Grades 7-8 (A), 9-12 (A)

Duration: 45 min

Tools: one 9 pcs Set / 1-2 student

Individual / Pair work

Keywords: Indirect proof

208 - Impossible Line

MATHS / LOGIC

LOGIFACES
METHODOLOGY
Erasmus+

TEACHER

Logifaces

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DESCRIPTION

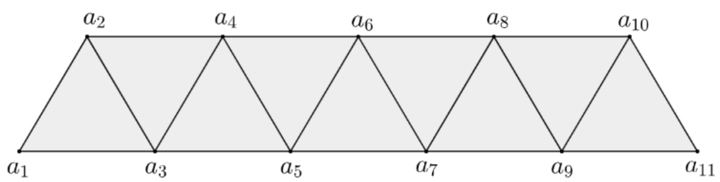
Students try to arrange the 9 pcs Set into a line with a continuous surface. If they do not succeed, they attempt to prove why it is impossible.

SOLUTIONS / EXAMPLES

The line arrangement is an impossible pattern using the 9 pcs Set. The proof of this statement presented here requires basic number theory (divisibility of numbers) and the logic of indirect proof.
Let us write the list of the blocks of the 9 pieces set, where every row represents a single block:

222
112 = 121 = 211
113 = 131 = 311
221 = 212 = 122
331 = 313 = 133
123 = 231 = 312
123 = 231 = 312
132 = 321 = 213
132 = 321 = 213

A proper line arrangement consisting of the blocks of the 9 pcs Set can be encoded by a sequence of length 11: $a_1a_2a_3...a_{10}a_{11}$, see the diagram.



We will show two facts that have to be true for any sequence given by a proper line arrangement, but they cannot be true at the same time. It follows that there can be no proper sequence, and hence there is no proper arrangement of the 9 pcs Set into a line.

FACT 1 The 222 block can be put only next to the block 122, hence in any proper line arrangement, the 222 block must be the first or the last element. Thus one of the followings holds for any proper sequence:

$a_1=2$, $a_2=2$, $a_3=2$, $a_4=1$ or
$a_8=1$, $a_9=2$, $a_{10}=2$, $a_{11}=2$

Reversing a proper sequence gives a proper sequence, hence if there is a proper line arrangement, then there is one with $a_1=2$, $a_2=2$, $a_3=2$, $a_4=1$.

FACT 2 Let us make the following list: write down every second triples of the sequence of a line arrangement starting from the first one: $a_1a_2a_3$, $a_3a_4a_5$, …,$a_9a_{10}a_{11}$ and write down all the remaining triples in the reversed order, i.e. the triples $a_4a_3a_2$, … , $a_{10}a_9a_8$.
In that way, exactly one code of each block of the 9 pcs Set is written, hence the following numbers are written down in the list (see the occurrence of the numbers in the list of the blocks of the 9 pcs Set):

10 lots of 1s
10 lots of 2s
7 lots of 3s

Seen from the point of view of the sequence $a_1a_2a_3...a_{10}a_{11}$, the following numbers are written down in the list:

a_1 and a_{11} once
a_2 and a_{10} twice
a_3,..., a_9 three times

We compare Fact 1 and the properties of the list given in Fact 2.
If a_1=2, a_2=2, a_3=2, a_4=1, then these numbers already imply writing down 6 times the number "2" in the list. The remaining 4 occurrences of "2" can happen only if one of them is a_5,...,a_8 or a_9 (written three times) and one of them is a_{11} (written once). Then the number a_{10} is either "1" or "3".

If a_{10}=1, then "3" occurs only among a_3,..., a_9, hence the number of 3's in the list must be divisible by 3.
If a_{10}=3, then "1" occurs only among a_3,..., a_9, hence the number of 1's in the list must be divisible by 3.

Both cases contradict Fact 2, hence Facts 1 and 2 can not hold simultaneously. This proves that there can be no line arrangement of the 9 pcs Set.

PRIOR KNOWLEDGE

Basic number theory, Indirect proof

RECOMMENDATIONS / COMMENTS

After laying out several patterns and counting the possible layouts of some pattern, this can be a more challenging question. The proof of the impossibility is often a hard exercise and needs experience in proofs.