Chapter 27

# The paradoxical nature of mathematics

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**Abstract** Mathematics is usually described as a deductive science. The set of axioms, with which we start, should be as economical as possible, hopefully consistent, and deductively strong with as many as possible "desirable" consequences. How do we achieve sufficient *deductive power* for an axiomatic system?

In this work we present the unorthodox thesis that the deductive strength in Mathematics comes, perhaps exclusively, from its paradoxical nature, namely from its proximity to the contradictory, a proximity that almost always takes the form of a Finitization of the Infinite. We support our thesis by examining Euclidean Geometry; Number Theory; Incommensurability and periodic anthyphairesis/continued fractions in the Mathematics and the Philosophy of the Pythagoreans, Zeno, Theaetetus, Plato; ratios of magnitudes and method of exhaustion in weakly finitized form by Eudoxus, and Real numbers and Calculus in the strongly finitized form by Dedekind completeness; Set theory axioms such as the axiom of choice, with special reference to compactness and ultrafilters, and Gödel's program with axioms of large cardinals. In the last two sections we argue that Beauty in Mathematics and we suggest that the "Unreasonable Effectiveness of Mathematics in the Natural Sciences" are both manifestations of the Paradoxical Nature of Mathematics.

## 1 The paradoxical nature of mathematics

**1.1 Truth table of the conditional.** The starting point of our idea is extremely simple, the *truth table* of the *Conditional* "if P, then Q". It is *false* only in one case, if P is true and Q is false, and it is *true* in the three other cases. The *canonical case* where the conditional is *true* is when P and Q are true. By taking the *contrapositive*, we are forced to conclude that if P and Q are false, then *surprisingly* the conditional must be true.

We cannot think of any adequate justification for the last remaining True case, the conditional False→True. One might argue that since False→False is judged worthy to be declared True, then, *a fortiori*, this must be too; but it must remain the most controversial, even scandalous, assignment, since it appears to allow providing False cause for something otherwise True.

| P | Q | P→Q |
|---|---|---|
| True | True | True |
| True | False | False |
| False | False | True |
| False | True | True |

Let us note that this truth table was perfectly known to Philo, the logician, who lived about the time of Euclid (Sextus Empiricus, *Adversus Mathematicos* Book 8, Sect. 112, line 4 to Sect. 117, line 7 of the original text, cf. [70]). Philo explained the truth table with convincing examples:

| | | |
|---|---|---|
| If it is day<br>True | Then there is light<br>True | True |
| If it is day<br>True | Then it is night<br>False | False |
| If earth flies<br>False | Then earth has wings<br>False | True |
| If earth flies<br>False | Then earth exists<br>True | True |

Thus Philo's example "If earth flies, then earth exists" is a True conditional.

**1.2 Contradictory hypotheses are most powerful.** By *Modus Ponens*, if a statement P, although False, is nevertheless assumed as an axiom, we conclude that every statement Q, irrespective of it being true or false, is necessarily a consequence of P.

If a statement P is a *contradictory* one, of the form P = (X & not X) for some statement X, then P is False. Thus, the strongest possible axioms or hypotheses, with *unlimited deductive power*, are the *contradictory* ones.

**1.3 Consistent approximations of the contradictory.** *The paradoxical nature of Mathematics: the deductive power of Mathematics is due to (a hopefully consistent) approximation of the contradictory.*

Even though contradictory statements are full of consequences, we nevertheless consider them as *unacceptable*, precisely because they are contradictory.

We will argue that substantial parts of Mathematics (perhaps even all of Mathematics) are derived from assumptions and axioms that may be described as *(hopefully consistent) approximations of the contradictory*, and which by being *consistent* are *acceptable* as axioms or constructions based on axioms, and by being *approximation to the contradictory* inherit some of its unlimited *deductive power*.

Thus, according to this view, we succeed in obtaining deeper and deeper mathematical "truths" by approaching more and more a contradiction, the absolute "false". This is what we mean by *the paradoxical nature of Mathematics*. As this might appear to be a *controversial thesis*, we will make every effort to justify it.

**1.4 Finitizing the infinite.** In fact, there is *one contradictory statement* that appears to be particularly important: the statement that an entity *infinite* in some sense is at the same time *finite*. We will examine some of the historically most important axiomatic systems in Mathematics, from Euclidean Geometry (in Section 2) and the Theory of Numbers (in Section 3) to the ancient and modern Real Numbers (in Sections 5, 6) and various aspects of Set Theory (in Sections 7, 8, 9), and we will find that, in each of these systems, some aspects of a built in *Infinity* is given, by the crucial introduction of an axiom or a principle or a definition, a near-contradictory *Finite* description, thus introducing a (hopefully consistent) *Finitization of the Infinity*. We will argue that the cause of the deductive power of each of these systems is precisely this crucial Finitization of Infinity (and the resulting approximation of the contradictory) that takes place in the system.

As we shall see in Section 4, the Pythagoreans already realized that the nature of Mathematics is the Finitization of the Infinite, a view that became central in Plato's philosophy; Hermann Weyl has expressed lucidly essentially the same thesis:[1]

> *Mathematics* is the science of the *infinite*, its goal the symbolic comprehension of the *infinite* with human, that is finite, means [85, p. 38].
>
> *Mathematics* has been called the science of the *infinite*. Indeed, the *mathematician* invents *finite* constructions by which questions are decided that by their very nature refer to the infinite. That is his *glory*. [84, p. 12].

## 2 The fifth postulate in Euclid's *Elements* is a finitization of the infinite in plane Euclidean Geometry without ratios

Plane Geometry without ratios is based on Postulates in Euclid's *Elements*.

*Postulate 2* states that any line segment can be extended to a greater line segment *ad infinitum*, and thus it introduces (potential) *infinity* into Euclidean Geometry.

**2.1 The impossibility to prove parallelness only with postulate 2.** Postulate 2 allows for the definition of parallelness.

**Definition of parallel lines.** Two lines $a$, $b$ are *parallel* if any extension of $a$ (by Postulate 2) does not meet any extension of $b$ (by Postulate 2).

**Impossibility of proof that lines are parallel from the definition alone.** From the definition alone, a proof that two lines are parallel *cannot be obtained*, since *an infinite number* of constructions (applications of Postulate 2) is needed. Indeed if after a finite number of extensions, according to Postulate 2, of lines $a$ and $b$, the extended lines meet, we immediately conclude that the lines are not parallel; but if they do not meet, then we would have to proceed to another extension of $a$ and $b$, and

[1] Here, and in other quotes, the emphasis is ours.

so on *ad infinitum*. It is clear that if the lines are indeed parallel, we will never be able to prove this from the definition alone.

*A criterion for parallelness* is given by Proposition I.16 of the *Elements*:

> The exterior angle of a triangle is greater than each interior opposite angle of the triangle.

The Proposition does not use Postulate 5, but only the first four Postulates, and, indeed a corollary of it provides an *effective criterion for parallelness*:

> If a third line $c$ cuts both $a$ and $b$ and the interior and alternate angles are equal, then $a$ and $b$ are parallel.

This criterion is used in Proposition I.30 to construct a line parallel to a line from a point not in the line.

**2.2 Postulate 5 finitizes the infinite number of steps needed to prove parallelness solely by the definition.** *Postulate 5*, stating that if two lines are parallel, and if cut by third, then the alternate internal angles thus formed are equal, rules that the *only possibility* for lines $a$ and $b$ to be *parallel* is given by corollary to I.16, thus excluding any other potential causes for parallelness.

*Postulate 5* is a (hopefully consistent) *Finitization of the Infinite*, since the *infinite* number of constuctions based on Postulate 2, described in Section 2.1 above, required for the verification that the two lines are parallel, are replaced by a *one step* verification of the equality of angles, and is thus a *near-contradictory statement*, replacing "infinity" with its contradictory "finite", hopefully in a way that avoids outright contradiction.

Byers in [6, pp. 95–96] has made much the same point:

> Parallel lines by definition never meet. But lines in Euclidean geometry are *infinitely* extendible. How then can one hope to prove that two lines will never meet *no matter how far they are extended?* Lines are *infinite* geometric objects. There is a certain intrinsic *incompleteness* about such objects, and yet mathematics wishes to make definitive statements about them.
>
> To prove that lines are parallel requires, at first glance, showing that no matter how far one extends the lines they will never meet.
>
> *This is a kind of infinite argument* in the same sense that showing that the sum of an odd number and an even number is always an odd number is also an *infinite argument* (it applies to an *infinite* number of cases). Mathematics characteristically deals with such "infinite" situations. In

> doing so it must replace an indefinite or infinite property with one that is *essentially finite*.
>
> In the proof, the *finite property* is the equality of the alternate angles.
>
> *In the parallel postulate the "infinite" condition of parallelism is replaced by the "finite" condition for nonparallelism.*
>
> The use of an argument by contradiction is a way of making this essential reduction.

Exactly because Postulate 5 is a finitization of the infinite, we expect that Postulate 5 has *great deductive power*.

Indeed, it is true that all plane Euclidean Geometry, without ratios of magnitudes, namely the second half of the first and the second, third and fourth Book of the *Elements* [65], are consequences of the Fifth Postulate.

For the same reason we expect that Postulate 5 has *paradoxical consequences*. Indeed the ancients had noticed that Postulate 5 has paradoxical consequences; Proclus, in his *Commentary to the first Book of the* Elements 395,16–397,12 and 403,4–404,26, has lengthy comments on the paradoxical nature of Propositions I.35 & 37 and III.20 of the *Elements*: Triangles with base on a line $a$ and third vertex on another line $b$ parallel to $a$ are finite, in fact they have fixed area, while increasing to infinity in length of perimeter.

**Note.** The axiom of parallelness in hyperbolic geometry is also a Finitization of Infinity, because the set of parallels from a point to a given line consists of two limiting lines. It follows then from a result of A. Papadopoulos and W. Su [59], that the Parallel Axiom for hyperbolic geometry produces an analogue of Proposition I.37, a corresponding paradoxical proposition. As mentioned in Section 12, below, Proposition I.37 was pivotal for Newton; in fact it is closely related to Kepler's Second Law and the conservation of angular momentum. We wonder if its hyperbolic analogue could play for Special Relativity an analogous role.

## 3 The finitization of the infinite in the theory of numbers. From incomplete induction to the principle of the least to the principle of mathematical induction

**3.1 The method of proof by induction in Greek mathematics vs. proofs by the principle of mathematical induction.** Ancient mathematical induction: Proof($n$) for every P($n$), but no proof of $(\forall n)$P($n$).

The Greeks, in fact already the Pythagoreans, had an incomplete version of mathematical induction. The best specimen, as far as we know, is the Pythagorean proof by induction of *Pell's Diophantine equation*

$$q_n^2 = 2p_n^2 + (-1)^n$$

for the double sequence of the *side* $p_n$ and *diameter* $q_n$ numbers, defined recursively by

$$p_1 = q_1 = 1,$$
$$p_{n+1} = p_n + q_n, \quad q_{n+1} = 2p_n + q_n \quad \text{for } n = 1, 2, \ldots$$

The Pythagorean proof of the Pell equation is recounted by Theon Smyrneus [72] 44,18–45,8, Iamblichus, *Comments to Nikomachus* [39] 92,23–93,6, and mainly Proclus, *Commentary to Politeia* [64] 2,24,16–25,13 and 2,27,1–29,4. Some scholars (Freudenthal [24]) support, but more recently other (Unguru [79], Acerbi [1]) have challenged, a proof by induction. In our work [52, Chapter 8], reading the sources meticulously, we have some novel arguments indicating an inductive proof, *but without the principle of mathematical induction*. Since this may well be the very first inductive proof, we are outlining here our reconstruction of the argument.

According to the sources cited, the Pythagoreans noted that each rational diameter by itself a not a true diameter, because its square is not equal to double the square of its side (something impossible for numbers, by the incommensurability of the diameter to the side of a square, cf. Section 4.1, below), but equal to double the square minus 1 or plus 1; however, they found encouragement into the fact that the square of the diameter is double the square minus 1 or plus 1 in an *alternating* (*enallax*, Theon, Iamblichus) manner. In consequence, in order to ensure exact diameter type equality to side (*apisosis, isotes, apisosei* in Iamblichus; *isoteta* in Theon) they were led to form the sum of the square of a rational diameter *together with* (*meta*) the square of the next diameter, which sum is then exactly equal to the double (*ontos diplasion* in Proclus 2,25,11) of the square of the side *and* (*kai*, a connective different from *meta*) the square of the next side This led them to the discovery of the geometric Proposition II.10 of the *Elements* (the identity $(a + 2b)^2 + a^2 = 2((a + b)^2 + b^2)$ for any lines $a, b$).

Proposition II.10 is now employed twice.

At first geometrically (*grammikos*, linearly), in order to prove the so-called *elegant* (*glaphuron*) theorem, clearly turning II.10 into a proof of a (geometric) inductive step (if $a$ is diameter and $b$ side, then $a + 2b$ is diameter and $a + b$ side). The peculiar language which was used by Proclus to describe the basic heuristic idea of the Pythagoreans (first the word *meta* and next the word *kai*, Proclus 2,25,9–13), is now used in the application of Proposition II.10 for *the proof of the elegant theorem* (2,27,24–2,28,4), thus revealing that the Pythagorean heuristics was in fact motivated by an inductive argument. We conjecture that the Pythagorean attempt to turn the rational diameter into a real, geometric one marked the Birth of Induction.

Finally, II.10, in an arithmetic form, immediately deduced from the geometrical, is used for the proof of every inductive step of the Pell equation. Again the fact that every inductive step of Pell's equation (2,28,17,21; 2,28,22–24; 2,28,27–29,1) is phrased in language identical with the *statement of the elegant theorem* (2,27,13–16), indicates that the elegant theorem is the model for every inductive step of Pell's equation. Thus the Pythagoreans proved the following

**Proposition P($n$).** *Let $n$ be a given natural number. The $n$th side number $p_n$ and diameter number $q_n$ satisfy Pell's Diophantine equation*

$$q_n^2 = 2p_n^2 + (-1)^n.$$

The Pythagorean proof, reconstructed according to above arguments, now runs smoothly as follows:

Proposition P(1) clearly holds.

We obtain a proof Proof(2) for Proposition P(2), by applying Proposition II.10 with $a = p_1$, $b = q_1$, using the recursive definition, and using the fact that Proposition P(1) holds.

We next obtain Proof(3) for Proposition P(3) by applying Proposition II.10 with $a = p_2$, $b = q_2$, using the recursive definition, and using the fact that Proposition P(2) holds. It is clear that continuing in this way we can obtain Proof($n$) for Proposition P($n$).

The number of steps of these separate proofs Proof($n$) is going to infinity as $n$ is going to infinity (and thus ad infinitum, *kai aei outos*, 2,29,4). So the ancients had no one proof for all $n$ *simultaneously*.

They had a proof Proof($n$) for P($n$) for every $n$, but they had no proof for the Proposition $(\forall n)$P($n$), because this proof would have infinite number of steps.

We may call the ancient induction *unfinitized induction*. Greeks never finitized mathematical induction.[2]

The situation is akin to the situation in geometry, with Postulate 2 in the absence of Postulate 5.

**3.2 The principle of the least in the arithmetical book VII of the *Elements*.** The Greeks never discovered complete induction, but nevertheless found a satisfactory solution to the problem by switching, consciously or unconsciously, to the *Principle of the Least* (or, of the well-ordering of the natural numbers): every non-empty subset of $\mathbb{N}$ has a least element.

The Principle of well-ordering of natural numbers is used twice in Book VII of the *Elements*,

[2] Plato's *Parmenides* 149a7c3 has been proposed, in [83] and also in [1], as a perfect specimen of ancient proof by mathematical induction; however our analysis in [47, Section 3.6] shows that Plato means this to be an argument involving *only finite* induction, since there are only finitely many "hapseis"/logoi in every true Being, forming a period.

(a) for the proof that every number is a prime or divided by a prime (Proposition VII.31), and

(b) for the proof that the anthyphairesis/Euclidean algorithm of two numbers is finite, with last remainder the Greatest Common Divisor of the two numbers (Proposition VII.2).

The Principle of the Least is very clearly expressed in the proof of Proposition VII.31:

> for if it [a prime number] is not found, an infinite sequence of numbers will measure the number A, each of which is less than the other: which is impossible in numbers (Euclid's *Elements* VII.31, 14–16).

From Proposition VII.31, the second application of the Principle of the Least, it is straightforward that every number is equal to a product of prime numbers, from Proposition VII.30, using Proposition VII.20 (if $a/b = c/d$ and $a$, $b$ are relatively prime, then there is $k$ such that $c = ka$, $d = kb$), itself a consequence of Propositions VII.1 & 2, the first application of the Principle of the Least, it follows that the analysis in primes is unique. Thus, although not stated explicitly, these two propositions essentially contain the Fundamental Theorem of Arithmetic (every number is a product of primes in a unique way).

The Principle of the Least is a principle *Finitizing the Infinite*, since there would be no reason *a priori* to exclude an infinite strictly descending sequence of the infinite set of natural numbers. It is perhaps best seen that it is in fact a Finitizing principle if we consider that, as it is known today, it is equivalent to Peano's Principle of Mathematical Induction, which is clearly a Finitizing Principle, as explained in Section 3.3.

The power of the Principle of the Least is evident, as all elementary Theory of Numbers is a consequence of this Principle.

**3.3 The principle of mathematical induction is a *finitization of the infinite* in arithmetic.** The *Principle of Mathematical Induction*: If a subset $A$ of the set of natural numbers contains the number 1, and if $n$ is an element of $A$ then $n + 1$ is also an element of $A$, then $A$ coincides with the set of natural numbers.

It has been introduced in modern Mathematics by Pascal [61] and formally by Peano. The Principle of Mathematical Induction is equivalent to the Principle of the Least. It is a *Finitization of the Infinite*, since the infinite number of steps required to prove an arithmetical statement with mathematical induction are reduced to *just two* steps. As such we expect that the principle is indeed extremely powerful, with great deductive power. Indeed, all elementary Number Theory, including the Fundamental Theory of Arithmetic is a consequence of the principle of Mathematical Induction.

**3.4 Is mathematical induction contradictory?** Since Mathematical Induction is, being a Finitization of the Infinite, an approximation of the Contradictory, we are in danger of introducing a contradiction in Mathematics. Is Mathematical Induction a consistent axiom? Edward Nelson [54] believed that inductive reasoning on large exponentials of numbers hides circular reasoning. Although his views have not been widely accepted, and although we do not believe that mathematical induction introduces some kind of contradiction, nevertheless Nelson's views do show that flirting with finitizations of infinity could be dangerous, and not necessarily routine or "innocent".

## 4 The Pythagoreans–Theaetetus finitization of the infinite in incommensurability with periodic anthyphairesis and their shaping of the philosophy of the Pythagoreans, Zeno and Plato

**4.1 The Pythagorean discovery of the incommensurables.** The great mathematical discovery of the incommensurability by the Pythagoreans was a momentous event for Greek Mathematics.

**Definition X.1** of the *Elements*. Two magnitudes $a, b$ are *commensurable* if there are natural numbers $m$, $n$, and a magnitude $c$, such that $a = mc$ and $b = nc$; and, *incommensurable* if they are not commensurable.

The greatest mathematical discovery of the Pythagoreans was the discovery of the incommensurability of the diameter/diagonal to the side of a square.

**Definition** (implicitly given in Propositions VII.1 & 2, X.2 of the *Elements*). If lines $a$, $b$, and $a > b$, then the anthyphairesis of $a$ to $b$ is defined as follows:

$$\begin{aligned} a &= k_1 b + c_1, \quad c_1 < b, \\ b &= k_2 c_1 + c_2, \quad c_2 < c_1, \\ c_1 &= k_3 c_2 + c_3, \quad c_3 < c_2, \\ &\dots \end{aligned}$$

If this division process never ends, then the anthyphairesis of $a$ to $b$ is *infinite*, otherwise *finite*.

According to Proposition X.2 of the *Elements*, if the anthyphairesis of $a$ to $b$ is infinite, then $a$ and $b$ are incommensurable. (Anthyphairesis is the precursor of modern *continued fractions*, incommensurability of modern *irrationality*, and Proposition X.2 of the statement: if the continued fraction of a real number is infinite, then the real number is irrational).

The Pythagoreans proved the incommensurability of the diameter to the side of a square by proving that the anthyphairesis of $a$ to $b$ is infinite and then employing Proposition X.2.

The incommensurability cannot be obtained simply from the definitions, since an infinite number of constructions are needed to verify the hypothesis of Proposition X.2, namely that the anthyphairesis of $a$ to $b$ is infinite ($a = b+c_1$, $b = 2c_1+c_2$, $c_1 = 2c_2+c_3$, and so on *ad infinitum*). The Pythagoreans overcame this difficulty and succeeded in *finitizing the infinite* anthyphairetic procedure, by means of the preservation of the Gnomons, equivalently, the preservation of the application of areas in the anthyphairesis of the diameter to the side of a square:

$$a^2 = 2b^2;$$
$$a = b + a_1, \text{ hence } (b + a_1)^2 = 2b^2, \text{ namely } a_1^2 + 2ba_1 = b^2,$$
application of areas in excess;
$$b = 2a_1 + b_1, \text{ hence } a_1^2 + 2(2a_1 + b_1)a_1 = (2a_1 + b_1)^2,$$
$$\text{namely } a_1^2 = b_1^2 + 2a_1b_1, \text{ application of areas in excess,}$$
identical to the previous one.

It follows that all steps from now on will be identical. This is the precursor of the Logos criterion for the *periodicity of the anthyphairesis* (here $b/c_1 = c_1/c_2$). Hence, $\text{Anth}(a, b) = [1, 2, 2, 2, \ldots]$.

### 4.2 Theaetetus' great theorem on palindromically periodic incommensurability and the first restricted model of the reals, and his theory of ratios of magnitudes based on equality of anthyphairesis, the first restricted model of the reals.

#### 4.2.1 Theaetetus theorem.

We start by stating this theorem:

**Theaetetus' theorem.** *If $a$, $b$ are lines, such that $a$, $b$ are incommensurable, but $a^2$, $b^2$ are commensurable, then the anthyphairesis of $a$ to $b$ is palindromically periodic. (In modern terminology: if $N$ is a non-square natural number, then the continued fraction of $\sqrt{N}$ is palindromically periodic).*

There is no ancient source attributing this remarkable theorem to Theaetetus, or to any ancient mathematician.

That Theaetetus has indeed proved this theorem follows:

- on the one hand from our analysis of the Platonic dialogues *Theaetetus*, in which Plato describes Theaetetus' mathematical contributions and then declares his will to imitate his method in philosophy; *Sophistes*, in which Plato defines the Angler and the Sophist by a philosophical method that clearly imitates periodic anthyphairesis; and especially *Politicus*, in which Plato defines the Statesman by a philosophic analogue of a palindromically periodic anthyphairetic division, and then essentially states the theorem when he speaks of two measurement of an entity one against its opposite, second against the [geometric] mean (283b–287b); and

- on the other hand, because the contents of the Theaetetean Book X of the *Elements* (especially the two central definitions of lines, the apotome and the binomial line, and Propositions X.112–114 on the conjugation of the apotome and the binomial lines) are all the tools needed to reconstruct its proof [48, 51].

The proof employs the Principle of the Least and the Fifth postulate. The mixture of incommensurability in length and commensurabilty in square produces pigeonhole principles from which both periodicity and palindromicity follow. Plato in the *Politicus* 272d6–e6 clearly takes note of the application of pigeonhole to go from the Cronus era (the first half of the period) to the Zeus era (the second half of the period, palindromic to the first).

Theaetetus' theorem may be thought of as a far-reaching generalization of the Euclidean algorithm for pairs of numbers (Proposition VII.1 & 2 of the *Elements*) or of pairs of commensurable lines (Proposition X.3) shown to have finite anthyphairesis, to pairs of incommensurable lines that have commensurable squares shown to have infinite but periodic anthyphairesis.

This highly non-trivial theorem was rediscovered in the 18th century by Euler [19] and Lagrange [42, 43].

#### 4.2.2 Theaetetus' theory of ratios of magnitudes based on equal anthyphairesis.

**Theaetetus' definition of equality of ratios for magnitudes.** Two pairs $a$, $b$ and $c$, $d$ of magnitudes are *analogous* ($a/b = c/d$) if $\mathrm{Anth}(a,b) = \mathrm{Anth}(c,d)$, where $\mathrm{Anth}(a,b)$ is the sequence of successive quotients of the anthyphairesis of $a$ to $b$.

That there was a pre-Eudoxian theory of ratios of magnitudes with this definition is mentioned by Aristotle in the *Topics* 158b24–35. The definition can be seen to be a direct generalization of the Pythagorean definition of analogy, employing the Euclidean algorithm (Definition VII.21 and Proposition VII.1 & 2 of the *Elements*):

> $a/b = c/d$ if and only if, setting $k =$ greatest common divisor of $a$, $b$ and $l =$ greatest common divisor of $c$, $d$, there are numbers $m$, $n$, such that $a = mk$, $b = nk$, and $c = ml$, $d = ml$, since from the Euclidean definition it is evident that $\mathrm{Anth}(a,b) = \mathrm{Anth}(c,d)$.

**4.2.3 Theaetetus' finitization of the infinite.** Theaetetus in Book X of the *Elements* introduced a *Postulate* in the form of

**Definition X.3:** A line a is *rational* ("rhetos", with logos, ratio), resp. *alogos* (without logos, ratio), with respect to a given ("protetheisa") line $b$ if $a^2$, $b^2$ are commensurable, resp. incommensurable.

The *basic Proposition*, proved by making essential use of the palindromic periodical anthyphairesis theorem and certainly making no use of Eudoxus principle, is the following pre-Eudoxian analogue of Proposition V.8 of the *Elements*:

> if the pairs $a, b$ and $a, b'$ have each periodic anthyphairesis,
> and $a/b = a/b'$, then $b = b'$.

The Theaetetus Postulate/Definition X.3 determines a very restrictive class of pairs of magnitudes for the Theaetetus theory of ratios of magnitudes, essentially those having periodic anthyphairesis.

Exactly because of the restrictive Definition X.3/Postulate, Theaetetus theory constitutes a *Finitization of the Infinite*, exactly in the sense that Postulate 5 in Euclidean Geometry constitutes a Finitization of the Infinite. Indeed, from the definition alone of the equality of ratios (when there is equality of anthyphairesis) a proof that the two pairs of lines $a$, $b$ and $c$, $d$ are analogous ($a/b = c/d$) cannot be obtained, since we might have to go through an infinite number of steps to verify this equality, if these anthyphaireses are infinite. But periodicity of the anthyphairesis for the pairs allowed by Theaetetus' definition of "rhetos", exactly on account of Theaetetus' theorem, turns the number of steps in the search finite – and this is exactly the Finitization of the Infinite effected by Theaetetus' Definition/Postulate. The situation described is analogous to the situation in Section 2 for parallelism: two lines cannot be shown to be parallel only from Postulate 2, since an infinite number of constructions are needed, and finitization there occurs with the introduction of Postulate 5.

**4.3 The philosophical use of periodic anthyphairesis by the Pythagoreans, Zeno, and Plato.** The Pythagorean incommensurabilities have shaped Pythagorean philosophy in terms of Finitizing the Infinite, Zeno exploited the Pythagorean Finitizing of the Infinite for arguments in favor of the existence of "intelligible", as his teacher Parmenides wished, and Plato, aided by Theaetetus' powerful mathematical discoveries, greatly expanded the philosophical scope of both the Pythagoreans and the Eleatics, and created a most potent philosophic system.

**4.3.1 The philosophy of the Pythagoreans shaped by the anthyphairetic proof of incommensurability of the diameter to the side of a square.** The mathematical achievements of the Pythagoreans produced a Pythagorean philosophy that was in terms of the Infinite and the Finitizing. However it was neither Arithmetic nor Euclidean Geometry in general that produced this philosophy, but the Mathematics of Incommensurability. The philosophy of the Pythagoreans was not shaped simply by the discovery of the incommensurability, but by the method they followed for their discovery, namely by showing that the anthyphairesis of the diameter to the side of a square is infinite, with the similarity of the Gnomons acting as its Finitizer. In consequence the Pythagoreans adopted as universal philosophical principles the Infinite and the Finitizer. Details are given in Negrepontis [50], Negrepontis–Farmaki, 2019 [52, Chapters 7 and 9].

**4.3.2 Zeno's paradoxes exploit the Pythagorean infinite and finitizing.** Zeno used the Pythagorean discoveries on incommensurability and their anthyphairetic proof as the model for constructing his paradoxes and arguments in support of the theory of his teacher Parmenides on the existence of, in Plato's subsequent language, "intelligible" Beings, separate and superior to the sensible entities. All of Zeno's famous paradoxes and arguments aim at showing, by contradiction, that the "real, intelligible Being", which has already taken the form of the philosophical analogue of a dyad of lines in periodic anthyphairesis, is distinguished from the sensible Beings.

The near-contradictory properties that Zeno uses to separate the intelligible from the sensibles, such as One (in sense of self-similarity) and Many (Fragment B1), Motion and Rest simultaneously (Third paradox of motion), Similar and Dissimilar simultaneously, infinite and finite simultaneously (Fragment B3), coincide with the properties of the intelligible One of the Second Hypothesis in the Platonic dialogue *Parmenides*, making clear that Zeno's true Beings are governed by the Principles of the (anthyphairetic) Infinite and the Finite, and rendering Zeno's arguments, both (a) philosophical arguments inspired from the Pythagoreans' discovery of incommensurability and (b) the principal precursor of Plato's philosophy (Negrepontis [47]).

**4.3.3 Plato's philosophy is centered on a philosophic analogue of the finitization of the infinite anthyphairesis of incommensurability.** *In earlier dialogues the Finitization is the Logos Crierion of anthyphairetic periodicity, in later the quadratic commensurability, leading by Theaetetus' theorem to anthyphairetic periodicity.*

Plato imitated philosophically Zeno and mathematically the Pythagoreans and Theaetetus, to form his theory of Platonic Ideas, a theory whose model is periodic anthyphairesis, and which dominated human thought for many centuries.

In the *Parmenides* the One of the Second Hypothesis, the paradigmatical intelligible Being, is the dyad One, Being, initially in the philosophic analogue of Infinite anthyphairesis, finitized by the Logos Criterion into a periodic one. The same form of the Finitization, an evolution from the Pythagorean Finitization in terms of Gnomons, can be found in the *Politeia, Meno, Phaedo, Sumposion, Sophistes.*

In the later dialogues Plato modified the Pythagorean Finitizing, keeping the same Infinite, exactly taking into account Theaetetus' theorem on quadratic incommensurabilities (Section 4.2.1). Thus now the Finitizing principle is the *second condition* in Theaetetus' Theorem, stating if $a$, $b$ are incommensurable lines, such that $a^2$, $b^2$ is commensurable, then the anthyphairesis of $a$, $b$ satisfies the *Logos Criterion of periodicity* and in fact the anthyphairesis is palindromically periodic; and by Theaetetus' Theorem, the new Theaetetean-type Finitizing principle "$a^2$, $b^2$ is commensurable" implies the old Pythagorean one, the *Logos Critrion of periodicity*. The later dialogues *Politicus*, *Philebus*, *Timaeus* bear the modifying influence of Theaetetean Mathematics.

A careful reading of the *Philebus* 23e–25e reveals that the Philebean Finite coincides with the commensurable, and the Philebean Infinite the incommensurable (Negrepontis [49]). In the Double Measurement (*Metretike*) passage 283b–287b of the *Politicus* it is made clear that the intelligible Platonic Being is a philosophic analogue of a dyad $a$, $b$ such that $a^2$, $b^2$ is commensurable and $a$, $b$ is incommensurable, exactly satisfying the hypotheses of Theaetetus' theorem (Section 4.2.1).

**4.3.4 Plato's paradoxes exploit the infinite and finitizing of periodic anthyphairesis.** The paradoxical Finitizing the Infinite nature of periodic anthyphairesis was well understood by Plato, who exploited it at the most in the dialogue *Parmenides*. He imitated some of the classical paradoxes of Zeno, for example, Zeno's third paradox of motion (the arrow paradox), stating that this true Being is both at rest and in motion, is repeated for the Platonic Ideas in the *Parmenides* 145e7–146a8. In fact, the *Parmenides* is a work full of scandalously paradoxical statements. An example will suffice:

> The One [of the second hypothesis, the paradigm of a Platonic Idea] is both younger and older than itself and the others, and is neither younger nor older than itself and the others (*Parmenides* 151e3–155c8)

These philosophical paradoxes have confused Platonic readers from antiquity down to the present day. Many modern scholars regard the whole of *Parmenides* as nothing but a bunch of contradictory statements, not realizing that they are perfectly consistent, but paradoxical, appearing to be contradictory, exactly because of their proximity to the contradictory.

## 5 The full model of the real numbers but with a weak finitization (principle of the least & Eudoxus principle) replaces the Theaetetus narrow model of the real numbers with strong finitization (anthyphairetic periodicity)

Even though Theaetetus had an internally near perfect mathematical system and Plato built on it the most powerful and influential philosophical system ever conceived, nevertheless it was soon realized that mathematically it was too narrow and inflexible.

The later part of Greek mathematics (Archytas, Eudoxus Archimedes) is best interpreted as an early preparation for the introduction of Dedekind cuts and the construction of the second full model of the real numbers and of Calculus.

Archytas moves away from proofs of incommensurability by periodic anthyphairesis, relying only on the Principle of the Least, producing incommensurabilities, not only quadratic as Theaetetus had, but also cubic as Theaetetus could not have, without any recourse to anthyphairesis, in fact employing only the arithmetical Principle of the Least, and the arithmetical Proposition VII.27. Book VIII of the *Elements* is essentially due to Archytas; his arithmetical non-anthyphairetic proofs of quadratic and cubic incommensurabilities are based on Propositions in Book VIII, themselves consequences of Proposition VII.27 (if $a, b$ are relatively prime numbers, then $a^2, b^2$ and $a^3, b^3$ are relatively prime).

Eudoxus, in order to develop his theory of ratios of magnitudes in Book V of the *Elements*, found necessary to introduce Eudoxus Principle (Definition V.4 of the *Elements*), essentially a Postulate, namely the statement

> For any two magnitudes $a$ and $b$, there is a natural number $n$ such that $a < nb$;

thus the set $\{n : n = 1, 2, \ldots, nb < a\}$ is always finite, thus finitizing infinity, since there is absolutely no reason for this set to be *a priori* always finite. The best way to realize the very weak finitization effected by Eudoxus Principle is to see this principle in action in Proposition X.1 of the *Elements*: it is shown there that Eudoxus Principle is equivalent to the statement that the sequence $(1/n)$, and therefore the sequence $(1/2^n)$, converges to zero, precisely in the modern "epsilontic" definition!

Eudoxus develops a general theory of ratios of magnitudes, in Book V of the *Elements*, the precursor of the real numbers, by Dedekind cuts on the rationals, relying only on the arithmetical Principle of the Least and a most weak condition of completeness, the Eudoxus Principle (Definition V.4), by which he is able to prove the fundamental Propositions V.8, and X.1.

Being a finitization of the infinite, Eudoxus' condition is expected, according to our approach, to have *some deductive power*. Indeed, Book VI of the *Elements* on geometric similarity is a remarkable application of two Finitizations, the Fifth Postulate acting in the straight direction and the combination of the Principle of the Least and Eudoxus Principle in the converse (as, e.g., with the so-called "Thales theorem", Propositions VI.2, 4, 5 of the *Elements*); and, the method of Exhaustion, in Book XII of the *Elements*, greatly extended by Archimedes, the precursor of modern Integral Calculus, is also an application of these finitizations.

On the other hand, remarkable as Eudoxus' theory is for its generality, as it has moved much further than the Theaetetus–Plato restricted model of the reals, it still is very weakly finitized; so the genius of Archimedes is needed to be able to achieve the quadrature of the parabola, namely to find the integral of $f(x) = x^2$. The Arabs (e.g., Ibn al-Haytham), trying to extend Archimedes method, expended great efforts for the integration of $f(x) = x^4$, while today a lowly student of Calculus, equipped with the Fundamental Theorem of Calculus, regards this as a simple exercise.

## 6 The full model of the real lines with strong finitization, the completeness axiom (Dedekind), calculus, mathematical analysis

In the 17th century Infinitesimal Calculus in its modern form was invented by Newton and Leibniz, with the contribution of many predecessors, and its fundamental importance, theoretical and in applications in Physics, was immediately recognized. Integral Calculus was related to the ancient method of Exhaustion, but for two centuries it was not realized that its foundation should be related to the foundation of the method of Exhaustion, namely Eudoxus theory of ratios. It was only in 1870, after the advent of Cantor's Set Theory that it was finally realized by Dedekind that the foundations of Calculus should be the real numbers, defined as Dedekind cuts on the rationals, a definition practically identical with the Eudoxian Definition V.5 of the *Elements*. The crucial new element, discovered by Dedekind, was the *Axiom of Completeness*, postulating the existence of the *supremum* of every nonempty bounded subset of the real numbers.

Completeness is a strong Finitization of the Infinity of real numbers. The most effective and convincing way to see this is by considering its topological equivalence:

Every closed and bounded interval of the real numbers is compact,

namely, every cover of the interval with any *infinite* collection of open intervals already has a *finite* subfamily that covers the interval. This certainly is a completely new and different type of Finitization of the Infinite, compared to the ancient ones. Whereas the Theaetetus–Plato finitization finitizes the infinity that may exist within each ratio, within every real number considered as a continued fraction, by periodicity, on the contrary Completeness finitizes whole intervals or even subsets of the reals by compactness.

Within a short time Calculus was based on mathematically firm foundations. The completeness of the real numbers has proved to have extremely strong deductive power, and filled with seeming paradoxes, indicating that it is a close approximation of the contradictory.

The deductive power of the completeness property has been proved legendary. Real numbers were used for the logically impeccable foundation of Calculus (continuous real-valued functions defined on the reals, Integral Calculus, Differential Calculus, and the Fundamental Theorem of Calculus) for the first time since Newton and Leibniz (Spivak [75]); for the foundation of Analytic functions (rigorous proof of Cauchy's integral theorem), Mathematical Analysis, Diferential Equations.

The Ascoli–Arzela theorem is an application of the completeness property of real numbers. The original deductive power of completeness is transferred to the Ascoli–Arzela theorem. Important applications of the Ascoli–Arzela theorem is the characterization of the compact subsets of Lp, and the Rellich–Kondrachov compactness theorem for Sobolev spaces. The reader would consult Brezis [5].

Together with its enormous deductive strength, the completeness property, as a near-contradictory statement is expected to have *paradoxical consequences*. There do indeed appear scandalously paradoxical consequences: such as Baire's category theorem, almost all reals are irrationals, in fact transcendentals, the exponential function is an exotic self-similar object ($f' = f$), almost all, in the sense of Baire category, continuous functions are nowhere differentials. Baire's Category Theorem is essential in the proof of basic theorems in the theory of Banach spaces, the open mapping theorem and the uniform boundedness principle.

The near contradictory Calculus, especially in its beginning, when it had no firm foundations, appeared outright contradictory; thus G. Berkeley, *The Analyst*, 1734, wrote:

> The *fallacious* way of proceeding to a certain Point on the Supposition of an Increment, and then at once shifting your Supposition to that of no Increment ... Since if this second Supposition had been made before the common Division by 0, all had vanished at once, and you must have got nothing by your Supposition. Whereas by this Artifice of first dividing, and then changing your Supposition, you retain 1 and $nx^{n-1}$. But, notwithstanding all this address to cover it, the *fallacy* is still the same [3, p. 25].

## 7 The axiom of choice is a finitization of the infinite in the axiomatization of Zermelo–Fraenkel set theory

The *Axiom of Choice*, an Axiom of Zermelo–Fraenkel Set Theory, is equivalent to the *well-ordering principle* (every set has a well-ordering, namely a total ordering such that every non-empty subset has a least element), and also equivalent to the Principle of Transfinite Induction (every set $X$ has a well ordering $\ll$, such that if $A$ is a subset of $X$, with (1) the smallest element of $X$ in the well-ordering $\ll$ is in $A$, and (2) if $x$ is any element of $X$ such that every $y \ll x$ is in $A$, then $x$ is in $A$, then $A = X$).

Of course the general well-ordering Principle is a generalization of the well-ordering of the natural numbers, and the principle of transfinite induction the corresponding generalization of the principle of mathematical induction.

The principle of mathematical induction is, as already mentioned in Section 3, equivalent to the statement that the natural order of the natural numbers is a well-ordering. Thus the Axiom of Choice is a vast generalization of the principle of mathematical induction to every set.

It is clear that with a reasoning analogous to the reasoning in Section 3 on the principle of mathematical induction, we conclude that an impossible problem of *infinite nature*, has been turned, by the principle of transfinite induction, into a perfectly feasible procedure in two steps; and, thus, the Axiom of Choice is a Finitization of the Infinite vastly stronger than the principle of mathematical Induction, and therefore a near-contradictory statement.

As expected, the Axiom of Choice has great deductive power (e.g., every linear space has a Hamel base, the Stone representation Theorem for Boolean Algebras, Tychonoff's Theorem for the product of compact spaces, Krein–Milman's Theorem on extreme points, the Hahn–Banach Theorem).

Being a near-contradictory statement, the Axiom of Choice does not only have great deductive power, but produces striking paradoxes, such as the existence of Lebesgue non-measurable set, the Hausdorff paradox, the Banach–Tarski paradox.

## 8 The Stone–Čech compactification $\beta\mathbb{N}$ of the natural numbers is finitizing the infinite of the natural numbers $\mathbb{N}$

The combination of two powerful finitizing principles, already employed in Antiquity (e.g., "Thales" theorem is the result of combining the Fifth Postulate with Eudoxus principle and the Least Principle) can be expected to have strong and paradoxical consequences. The combination of the Axiom of Choice and the completeness of the reals produces strong consequences in Functional Analysis, such as the Uniform Boundedness Principle, the Hahn–Banach Theorem, Alaoglu's Theorem.

**8.1 The space of the Stone–Čech compactification, the space of ultrafilters of natural numbers.** One especially powerful consequence of these two finitizing Principles is the *Stone–Čech compactification* $\beta\mathbb{N}$ *of the natural numbers*, arguably the most fascinating object in Mathematics,

The construction of $\beta\mathbb{N}$:

Let $I = [0, 1]^{\mathbb{N}}$, as an index set (of the cardinality of the continuum). By the Axiom of Choice (Tychonoff Theorem) and the completeness property ([0, 1] is compact), $[0, 1]^I$ is a compact space.

We define $\Phi\colon \mathbb{N} \to [0, 1]^I$ by $n \to (f(n))_{f \in I}$. Then $\Phi$ is an embedding of $\mathbb{N}$, the closure of $\Phi(\mathbb{N})$ in the product is compact, and this is, by definition, the Stone–Čech compactification of $\mathbb{N}$, denoted by $\beta\mathbb{N}$.

$\beta\mathbb{N}$ has, by its very construction above, a *universal extension property*: for every $f\colon \mathbb{N} \to \beta\mathbb{N}$, there is a unique continuous extension $f^*\colon \beta\mathbb{N} \to \beta\mathbb{N}$.

**Definition.** An ultrafilter on $\mathbb{N}$ is, by definition, a family of non-empty subsets of $\mathbb{N}$, closed under the Finite Intersection Property and under taking supersets.

**Proposition.** *$\beta\mathbb{N}$ is the space of all* ultrafilters *on $\mathbb{N}$ (Comfort–Negrepontis* [13]*).*

Thus $\beta\mathbb{N}$, because it is a compact space in which $\mathbb{N}$ is dense, can be seen as a *finitization of the infinite* set of natural numbers, and in fact a *gigantic finitization*.

Since $\beta\mathbb{N}$ is a very strong finitization of the infinite, we expect that $\beta\mathbb{N}$ will be an object of great deductive power, and since it is a near-contradictory object we expect that it will produce strong paradoxes. Here are some indications of the deductive and paradoxical power of ultrafilters.

**8.2 A non-trivial ultrafilter exists by the axiom of choice and it has the power to prove Ramsey's theorem.**

**Ramsey's Theorem** (1930 [66]). *For every n, and every finite partition (coloring) of $[\mathbb{N}]^n$ (= $\{F\colon F$ finite subset of $\mathbb{N}$ of cardinality $n\}$), there is an infinite set A such that $[A]^n$ is monochromatic.*

Ramsey's Theory is about partitions of large structures, and thus generalizes the pigeonhole principle. According to Terence Tao [77, p. 101],

> ultrafilters can be used to simplify a lot of infinitary Ramsey theory, as all the pigeonholing has been done for you in advance.

In fact, there is a simple proof of Ramsey's infinitary theorem, making use of just one non-trivial (non-principal) ultrafilter.

**8.3 An idempotent ultrafilter exists (Galvin, Glaser) by the axiom of choice and compactness.**

**Lemma 8.1.** *$\beta\mathbb{N}$ is a left-continuous compact semigroup [Applying this universal extension property to the operation of addition $+\colon \mathbb{N}\times\mathbb{N}\to\beta\mathbb{N}$, in every variable separately, we obtain in two steps the extension $^*$ of $+$ to $\beta\mathbb{N}\times\beta\mathbb{N}$. The operation $^*$ turns $\beta\mathbb{N}$ into a compact semigroup, and $^*$ is separately continuous in one of the two variables (say left continuous)].*

(Note here that addition $+$ does not extend to a fully continuous function on $\beta\mathbb{N}\times\beta\mathbb{N}$, since $\beta(\mathbb{N}\times\mathbb{N})$ is different, richer than $\beta\mathbb{N}\times\beta\mathbb{N}$.)

**Explicit definition of $p^*q$.** Let $p, q$ be ultrafilters on $\mathbb{N}$. We say that $A$ is in $p^*q$ if and only if $\{n\colon A-n \text{ is in } q\}$ is in $p$ (where $A-n$ is the translation of $A$ by $-n$), i.e., $p^*q = \{A\colon B \text{ in } q \text{ implies } A-B \text{ in } p\}$.

Fortunately separate continuity in one variable provides a gate, straight but just enough open, for the paradoxical object to be created. Exactly what was needed was a simple but remarkable result that already existed:

**Lemma 8.2** (Ellis [17]). *If $(K,^*)$ is a nonempty compact semigroup, such that the operation $^*$ is separately continuous in one of the two variables, then there is an idempotent element $p$ of $K$, namely $p^*p = p$.*

*Proof.* By Zorn's lemma (equivalent to the Axiom of Choice) there is a minimal nonempty compact subsemigroup. This must necessarily consist of one only element, and this element must be necessarily idempotent. ∎

**Theorem.** *There is an idempotent ultrafilter $p$, $p = p^*p$, on $\mathbb{N}$.*

Now an idempotent ultrafilter is an exotic and paradoxical, an almost contradictory object, and exactly for this reason, a most valuable object. Its existence makes essential use of the finitizing compactness of $\beta\mathbb{N}$.

**Remark.** Let $p$ be an idempotent ultrafilter. Then $p$ is clearly non-trivial and $A\in p$ if and only if $\{n\colon A-n\in p\}\in p$ if and only if $B\in p$ implies $A-B\in p$. For any $A\in p$, set $A^* = \{n\colon A-n\in p\}$, thus $A^*\in p$.

### 8.4 An idempotent ultrafilter has the power to prove Hindman's theorem.

**Hindman's theorem** (1974 [38]). *If $\mathbb{N}$ is finitely colored, then there is an infinite monochromatic subset $X$ of $\mathbb{N}$ such that all distinct finite sums of $X$ are monochromatic.*

Hindman's theorem was a remarkable result in infinitary combinatorics, but his proof was opaque. Then one of these miracles of Mathematics happened. It appears that Galvin had realized that what was needed for a proof of Hindman's theorem was the existence of an idempotent ultrafilter; and Glaser eventually noted that a proof of the existence of such an object was already essentially known, by Ellis' theorem.

*Proof (Galvin/Glaser, 1974, unpublished; appeared in* [12]*).* Let p be an idempotent ultrafilter on $\mathbb{N}$. Choose $A_0 \in p$ such that $A_0$ is monochromatic; choose $a_0 \in A_0 \cap A_0^*$.

Set $A_1 = A_0 \cap (A_0 - a_0) \setminus \{a_0\}$; choose $a_1 \in A_1 \cap A_1^*$.

In general, suppose $A_n \in p$, and $a_n \in A_n \cap A_n^*$.

Set $A_{n+1} = A_n \cap (A_n - a_n) \setminus (a_n)$; choose $a_{n+1} \in A_{n+1} \cap A_{n+1}^*$.

Set $X = \{a_0, a_1, a_2, \ldots\}$.

Then all finite (non-repeating) sums of $X$ are monochromatic.

E.g. $a_0 + a_2 + a_4$;

$$a_4 \in A_4 \subset A_3,$$
$$a_2 + a_4 \in A_2 \subset A_1$$
$$a_0 + a_2 + a_4 \in A_0. \qquad \blacksquare$$

More general Theorems: Milliken [45], Taylor [78], Farmaki–Negrepontis [20] (general Milliken–Taylor theorem for all Schreier sets).

**8.5 A minimal idempotent ultrafilter exists by repeated use of the axiom of choice and compactness and it has the power to prove van der Waerden's theorem.** But one can do even better, ascending to an even more paradoxical object.

**The Furstenberg–Katznelson theorem** (1989 [25]). *There is an idempotent ultrafilter on $\mathbb{N}$, minimal in the ordering among ultrafilters: $p \leq q$ if $p^*q = p$.*

The consequences of the existence of such an object are rich; some of them are (a) van der Waerden's theorem [80]: If $\mathbb{N}$ is partitioned (colored) in two or more finitely many sets, then one of them contains arbitrarily long arithmetical progressions; (b) Hales–Jewett's Theorem [34], an important combinatorial analogue of van der Waerden theorem; (c) Carlson [7] and Furstenberg–Katznelson [25] (Ramsey infinitary combinatorics); (d) Farmaki–Negrepontis [21] 2008 (general Ramsey theory for all Schreier sets).

For many more results about ultrafilters consider J. Konieczny [41].

## 9 Gödel's program for the continuum hypothesis suggests that near-contradictory axioms of very large cardinals are added to ZFC

Frege's inspired error, resulting in contradiction by assuming the existence of a class of elements that cannot in general be a set, and quickly corrected by Russell, may have eventually prompted Gödel to an exciting Finitization of the Infinite in Set Theory.

**9.1 Frege's inspired error and Russell's discovery.** The collection of all sets is (a proper class and) *not a set*. This was discovered when Frege committed his famous inspired mistake by introducing as an axiom in his logical system Basic Law V (Naive Comprehension Schema for Extensions): $\exists y \forall x (x \in y \equiv \Phi(x))$, namely for every formula $\Phi(x)$ there is a set $y$, the extension of $\Phi$, such that $y = \{x\colon \Phi(x) \text{ is satisfied}\}$.

Just as Frege's work [23] was about to go to press in 1903, Bertrand Russell [69] wrote to Frege, showing that Russell's paradox results from Frege's basic law V. Indeed apply Frege's comprehension law V for the self-referential, but perfectly well-defined formula $\Phi(x)$: "$x$ is not an element of $x$". By Frege's law, there is a set $A$ of all the sets $x$, such that $x$ is not an element of $x$. Then it can be seen that the set $A$ *both* is and is not an element of itself, and thus to assume its existence is contradictory and inconsistent.

Thus the hypothesis that there is a set of all sets is a contradictory statement.

**9.2 Cantor's obsession with the continuum hypothesis.** One of the outstanding problems of Set Theory is the *Continuum Hypothesis* (CH), the statement that every subset of the reals is either countable or with cardinality equal to that of the set of reals. Cantor believed the continuum hypothesis to be true, was obsessed by it, and tried for many years to prove it, in vain. It became the first on David Hilbert's list of important open questions that was presented at the International Congress of Mathematicians in the year 1900 in Paris. Kurt Gödel [29], 1938 proved Consistency with ZF. Paul Cohen [8–10] proved independence from ZF.

**9.3 Gödel's program on CH.** Gödel [30], in 1947, most surprisingly, suggested that the solution of CH might be decided by the introduction in set theory of axioms for *large cardinals*.

This came to be referred to as *Gödel's program*.

At first one is surprised: why should assumptions of existence of *very-very large, enormous sets*, have any bearing on the CH, a question about subsets of the real numbers? *The rationale for Gödel's program* was not explained by him, but within our framework it can be understood as follows:

**9.4 Our interpretation of Gödel's program: Large cardinals are near contradictory entities approximating the contradictory "set of all sets".** We can now explain Gödel's suggestion as follows: the continuum hypothesis can possibly be answered by postulating the existence of large cardinals (such as the strongly inaccessible), exactly because (sets with) these large cardinals are the (hopefully) consistent near-contradictory objects approximating the truly contradictory set of all sets. Thus, the principle of the near-contradictory that we propose assumes a fascinating turn with the appearance of an entirely new approximation of the contradictory in Set Theory, and specifically in the relation to the Continuum Hypothesis. Gödel in effect turns the rejectable Frege's outright contradictory statement into a fully acceptable creative near-contradictory one!

**9.5 The success of Gödel's program for the projective hierarchy (Solovay, Shelah and Woodin).**

- *The projective hierarchy*

Gödel's idea gave rich results in the *class of Projective subsets* of the reals. The class of projective sets, ramifies in an infinite hierarchy of length $\omega$ (= the first infinite ordinal), the *projective hierarchy*, consisting of the *Borel*, the *analytic* (A) (continuous images of Borel), the *co-analytic* (CA) (complements of analytic), PCA (continuous images of CA), CPCA (complements of PCA), and so on *ad infinitum*, classes of sets.

- *The perfect set property for analytic sets*:

*Souslin* (1917). Every Borel or even every analytic (by a result of Souslin) subset of the reals has the *perfect set property* (namely, it is either at most countable or else contain a homeomorphic copy of the Cantor set). Thus CH *is true for the class of analytic sets*.

- *Solovay's success on Gödel's program with measurable cardinals*:

The first success of Gödel's program was by Solovay [73, 74]: If there is a *measurable cardinal* [an extremely large hypothetical cardinal], then every PCA (continuous image of complement of analytic) set has the perfect set property, and thus CH *is true for the class of PCA sets*.

- *Shelah–Woodin's success on Gödel's program with supercompact cardinals*:

Solovay conjectured that other much larger cardinals could possibly settle CH for the higher levels of the projective hierarchy. This started the quest for much greater cardinals (Woodin, supercompact cardinals).

**Theorem** (Shelah–Woodin [71]; Woodin [88]). *If there is a* supercompact cardinal*, or, more generally, a proper class of Woodin cardinals, then every set of reals in L(R), the smallest transitive inner model of set theory that contains all the reals, in particular every projective set, has the perfect set property, and thus CH* is true for the whole class of projective hierarchy.

**9.6 On the verge of inconsistency: General CH & large cardinals.** At present there does not seem that still greater cardinals will decide the full CH hypothesis. But supercompact cardinals are here to stay in the search for the truth value of CH. Woodin [89] is looking for the so-called Axiom V =*Ultimate L*, an inner model of ZFC with a *supercompact cardinal*, which if realized will decide CH in the positive; and, the rival position of set theory, based on the so-called *Martin's Maximum* axiom by Foreman–Magidor–Shelah [22, 88], and deciding CH *in the negative*, can be shown to be consistent from the existence of a *supercompact cardinal*

It seems fair to say that at present Set Theory is trying to find solutions to open questions, such as the 140 year old CH, solely with near-contradictory axioms of very-very huge cardinals, flirting in a rather scandalous way with the contradictory. The title of a recent paper by some great experts on the field is indicative: *On the Verge of Inconsistency*, [27].

## 10 Conclusion: The near-contradictory nature of mathematics

We have shown that a substantial part of Mathematics (ancient, classical, current) receives its deductive power by approximating the all-powerful contradictory. It appears that the only source of deductive power in Mathematics is some suitable approximation of the contradictory, most of the time in the form of Finitizing an Infinity.

We tend to believe that the basic mathematical axioms express the ultimate *truth* in Mathematics and that their deductive power leads to the discovery of mathematical *truths* (e.g., the completeness property of the reals leads to the proof of the Fundamental Theorem of Calculus); and we also tend to believe that the contradictory is *false*, lying at the antipodes of mathematical truths. But in fact our examination has led us to a more complex and disquieting and even untenable position: the deeper and stronger mathematical truths are approximations, surely suitable and genial but approximations all the same, of the contradictory, that finitize the infinite, barely escaping contradiction.

**10.1 Gödel's incompleteness theorem.** In this general scheme, Gödel's Incompleteness Theorem, and more generally the existence of undecidable statements, such as the Continuum Hypothesis in Set Theory, can be seen as a positive, creative force, contributing in pushing Mathematics towards more powerful and near-contradictory approximations.

We have seen, in Section 3.1, that Greek mathematics had proofs by *unfinitized induction*. Thus the Pell statement P($n$): $q_n^2 = 2p_n^2 + (-1)^n$ (where $p_n$, $q_n$ are the side and diameter numbers) had a proof for each n, but the statement $(\forall n)$P($n$) had no proof. Gödel [28], in 1931, proved that even in Peano's Arithmetic (assuming the fully finitized Principle of mathematical Induction), there are (Gödel) statements

G, such that G($n$) has a proof for each $n$ but $(\forall n)$G($n$) has no proof. Such statements were shrouded in mystery, until Paris and Harrington [60] exhibited a Ramsey type Gödel statement. A mathematical, rather than a model-theoretic, proof was given by Ketonen, Solovay [40], and can be found in R. L. Graham, B. L. Rothschild, J. H. Spencer [32]). Thus Gödel's statements, and in general statements that are *undecidable*, namely neither provable nor refutable, in an axiomatic system, and thus *proof-unfinitized*, make imperative the need for a more powerful system, namely for a closer approximation of the contradictory, in which they have a proof, and are thus proof finitized.

## 11 Beauty in mathematics and beauty in general

Many mathematicians almost instinctively describe either a mathematical discovery of their own or a mathematical result that they studied and understood as "beautiful"; when pressed to explain they are often at a loss to provide a reasonable explanation of their feeling. Since our present study has revealed an underlying feature of the nature of Mathematics, the finitization of the infinite, the approximation of the contradictory, it would be expected that the reason for Beauty in Mathematics lies with its paradoxical nature, as well.

**11.1 Beauty itself in Plato's *Sumposion* 210a4–212a7 in terms of "ephexes" and "exaiphnes".** We return to Plato's Finitization of the Infinite (Section 4.3), which according to our interpretation corresponds to the description of the Platonic Idea as the philosophic analogue of periodic anthyphairesis. Plato, in the *Sumposion* 210a–212a, describes the ascent to ideal Beauty as at first "ephexes", a process gradual, sequential, and painful, suddenly and unexpectedly "exaiphnes" culminating with the revelation of knowledge and Beauty.

> He who would proceed in *true opinion (ton orthos ionta)* towards *Beauty* must not merely begin from his youth to proceed (*ienai*) to beautiful bodies (*epi ta kala somata*). *In the first place*, indeed, if his conductor guides him to *true opinion (orthos hegetai)*, he must be in love with one particular body, and beget beautiful "logoi" (gennan kalous logous) therein;
>
> *but next* he must remark how the beauty attached to this or that body is cognate to that which is attached to any other, and that if he means to ensue beauty in form, it is gross folly not to be of the *opinion (hegesthai)* as one and the same the beauty belonging to all; and so, having grasped this truth, he must make himself a lover of all beautiful bodies, and

slacken the stress of his feeling for one by despising it and *being of the opinion (hegesamenon)* that it is a trifle.

*But his next advance* will be to set *a higher opinion (timioteron hegesasthai)* on the beauty of souls than on that of the body, so that however little the grace that may bloom in any likely soul it shall suffice him for loving and caring, and for *begetting (tiktein)* and soliciting such "logoi" as will tend to the betterment of the young; and that finally he may be constrained to contemplate the beautiful (*kalon*) as appearing in our customs (*epitedeumasi*) and our laws (*nomois*), and to behold it all bound together in kinship and so be of the opnion (*hegesetai*) the body's beauty as a slight affair.

From customs he should be led on to the branches of knowledge, that there also he may behold a province of beauty, and by looking thus on beauty in the mass may escape from the mean, meticulous slavery of a single instance, where he must center all his care, like a lackey, upon the beauty of a particular child or man or single customs; and turning rather towards the vast sea (*to polu pelagos*) of the beautiful may by contemplation of this bring forth in all their splendor many fair fruits of discourse and meditation in a plenteous crop of philosophy; until with the strength and increase there acquired he descries a certain single knowledge connected with a beauty which has yet to be told. And here, I pray you, said she, give me the very best of your attention.

When a man has been thus far taught (*paidagogethei*) in the matters of love (*erotika*), passing from view to view of beautiful things, in the *right, true sequential, succeeding order* (*ephexes te kai orthos*, 210e3), as he draws to the close of his dealings in love, *suddenly* (*exaiphnes*, 210e4) *he will have revealed to him, a wondrous (thaumaston) vision, beautiful in its nature*; and this, Socrates, is the final object of all those previous pains. 210a4–e6

Do but consider, she said, that there only will it befall him, as he sees the beautiful through *the cause* that makes it visible, to generate not illusions of virtue (eidola aretes), since his contact is not with illusion, but true examples of virtue, since his contact is with truth. So when he has begotten (tekonti) a true virtue and has reared it up (threpsamenoi) he is destined to win the friendship of Heaven; he, above all men, is immortal (athanatoi). 212a2–7 [translation by H. N. Fowler [62, volume 9], with modifications by the authors]

**11.2 The interpretation of Plato's nature of beauty in terms of mathematical finitization of the infinite.** Now we will interpret Plato's vision of the Idea of Beauty and the immortality it bestows on the one who obtains the knowledge and begets such beauty. The key description is the terms of "ephexes" and "exaiphnes". The "ephexes" describes the successive stages of true opinion (doxa, hegetai, ienai, ...), describes the infinite anthyphairetic Division of an intelligible Being, while the "exaiphnes" is the Logos Criterion, namely the Logos of the periodicity of the intelligible anthyphairesis, equivalently, the Collection, namely, the equalization of parts resulting from periodicity.

Next, we realize that the dramatic description given in the *Sumposion* is valid not only for the Idea of Beauty, but for every intelligible Being. For example in the *Meno*, the "ephexes" is likened to a travel from Athens to Larisa (97a), and the "exaiphnes", the sudden and unexpected moment that knowledge is attained by the *recollection* (*anamnesis*) (98a), by the repetition of Logos.

In general, an intelligible Being is a philosophic analogue of the mathematical periodic anthyphairesis, and "ephexes" is an initial segment of the Infinite anthyphairetic division, while "exaiphnes" is the Finitization of the Infinite, occurring exactly at the moment of the completion of the anthyphairetic period, and the appearance of the repetition (recollection) of the Logos, namely the establishment of the Logos Criterion.

The infinite anthyphairetic division is finitized by the finiteness of the period of the "logoi"/ratios of successive remainders of the anthyphairesis. Thus Plato's model for Ideal Beauty is mathematical, periodic anthyphairesis. Now it becomes clear that we sense Beauty and attain Knowledge "exaiphnes", abruptly, exactly at the moment that we achieve logos of periodicity, namely exactly at the moment of finitization of the infinite. So at the end, *general beauty*, according to Plato, is described in terms of a philosophical analogue of a purely mathematical finitization of the infinite, namely periodic anthyphairesis. This finitization coincides with our sudden and unexpected acquisition of knowledge.

The feeling of achieving, of participating in, *Immortality* that Plato mentions at *Sumposion* 212a results presumably because of the finitization, which brings within human accessibility the hitherto unknown and inaccessible infinite.

**11.3 Mathematicians describe mathematical discovery and Beauty in terms of the mathematically inspired Platonic "ephexes/exaiphnes" model.** We will now examine and try to explain the aesthetic pleasure and enjoyment felt when we study and try to understand the Mathematics created by others, or when Mathematics is created; and the way various mathematicians have described mathematical discovery, and Beauty in Mathematics. Since, according to our proposal, the fundamental feature of Mathematics is the paradoxical, near-contradictory nature of its basic hypotheses, we expect that Beauty in Mathematics must be a result of this near-contradictory power of Mathematics.

Theaetetus' inspiration to Finitize the Infinite anthyphairetic division by "Logos" and periodicity:

> *Theaetetus*: Theodorus here was drawing some figures for us in demonstration about powers (*dunameon*), showing that squares containing three square feet and five square feet are not commensurable (*mekei ou summetroi*) in length with the one foot, and so, *proceeding with each one in its turn separately (kata mian hekasten proairoumenos)*, up to the square containing seventeen square feet, and at that he stopped.
>
> *Now*, since each of the powers appeared to be infinite in multitude, *we had an inspiration* (hemin eiselthe ti toiouton) to attempt to collect this multitude into one (sullabein eis hen), by which we could call each of the powers (Plato, *Theaetetus* 147d3–e1).[3]

Archimedes' sudden revelation, "Eureka", according to Vitruvius (De Architectura [82, Book IX, 10]):

> But a report having been circulated, that some of the gold had been abstracted, and that the deficiency thus caused had been supplied with silver, Hiero was indignant at the fraud, and, unacquainted with the method by which the theft might be detected, requested Archimedes would undertake to give it his attention. Charged with this commission, he by chance went to a bath, and being in the vessel, perceived that, as his body became immersed, the water ran out of the vessel.
>
> Whence, *catching at the method to be adopted for the solution of the proposition*, he *immediately* followed it up, *leapt out of the vessel in joy*, and, returning home naked, cried out with a loud voice that *he had found that of which he was in search*, for he continued exclaiming, in Greek, "Eureka", (I have found it out)

Karl Friedrich Gauss spoke of "a sudden flash of lightning" [33, p. 15]:

> Thus Gauss, referring to an arithmetical theorem which he had unsuccessfully tried to prove for years, writes:
>
> Finally, two days ago, I succeeded, not on account of my painful efforts, but by the grace of God. *Like a sudden flash of lightning*, the riddle happened to be solved. I myself cannot say what was the conducting thread which connected what I previously knew with what made my success possible.

[3] [translation by H. N. Fowler [62], with modifications by the authors]

Henri Poincaré spoke of "suddenness and immediate certainty" [33, pp. 13–14]:

> Then I turned my attention to the study of some arithmetical questions apparently without much success and without a suspicion of any connection with my preceding researches. Disgusted with my failure, I went to spend a few days at the seaside and thought of something else.
>
> One morning, walking on the bluff, the *idea came to me*, with just the same characteristics of *brevity, suddenness and immediate certainty*, that the arithmetic transformations of indefinite ternary quadratic forms were identical with those of non-Euclidean geometry.

Godfrey Harold Hardy:

> The mathematician's patterns, like the painter's or the poet's must be beautiful; the ideas like the colours or the words, must fit together in a harmonious way. Beauty is the first test: there is no permanent place in the world for ugly mathematics [37, p. 14].
>
> What "purely aesthetic" qualities can we distinguish in such theorems as Euclid's[4] or Pythagoras's?[5] I will not risk more than a few disjointed remarks. In both theorems (and in the theorems, of course, I include the proofs) there is *a very high degree of unexpectedness*, combined with inevitability and economy. *The arguments take so odd and surprising a form*; the weapons used seem so childishly simple when compared with the far-reaching results; but there is no escape from the conclusions [37, p. 29].

Thus Hardy speaks of "a very high degree of unexpectedness" and for "arguments that take so odd and surprising a form." Hardy [37] regards as a basic feature of a beautiful theorem and proof in Mathematics its "unexpectedness", the "surprise", the sudden revelation with which "truth" and understanding is attained, and everything is clarified.

Timothy Gowers [31, p. 51]:

> It is a notable feature of the [second] argument that it depends on a single idea, which, though *unexpected*, seems very natural as soon as one has understood it.
>
> It often puzzles people when mathematicians use words like *"elegant", "beautiful", or even "witty"* to describe proofs, but an example such as this gives an idea of what they mean.

[4] Given in 11.4, Example 3
[5] Given in 11.4, Example 1

> Music provides a useful analogy: we may be entranced when a piece moves in an unexpected harmonic direction that later comes to seem wonderfully appropriate, or when an orchestral texture appears to be more than the sum of its parts in a way that we do not fully understand.
>
> Mathematical proofs can provide a similar pleasure with *sudden revelations, unexpected* yet natural ideas, and *intriguing hints* that there is more to be discovered.
>
> Of course, *beauty in mathematics* is not the same as *beauty in music*, but then neither is musical *beauty* the same as the *beauty* of a painting, or a poem, or a human face.

According to Timothy Gowers the most important element that contributes to beauty in Mathematics is the sudden, unexpected, the surprise, with which the revelation of a proof, that leads to understanding and knowledge, is realized.

Andrew Wiles [87]:

> Andrew Wiles likened the mathematical equivalent of experiencing the rapture of beauty to walking down a path to explore a garden by the great landscape architect Capability Brown, *when a breathtaking vista suddenly beckons*. In other words, elegance in mathematics 'is this *surprise* element of *suddenly* see everything clarified and beautiful.'

Exactly in the same spirit Andrew Wiles likens mathematical beauty, employing the image of someone walking in the road of one of the royal gardens created by the great architect of landscapes Capability Brown, when suddenly, surprisingly, and unexpectedly find himself in a view with which hitherto unclear and dark matters clear up and reveal their structure.

We find that there is a wide and impressive agreement with the "ephexes/exaiphnes" Platonic model. The Beauty of Mathematics is almost always connected with the sudden and unexpected understanding. Since Plato was referring to periodic anthyphairesis, a specific Finitization of Infinity, and since Mathematics, as we just saw, is grounded on various different Finitizations of Infinity, it is difficult to avoid the conclusion that Plato and the ancient and modern Mathematicians have the Finitization of Infinity as their common ground. We then associate mathematical Beauty with Finitization of Infinity.

Gian Carlo Rota [68] expresses a dissenting opinion:

> Hardy's opinion (expressed in his essay *A mathematician's apology*) that much of the beauty of a mathematical statement, or of a mathematical proof, depends on an element of *surprise*, is, in my opinion, mistaken.
>
> Nonetheless, despite the fact that most proofs are long, despite our awareness of the need for an extensive background in order to appre-

> ciate a beautiful theorem, we think back to instances of mathematical beauty as if they had been perceived by an instantaneous realization, in a moment of truth, like a light-bulb suddenly being lit.
>
> *All the effort* that went in understanding the proof of a beautiful theorem, all the background material that is needed if the statement is to make any sense, all the difficulties we met in following an intricate sequence of logical inferences, all these features disappear once we become aware of the beauty of a mathematical theorem, and what will remain in our memory of our process of learning is the image of an instant flash of insight, of a sudden light in the darkness. We would like mathematical beauty to consist of such a sudden flash.
>
> The phenomenon of enlightenment is seldom explicitly acknowledged among mathematicians, for at least two reasons: First, enlightenment is not easily formalized, like truth or falsehood. Second, enlightenment admits degrees; some statements are more enlightening than others.

It seems to us that Rota's dissent may well be accommodated in the "ephexes" and "exaiphnes" *Sumposium* scheme. Indeed, Rota's process of enlightenment does not correspond to the "exaiphnes", but since enlightenment includes the pains (cf. "ponoi" in *Sumposion* 210e6) and efforts that are needed for attaining understanding of Mathematics, and since it has "degrees" (like True Opinion in the *Sumposion*), Rota's enlightenment includes both the "ephexes" and the "exaiphnes" stage.

### 11.4 Mathematical proofs exhibit their beauty at the instant of finitization.

Hardy [37] examined two examples of proofs he considers beautiful from Greek Mathematics, which we will now consider in **I** and **III** below; we add a few more.

**I. *Proposition X.117* of the *Elements*.** The first example of a mathematical proof that Hardy [37, pp. 19–20] regards as beautiful is the proof of the incommensurability of a diameter to the side of a square. Hardy erroneously believes that this is the original Pythagorean proof, but as we saw (in § 4.1) the Pythagorean proof was anthyphairetic. The proof that Hardy has in mind was known in antiquity, in fact appears as a later addition to the *Elements* as Proposition X.117, and is the following: Let $a$, $d$ be the side and the diameter of a square, so that $d^2 = 2a^2$, and suppose that $d$, $a$ are commensurable. Then there are numbers $m$, $n$ and a line segment $c$, such that $a = mc$, $d = nc$. We then get the equation $n^2 = 2m^2$. Now, the basic point of the proof is the possibility that we have to assume that additionally the numbers $m$, $n$ are relatively prime. Why we may assume that $m, n$ are relatively prime? This is not a matter of routine, and if it is presented as such all essence and beauty of the proof evaporates. It is based on the construction of the greatest common divisor

(Propositions VII.1 & 2 in the *Elements*); but this construction as we saw was based on the *Principle of the Least*, a principle Finitizing the Infinite. Thus we may, and do, assume that $n^2 = 2m^2$ and that $m, n$ are relatively prime numbers. The rest is routine: the equation $m^2 = 2n^2$ implies that $m^2$ is even, hence $m$ is even, say $m = 2k$ for some $k$, hence $(2k)^2 = 2n^2$, hence $2k^2 = n^2$, hence $n^2$ is even, hence $n$ is even, finally a contradiction!

The knowledge of incommensurabilty, acquired by a surprise, by the "lighting of a bulb" and generating a feeling of beauty, is clearly due to a Finitization of the Infinite.

**II. *Tennenbaum's proof of incommensurability*.** There is a still more beautiful proof of the incommensurability of the diameter to the side of a square, in our opinion the most beautiful there is (and we feel certain that Hardy would agree). In its modern version it is due to (the unforgettable friend) Stanley Tennenbaum, who discovered it at about 1950. It remained unpublished and became widely known much later by John Conway [14, 15].

Let $a, d$ be the side and the diameter of a square, so that $d^2 = 2a^2$, and suppose that $d, a$ are commensurable. Then there are numbers $m, n$ and a line segment $c$, such that $a = mc$, $d = nc$. As in **I**, we get to the equation $n^2 = 2m^2$. Now employing nothing more than the Principle of the Least, a principle Finitizing the Infinite, we may, and do, assume that $m, n$ are the least numbers, such that $n^2 = 2m^2$. Now the secret of this new proof is to form the great square with side $n$, and to place inside it the two smaller squares with side $m$ below left and above right. Since $n^2 = 2m^2$, it is clear that the intersection of the two smaller squares must be equal to the sum of the two remaining squares located above left and below right.

It is easy to see that this equality takes the form

$$(2m - n)^2 = 2(n - m)^2,$$

and

$$0 < 2m - n < n, \quad 0 < n - m < m,$$

a contradiction, since we have found numbers that express the equality $d^2 = 2a^2$ and which are strictly smaller than the numbers $m, n$, respectively. The idea of this proof is completely different from the idea we used in **I**. We have talked of the modern version of this proof, because the proof just given is closely connected to, and results from, Proposition II.9 of the *Elements*. (Details can be found in our book [52]).

**III. Proposition IX.20 of the *Elements*.** A second example of a beautiful mathematical proof given by Hardy [37, pp. 18–19] is the following Proposition IX.20 of the *Elements*: *The multitude of the prime numbers is infinite*. Let us see the proof.

Let us assume that this multitude is finite, say $p_1, p_2, \ldots, p_N$. We set $M = 1 + p_1.p_2 \ldots, p_N$. At this point we invoke the seemingly "innocent" Proposition VII.31

of the *Elements*, according to which every natural number is divided by a prime number. But "innocence" is only apparent. Proposition VII.31 employs the *Principle of the Least*, a Principle that Finitizes the Infinite. The proof is now completed as follows: Let $p$ be a prime number that divides $M$, namely $M = kp$ for some natural number $k$. Clearly $p$ is one of $p_1, p_2, \ldots, p_N$, e.g., $p = p_7$. Then

$$1 = M - p_1.p_2 \ldots .p_N = p_7(k - p_1.p_2 \ldots p_6.p_8 \ldots p_N),$$

impossible! The final surprise and the "lightning of the bulb" is clearly due to the Finitization of the Infinite. The Pythagorean proof, briefly outlined in Section 4.1, is beautiful as well due the identical application of areas, a Pythagorean precursor of the Logos Criterion for periodicity, and thus Finitization of the infinite anthyphairesis.

**IV. Propositions I.35, 37, III.20 of the *Elements*.** Propositions I.35, 37 (every parallelogram, resp. triangle, with fixed base in one of the two parallel lines, and moving parallel side, resp. moving third vertex, in the other parallel line has constant finite area and perimeter increasing to infinity) and Proposition III.20 (all angles formed by a chord of a circle are constant) of the *Elements* are not described as "beautiful", but as "paradoxical" by the ancient scholiast Proclus (as we saw in Section 2.2), in the sense that they are a mixture of the Infinite and the Finite, and thus resembling a Platonic Idea. Since these Propositions are consequences of the Fifth Postulate, Proclus' comments in effect point at the fact that both the Fifth Postulate and Platonic Ideas are Finitizations of the Infinite.

**V. *The Bolzano–Weierstrass Theorem*.** The Bolzano–Weierstrass theorem is certainly one of the most powerful and beautiful in real numbers. The theorem, stating that every bounded sequence of real numbers has a convergent subsequence, can be proved as follows: An index $n$ is a peak point of the sequence $(a_n)$ if $a_m < a_n$ for every $m$ such that $m > n$.

*Claim 1*. If the set of peak points of the sequence $(a_n)$ is infinite, then the subsequence of $(a_n)$ determined by this infinite set is decreasing.

*Claim 2*. If the set of peak points of the sequence $(a_n)$ is finite, then, with mathematical induction, we can define an increasing subsequence of $(a_n)$.

*Claim 3*. Every sequence of real numbers has a monotonic subsequence [Immediate from Claims 1 & 2].

*Claim 4*. Every bounded monotonic sequence converges. [Indeed, if $(b_n)$ is a bounded increasing sequence, then the supremum of the set $\{b_n : n = 1, 2, \ldots\}$ exists, by the Completeness property of real numbers, and is clearly the limit of the sequence $(b_n)$. Dually, if $(b_n)$ is decreasing.]

Thus the surprising indeed final result is a consequence of mathematical induction (in Claim 2) and mainly of the Completeness property of the real numbers, both Finitizations of the Infinite.

Claims 1 & 2 form a crypto-Ramsey type argument, and so their use and the use of mathematical induction, may be replaced by an appeal to Ramsey's theorem, itself based on stronger Finitizations of the Infinite (as we have seen in Section 8).

**VI.** ***The definition of the exponential function.***

> Qui n'a été étonné en apprenant que la function: $y = e^x$, telle un phenix renaissant de ses cendres, est à elle-même sa propre derivée?[6] (F. Le Lionnais [44, p. 441])

The surprising and unexpected beauty of the exponential function is closely related with the employment of the Completeness Axiom in its definition. Indeed the definition of the number $e$ conforms to Plato's description of the approach to the beautiful:

"ephexes": $(1+1)^1 < (1+1/2)^2 < (1+1/3)^3 < \ldots < (1+1/n)^n < \ldots$, the Infinite, and

"exaiphnes": the limit, the exponential number $e$, by Finitizing the infinite;

a different Infinite from the one that Plato had in mind, and certainly a different Finitization from the one that Plato had in mind, but equally beautiful, producing, as in Plato's case, a kind of paradoxical self-similarity, in the sense that the derivative of the exponential function is itself.

**VII.** ***Pell's Equation: Finitization of quadratic incommensurability.*** A notable consequence of the Theaetetus palindromically periodic anthyphairesis of quadratic irrationals is the complete solution of the Pell equation for every non-square number $N$, namely, the finding of the pairs $p, q$ of natural numbers such that $q^2 = Np^2 + 1$.

The Hindu mathematicians Brahmagupta, Bhaskara II, and others had methods of solving Pell's equation. Van der Waerden [81], in 1976, believed that these Hindu methods go back to Greek sources, but his arguments were inconclusive. In our forthcoming work Negrepontis, Farmaki, Brokou [53], we present Theaetetus' contribution (as deduced from Plato's *Politicus* and the Theaetetean Book X of the *Elements*), compare it with Hindu sources, and conclude that Theaetetus had made substantial progress on, and even achieved a full proof of, Pell's equation.

[6] Who has not been amazed to learn that the function $y = e^x$, like a phoenix rising again from its own ashes, is its own derivative?

The existence, given a non-square number $N$, of a pair $p, q$ of natural numbers such that $q^2 = Np^2 + 1$ makes $\sqrt{N}$ almost rational, by $+1$, and is thus a highly a paradoxical situation. How can we find these solutions?

Every time the *convergents* $q_n$, $p_n$ of the infinite anthyphairesis/continued fraction of $\sqrt{N}$ completes a period, and thus every time that the infinite anthyphairesis is finitized by periodicity, we have a solution and the paradoxical validity of Pell's equation.

In conclusion, the feeling, that qualifies as Beauty in Mathematics, of suddenly and unexpectedly reaching deep understanding of a problem, of a theorem and its proof sits well with the paradoxical, near-contradictory deductive power of Mathematics: we obtain this sudden deep understanding exactly when, and only when, we connect the givens of the problem with the paradoxical, near contradictory axiom or basic hypothesis. Thus mathematical beauty is inextricably linked to the near-contradictory nature of Mathematics.

**11.5 The Platonic, hence ultimately mathematical, principles infinite and finite, are described by Proclus as the Dionysiac and the Apollonian, respectively. Most surprisingly, Nietzsche sets the Dionysiac and the Apollonian as the supreme principles of poetry and art.** The Platonic Scholiast Proclus, in the *In Timaeus*, surprisingly describes the opposition Dionysiac and Apollonian as one manifestation of the Platonic principles of Infinite and Finite:

> he [the Demiurgus] divides the soul into parts (diairei kata moiras), harmonizes the divided parts (ta dieiremena), and renders them *concordant with each other (sumphona allelois).* But in effecting these things, he acts at one and the same time *Dionysiacally and Apolloniacally (hama men Dionysiakos, hama men Appoloniakos).* For *to divide* and produce wholes *into parts*, and to preside over the distribution (dianomes) of kinds, is *Dionysiacal*; but to perfect *all things harmonically*, is *Apolloniacal*. As the Demiurgus, therefore, comprehends in himself the cause of both these gods, he *both divides and harmonizes* the soul (Proclus, *In Timaeus* 2, 197,13–23).
>
> The soul may have a signature (sunthema) of the *Dionysiacal* series (seiras), and of the *fabulous lacerations (mutheuomenou sparagmou)* of Bacchus. For it is necessary that it should participate of the *Dionysiacal* intellect; and as Orpheus say, that bearing of god on its head, it should be divided conformably to him.
>
> But it possesses *harmony* in these parts, as a symbol (sumbolon) of the *Apollonian* order (taxeos). For in the *lacerations* of Bacchus, it is *Apollo* who *collects (sunagon) and unites (henizon) the divided (meristhenta) parts* of Bacchus (Proclus, *In Timaeus* 2,198,7–14). [translation

by Thomas Taylor, [63, vol. II, pp. 77–78], with minor modifications by the authors].

On the other hand, Nietzsche, in *The Birth of Tragedy* [57] described these same two principles, the Dionysiac and the Apollonian, as the principles whose mixture produced Greek Tragedy and in fact all Greek civilization.

Nietzsche was not a friend of Plato (perhaps because of a basic misunderstanding on his part), and neither had much if any relation to Mathematics, and Plato was not a friend of Tragedy or even of Homer; so it is remarkable that these exact principles that Proclus describes as the Platonic Infinite and Finite, and that we interpreted as eventually mathematical, Nietzsche considers supreme principles for Art and Poetry.

Thus, Beauty in the Arts and in Poetry is being described by Nietzsche in terms of principles ultimately closely related to the mathematical principles of Infinite and Finite. (We have no idea if Nietzsche knew of Proclus' comments when he evoked these principles.)

**Note.** Our approach to Mathematical Beauty in terms of Finitization of the Infinite has a surprising connection, not only with Nietzsche's philosophy, but, as we now realize, with the wider approach to Beauty in the Arts by means of the Sublime and its association with Infinity, in the philosophical works, among others, of Shaftesbury, Burke, and Kant (cf. T. M. Costelloe (editor) [16, Chapter 4 by T. M. Costelloe, Chapter 2 by R. Gasche, Chapter 3 by M. McBay Merritt, and Chapter 7 by P. Guyer]. There seems to be a fascinating similarity of the attractive aspect of the initially menacing Sublime/Infinite, with our Finitized Infinity, both achieved with surprise and astonishment. Further study is needed to examine the nature of this similarity.

**11.6 M. Atiyah and S. Zeki experimental discovery: Beauty in the arts is closely related with beauty in mathematics.** As we saw, Plato in the *Sumposion* described Ideal Beauty in terms of a very special mathematical in nature finitization of the infinite, namely periodic anthyphairesis.

Nietzsche invokes as principles of the Arts the Dionysiac and the Apollonian, which, even if not identically interpreted, are certainly related to the Platonic Infinite and Finite, and, mathematicians describe mathematical Beauty in terms that are closely related to a finitization of infinity, one of the several modern mathematical ones, some of which we saw in earlier sections. These are thus strong indications of relations between mathematical Beauty and Beauty of the Arts in general.

Michael Atiyah co-operated with neuro-biologists in an experimental investigation of the relation of areas of the human brain activated by Beauty in the Arts and correspondingly with Beauty in Mathematics, and came up, in their paper [90], with the unexpected and remarkable conclusion that these two areas of the brain coincide:

> Results showed that the experience of mathematical beauty correlates parametrically with activity in the same part of the emotional brain, namely field A1 of the medial orbito-frontal cortex (mOFC), as the experience of beauty derived from other sources.

This is how Atiyah described this discovery to S. Roberts [67]:

> *Question*: Not too long ago you published a study, with Semir Zeki, a neurobiologist at University College London, and other collaborators, on The Experience of Mathematical Beauty and Its Neural Correlates.
> *Atiyah*: That's the most-read article I've ever written! It's been known for a long time that some part of the brain lights up when you listen to nice music, or read nice poetry, or look at nice pictures – and all of those reactions happen in the same place [the "emotional brain," specifically the medial orbitofrontal cortex]. And the question was: *Is the appreciation of mathematical beauty the same, or is it different? And the conclusion was, it is the same.* The same bit of the brain that appreciates beauty in music, art and poetry is also involved in the appreciation of mathematical beauty. *And that was a big discovery.*

A successful explanation of Beauty in Mathematics might open the way for an explanation of the beautiful in the Arts. Indeed, this correlation, or even identification, suggests that the process of attaining knowledge and producing Beauty in Mathematics by approximating the contradictory and finitizing the infinite is not unique in Mathematics, but is relevant, has analogous counterparts and must have consequences for the understanding of the nature of Beauty in general in the Arts and in Poetry, and other areas of human creativity.

## 12 A preliminary note on "the Unreasonable Effectiveness of Mathematics in the Natural Sciences"

The famous paper by Eugene Wigner [86] with the provocative title *The Unreasonable Effectiveness of Mathematics in the Natural Sciences*, in 1960, and his conclusion that this effectiveness is a "complete mystery", has been a strong impetus for our present investigation, although an indirect one (as we deal here only with pure mathematics itself). Our treatment of the Unreasonable Effectiveness will be preliminary and sketchy. Wigner's question has been studied among others, by R. W. Hamming [36]; M. Steiner, [76]; M. Colyvan [11]; J. Y. Halpern, R. Harper, N. Immerman, P. G. Kolaitis, M. Vardi, and V. Vianu [35], S. Bangu [2].

If the nature of Mathematics is indeed paradoxical, in the sense we have described, then we expect that this nature will account for its "unreasonable effectiveness", as well. In this preliminary and sketchy section on the "unreasonable effectiveness" of Mathematics in applications to Physics and elsewhere, we confine ourselves to argue that the hypothesis in Noether's theorem, a theorem fundamental for modern Theoretical Physics, namely the symmetry assumption, is analogous to the Fifth Postulate, and thus a powerful Finitization of the Infinite. It is our hope that other mathematicians or physicists more versed than we are in Mathematical Physics will greatly complete the picture.

As we saw in Section 2, Propositions I.35, I.37 of the *Elements* were considered paradoxical in antiquity; we were able to explain this paradoxical sense by the fact that they are consequences of the Fifth Postulate.

Newton derived Kepler's second law (equal areas in equal time) employing Proposition I.37 and a limiting infinitesimal argument (Newton, *De Motu* [55, Theorem 1]; cf. Brackenridge [4, pp. 78–85]; Newton, *Principia* [56, Proposition 1]; Erlichson [18]). As M. Nauenberg [46] 2003 writes, this Proposition is "justifiably regarded as the cornerstone" of Newton's *Principia*:

> In Prop. 1 of the *Principia* Newton gave a proof that Kepler's empirical area law for planetary orbits and the confinement of these orbits to a plane are consequences of his laws of motion for the special case of central forces. In his words,
>
> The areas which bodies made to move in orbits described by radii drawn to an unmoving center of force lie in unmoving planes and are proportional to the times.
>
> This proposition is justifiably regarded as a cornerstone of the *Principia*, because the proportionality between the area swept out by the radius vector of the orbit and the elapsed time enabled Newton to solve dynamical problems by purely geometrical methods supplemented by continuum limit arguments which he had developed.

The fact that Newton's Second Law implies Kepler's Second Law, which is equivalent to the conservation of angular momentum, suggests that Newton's Second Law contains some physical principle Finitizing Infinity.

Returning to Proposition I.35 we note that we can derive it in a different way, as follows: instead of using the Fifth Postulate itself, we replace it by its symmetry equivalent,

> every length, every angle, hence every orthogonal parallelogram is invariant under Euclidean translation.

Then, one can prove I.35, namely that the area of the orthogonal parallelogram with base the interval $[(a, 0), (b, 0)]$, height $h$, and opposite side the interval $[(a, h), (b, h)]$ is equal to the parallelogram with base the interval $[(a, 0), (b, 0)]$ and opposite side the interval

$$[(a + t, h), (b + t, h)]$$

by Calculus: consider for every partition

$$P = \{0 = h_0, h_1, \ldots, h_n = h\}$$

of the interval $h$, the sum of the horizontal slices, take the limit as the norm of the partitions $P$ goes to zero, so that the integral from 0 to $h$ is equal to the area of the second parallelogram which, employing the Euclidean invariance with respect to translation, is equal to $h.(b - a)$, the area of the orthogonal parallelogram.

With the symmetry formulation of the Fifth Principle in deriving I.35 & 37 and with the central importance of I.37 in the development of Newton's *Principia*, we now realize that Noether's theorem [58], in 1918, (every differentiable *symmetry* of the action of a physical system has a corresponding *conservation law*) is in some sense analogous to Proposition I.35, and its hypothesis, a differentiable symmetry of the action of a physical system, is a powerful far-ranging physical *Finitization of the Infinite* analogous to the simple Fifth Postulate Finitization. Indeed a "symmetry" condition intuitively states that the form of the "Law of Nature" conforming with the symmetry is the same everywhere or always as is in any one instance, similarly with the Fifth Postulate.

Considering the central position that Noether's theorem has acquired in Theoretical Physics, and that in every theory in Physics, be it Classical, or Quantum, or Relativity, or other, conservation laws are fundamental, we now might conjecture that the Unreasonable Effectiveness of Mathematics in the Physical Sciences is due to its paradoxical nature. The True value of the conditional False→True makes it possible for a hypothesis that is approximating the contradictory and is thus approximating Falsity to provide a satisfactory explanation for any experimental data. These considerations fit well with Wigner's original misgivings:

> Heisenberg's rules presupposed that the classical equations of motion had solutions with certain *periodicity properties*; and the equations of motion of the two electrons of the helium atom, or of the even greater number of electrons of heavier atoms, *simply do not have these properties*, so that Heisenberg's rules cannot be applied to these cases.
>
> Nevertheless, the calculation of the lowest energy level of helium, as carried out a few months ago by Kinoshita at Cornell and by Bazley at the Bureau of Standards, agree with the experimental data within the accuracy of the observations, which is one part in ten millions. Surely in this case we "got something out" of the equations that we did not put in [86, pp. 8–11].

If then Mathematics is being used in Physics, not so much to discover eternal laws of nature, but rather to devise better approximations of the contradictory that will provide a better account of the more and more exact experimental data, as admittedly it has the power to do, then the famous Galileo dictum, "[il grande libro della natura] e scritto in lingua matematica" [26] might be modified into "The great book of nature is being written in mathematical language."

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