 2. Jada says, "Since the side lengths go down by 1 in this scaling, the perimeter goes down by 4 in total." Do you agree with Jada? Explain your reasoning.

Solution

No. The side lengths of B are not each 1 less than those of A. The side lengths of B are $\frac{2}{3}$ of those of A, so they must be 2, 4, 4, and 6. The perimeter of A is 24 and the perimeter of B is 16, which is 8 less in total.

Problem 5 Statement

Polygon B is a scaled copy of polygon A using a scale factor of 5. Polygon A’s area is what fraction of polygon B’s area?

Solution

$\frac{1}{25}$

Problem 6 Statement

Shapes R, S, and T are all scaled copies of one another. Shape S is a scaled copy of R using a scale factor of 3. Shape T is a scaled copy of S using a scale factor of 2. Find the scale factors for each of the following:

a. From T to S

b. From S to R

c. From R to T

d. From T to R

Solution

a. $\frac{1}{2}$

b. $\frac{1}{3}$

c. 6

d. $\frac{1}{6}$