Lesson 5: How crowded is this neighbourhood?

Goals

- Compare and contrast the density of uniformly distributed dots in squares.
- Create an equation and a graph that represent the proportional relationship between the area of a square and the number of dots enclosed by the square.
- Interpret the constant of proportionality in models of housing per square kilometre or population of people per square kilometre.

Lesson Narrative

This lesson is optional. This lesson involves a sequence of four activities that prepare and introduce students to the concept of population density. The lesson can be adjusted depending on available time and teacher-identified goals from 1 to 2 class days.

Contexts involving population density are useful for helping students understand how derived units arise from a proportional relationship. Population density arises from two familiar quantities, number of people and area. The way the lesson develops helps students make sense of the somewhat abstract idea of density in very concrete terms: They start by comparing the number of dots distributed in squares and move on to houses in different neighbourhoods. Finally they compare the number of people living in different cities. Unlike speed or unit pricing, density is not likely to be familiar to students, so it provides an opportunity to make sense of an unfamiliar situation by thinking about familiar quantities in a new way.

This lesson engages students in important aspects of modelling. In particular, the rates are used to model rather than represent. For example, houses may not be uniformly distributed in any given area, but rates for houses per square mile characterise differences between rural and urban areas. This lesson begins students' transition from contexts that involve constant rates to contexts that involve average rates of change.

As with all lessons in this unit, all related standards have been addressed in prior units. This lesson provides an *optional* opportunity to go deeper and make connections between domains.

Building On

- Recognise and represent proportional relationships between quantities.

Addressing

- Solve real-world and mathematical problems involving area, volume and surface area of two- and three-dimensional objects composed of triangles, quadrilaterals, polygons, cubes, and cuboids.
- Calculate unit rates associated with ratios of fractions, including ratios of lengths, areas and other quantities measured in like or different units. For example, if a person

walks $\frac{1}{2}$ mile in each $\frac{1}{4}$ hour, compute the unit rate as the complex fraction $\frac{\frac{1}{2}}{\frac{1}{4}}$ miles per hour, equivalently 2 miles per hour.

- Recognise and represent proportional relationships between quantities.

Instructional Routines

- Co-Craft Questions
- Compare and Connect
- Discussion Supports
- Notice and Wonder
- Think Pair Share

Required Materials

Four-function calculators

Required Preparation

If desired, prepare to display satellite images that show the housing density in different neighbourhoods of your city, New York City and Los Angeles.

Provide access to four-function calculators.

Student Learning Goals

Let’s see how proportional relationships apply to where people live.

5.1 Dot Density

Optional: 5 minutes

In this activity, students compare dot densities when dots are uniformly distributed. The squares are sized so that students can compare dot density in large and small squares by drawing a partition of the larger square into four smaller squares and comparing the number of dots in squares of the same size. In the next activity, the dots are not uniformly distributed, so students need to think more about the meaning of “dots per square inch.”

Instructional Routines

- Discussion Supports
- Notice and Wonder

Launch

Display the image of the four squares with dots. Invite students to share what they notice and what they wonder.

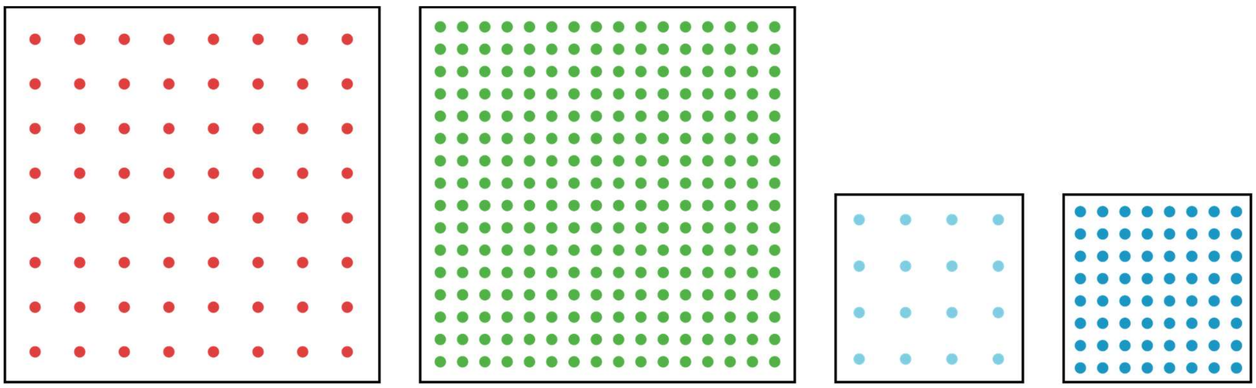
Give students 5 minutes of quiet work time followed by whole-class discussion.

Anticipated Misconceptions

Some students might not understand what the last column in the table is asking them for. Remind them that the word *per* means “for each” or “for one.”

Student Task Statement

The figure shows four squares. Each square encloses an array of dots. Squares A and B have side length 2 inches. Squares C and D have side length 1 inch.



A B C D

1. Complete the table with information about each square.

square	area of the square in square inches	number of dots	number of dots per square inch
A			
B			
C			
D			

2. Compare each square to the others. What is the same and what is different?

Student Response

1. Completed table:

square	area of the square in square inches	number of dots	number of dots per square inch
A	4	64	16
B	4	256	64
C	1	16	16

D	1	64	64

- Squares A and B have the same area. Squares C and D have the same area.
- The number of dots in square A is the same as the number of dots in square D. The two other squares have different numbers of dots.
- The number of dots per square inch is the same for squares A and C. The number of dots per square inch is the same for squares B and D.

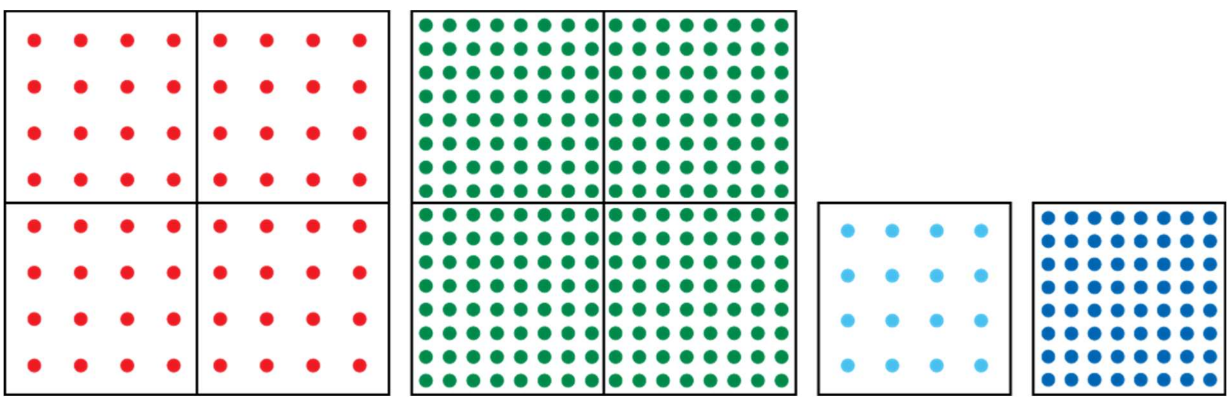
Activity Synthesis

Invite students to share what is similar and what is different about the arrays.

Define *density* as "things per square inch," in this case dots per square inch. Demonstrate the correct use of "dense" and "density" by saying things like:

- "The green dots in square B are more densely packed than the red dots in square A and the blue dots in square C."
- "The density of the red dots in square A and the blue dots in square C is the same."

If students haven't noted it already, point out that the large square A can be partitioned into four smaller squares. Each has an array of red dots identical (except for the colour) to the array of blue dots in square C. The same is true for squares B and D.

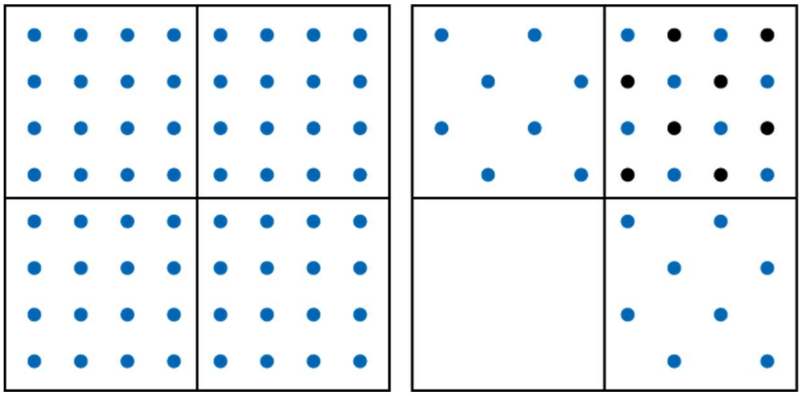


Speaking: Discussion Supports. Use this routine to support whole-class discussion. For each observation or response that is shared, ask students to restate and/or revoice what they heard using precise mathematical language. Consider providing students time to restate what they hear to a partner, before selecting one or two students to share with the class. Ask the original speaker if their peer was accurately able to restate their thinking. Call students' attention to any words or phrases that helped to clarify the original statement. This will provide more students with an opportunity to produce language as they interpret the reasoning of others.
Design Principle(s): Support sense-making

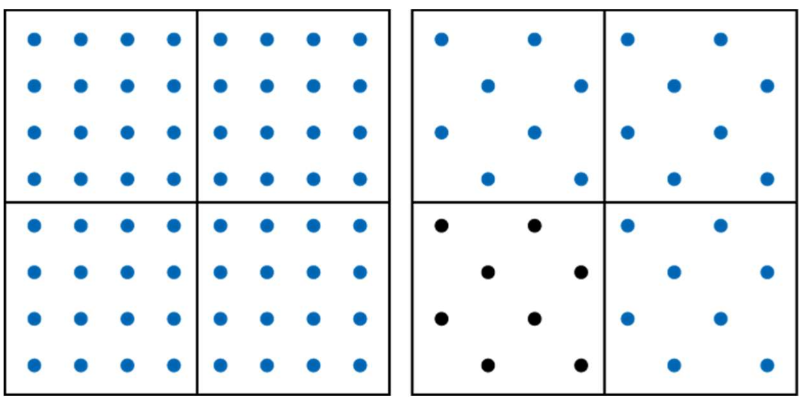
5.2 Dot Density with a Twist

Optional: 10 minutes (there is a digital version of this activity)

In this activity, the dots are distributed uniformly in the first square but not in the second square:



However, these dots are drawn so it is not too hard to see that if they were redistributed, each square inch would have 8 dots:



The fact that we have 8 dots per square inch means that if we distributed the dots uniformly throughout the square in the right way, we would see 8 dots in each square inch. This prepares students to be able to interpret the constant of proportionality in the next two activities when working with actual houses per square mile or people per square kilometre. When we speak of "500 houses per square kilometre," we can think of this as, "If we took all of the houses in the region and spread them out uniformly, then we would see 500 houses in every square kilometre."

Instructional Routines

- Co-Craft Questions
- Think Pair Share

Launch

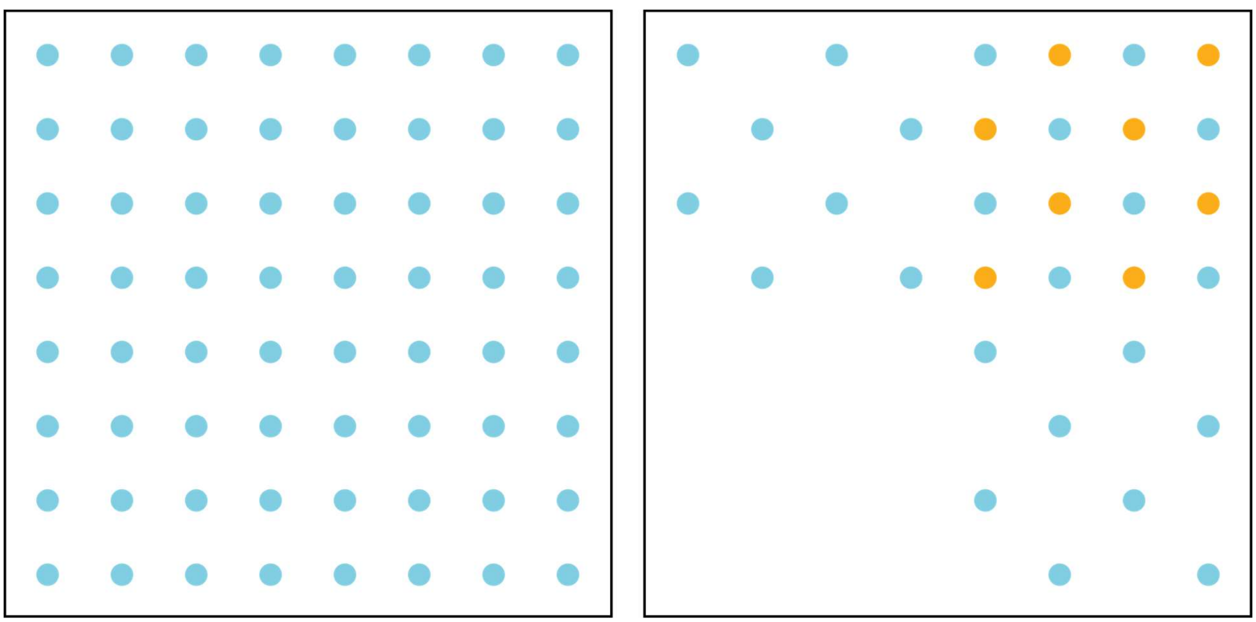
Display the image of the two squares with dots. Ask students to describe what is the same and what is different about these squares.

Give students 2–3 minutes of quiet work time, followed by 4–5 minutes of partner work time and whole-class discussion.

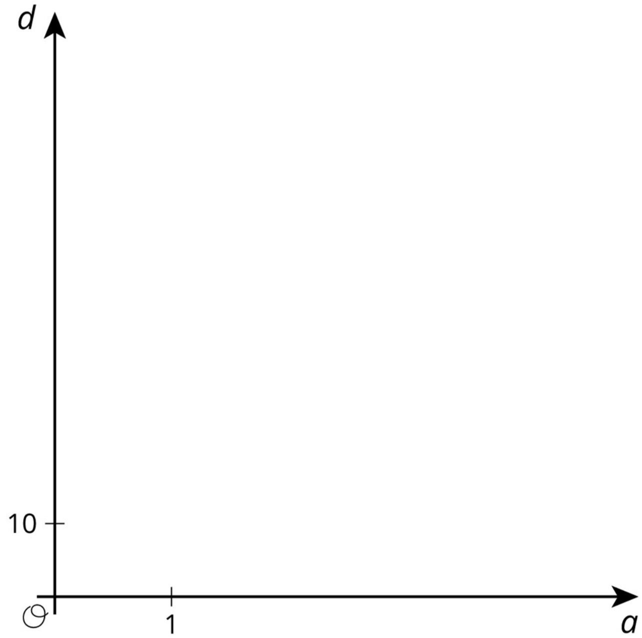
Action and Expression: Internalise Executive Functions. Chunk this task into more manageable parts to support students who benefit from support with organisational skills in problem solving. For example, pause to check for understanding after the first 2–3 minutes of work time.
Supports accessibility for: Organisation; Attention Writing, Conversing: Co-craft Questions. Display only the image of the four dot arrays without revealing any of the questions that follow. Give students 1–2 minutes to write a list of possible mathematical questions they could ask about the arrays. Invite students to share their questions with a partner, and then select 2–3 students to share their questions with the whole class. Highlight any questions that refer to how dots are "distributed," even if students do not use that particular phrase. Finally, reveal the whole problem with text so that students may begin addressing the questions. This helps amplify language related to the distribution of dots.
Design Principle(s): Cultivate conversation; Maximise meta-awareness

Student Task Statement

The figure shows two arrays, each enclosed by a square that is 2 inches wide.



1. Let a be the area of the square and d be the number of dots enclosed by the square. For each square, plot a point that represents its values of a and d.



2. Draw lines from (0,0) to each point. For each line, write an equation that represents the proportional relationship.

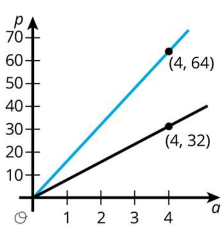
3. What is the constant of proportionality for each relationship? What do the constants of proportionality tell us about the dots and squares?

Student Response

1. See figure.

2. See figure. The equations are $d = 16a$ and $d = 8a$ respectively.

3. The constants of proportionality indicate the number of dots per square inch, 16 and 8, respectively. In the first case, they tell us that if we partition the square into square inches, there will be 16 dots in each. In the second case, they tell us if we were to redistribute the dots uniformly, there would be 8 dots per square inch.



Activity Synthesis

The goal of this discussion is for students to make sense of the constant of proportionality in the case where the dots are not uniformly distributed. Invite students to share their interpretations of the constants of proportionality.

Consider asking questions like:

- "Why is one of the constants of proportionality larger than the other? How can we see this in the picture of the squares?" (There are more dots in the same area.)
- "What are the units of the constants of proportionality?" (The number of dots per square inch).

Students will have more opportunities to think about this when working on the activities that follow.

5.3 Housing Density

Optional: 15 minutes

This activity starts to transition students from arrays of dots to real-world objects distributed over the surface of Earth. This task concerns housing density, which is very similar to dot density. The two images in the activity have the following properties:

- It is fairly easy to distinguish the houses and not too tedious to count them all.
- The scale of the images is similar, but the size of each image is not the same, nor is the number of houses in each image.
- In the first image, the houses look fairly uniformly distributed, and in the other, they look less uniformly distributed but not to the point that it is hard to interpret the image.

This task can be customised to any location, for example different neighbourhoods in your city. Care should be taken in selecting the images to include noticeably different housing densities and easy-to-count houses.

Launch

Give students 5 minutes of quiet work time followed by partner and whole-class discussion. Provide access to four-function calculators.

Representation: Internalise Comprehension. Activate or supply background knowledge about multiplication of fractions. Allow students to use calculators to ensure inclusive participation in the activity.
Supports accessibility for: Memory; Conceptual processing

Anticipated Misconceptions

Some students may struggle interpreting the decimal numbers in the activity. A length of 0.1 km is $\frac{1}{10}$ of a kilometre. This means that 10 of these lengths will make a kilometre.

Students are accustomed to an area (of a rectangle) being *numerically* bigger than its length or width. Ask students to think about multiplication of fractions: a square that is $\frac{1}{2}$ inch by $\frac{1}{2}$ inch has area $\frac{1}{4}$ square inch because four $\frac{1}{2}$ inch by $\frac{1}{2}$ inch squares compose a square inch.

Student Task Statement

Here are pictures of two different neighbourhoods.

This image depicts an area that is 0.3 kilometres long and 0.2 kilometres wide.

0.1 km

This image depicts an area that is 0.4 kilometres long and 0.2 kilometres wide.

0.1 km

For each neighbourhood, find the number of houses per square kilometre.

Student Response

The first image shows 48 houses. It depicts an area that is 0.06 square kilometres, so the housing density is 800 houses per square kilometre.

The second image shows 9 (or 10) houses. It depicts an area that is 0.08 square kilometres, so the housing density is 112.5 (or 125) houses per square kilometre.

Activity Synthesis

Poll students' answers for the two densities. Invite students to share their reasoning. Consider asking the following questions:

- One number is a lot bigger than the other. How can we see this in the images? (There are a lot more houses on the same area.)
- You only counted 48 houses in the first image but you say that there are 800 houses per square kilometre. Why is that happening? (The area in the image is just a fraction of a square kilometre. If we looked at a square kilometre with the same density, there would be 800 houses in it.)
- Some of you said that there are 112.5 houses per square kilometre in the second neighbourhood. How can there be half houses? (If we had an image with an area of 2 square kilometres, there would be 225 houses in the image, or if they are uniformly distributed, the image might contain a part of a house.)

If desired, display this image to help students make sense of their answers.

The map shows a rectangle 0.3 km by 0.2 km. This means that any one of the six squares is 0.1 km by 0.1 km, which has area 0.01 square kilometre.

If you lined up 10 of these, you would have a strip 1 km long. 10 of these strips would make a square 1 km by 1 km; that is, 1 square kilometre. Now it's clear that it takes 100 of these small squares to make a square kilometre, so the small square indeed has area $\frac{1}{100}$ of a square kilometre. In fact, there are 8 houses in this square, so if the entire square kilometre were filled the same way, there would be 800 houses.

5.4 Population Density

Optional: 15 minutes

The purpose of this activity is to introduce the concept of *population density*. One added step going from houses in a neighbourhood to people in a location is the fact that people do not stay at a fixed spot but rather move around. In this activity, students make sense of what it means to say there are 42.3 people per square kilometre in some location.

This activity gives some information about New York City and Los Angeles and ultimately asks students to decide which city is more crowded. Students may benefit from a demonstration of situations that feel more crowded vs. less crowded.

In the data for this task, people are given in "blocks" of 1000 people. It's perfectly fine to make up a new unit customised to the situation, even if it doesn't have an official name. This is a more sophisticated use of nonstandard units. In earlier years, nonstandard units tend to be the length of a paper clip, or the length of your shoe. When dealing with any units, it's important to list the units: in the heading of a table, in labels for the axes of a graph, or in writing numbers.

Instructional Routines

- Compare and Connect

Launch

Arrange students in groups of 2–4. Provide access to calculators.

Consider demonstrating situations that feel more crowded or less crowded by having a certain number of students stand in an area. "What would you have to do to feel more crowded with the same number of people?" (Stand in a smaller space, which would require people to stand closer to each other.) "Less crowded?" (Take up more space, so that people are further apart.) Then, mark off a region on the classroom floor with tape. Ask some students to stand inside it, and then ask, "What would make the space feel more crowded?" (If more people stood in the same space.) "Less crowded?" (Fewer people.)

If desired, provide some background information about New York City and Los Angeles and display satellite images:

- New York City is the U.S. city with the largest population. The city has five parts (called boroughs): Manhattan, Brooklyn, Bronx, Queens, and Staten Island. In Manhattan, most people live in apartment buildings, many in high-rises. In the other boroughs, many people also live in single-family houses. Staten Island is quite different, almost suburban.

- Los Angeles is the U.S. city with the second largest population. Although there are some high-rise apartment buildings, many people live in single-family houses, and many of these houses are single-story.

Give students 5 minutes of quiet work time, followed by small-group and whole-class discussion.

Action and Expression: Develop Expression and Communication. Activate or supply background knowledge. During the launch, take time to review the following terms from

previous lessons that students will need to access for this activity: constant of proportionality, writing equations.
Supports accessibility for: Memory; Language

Anticipated Misconceptions

Be on the lookout for these areas of potential difficulty:

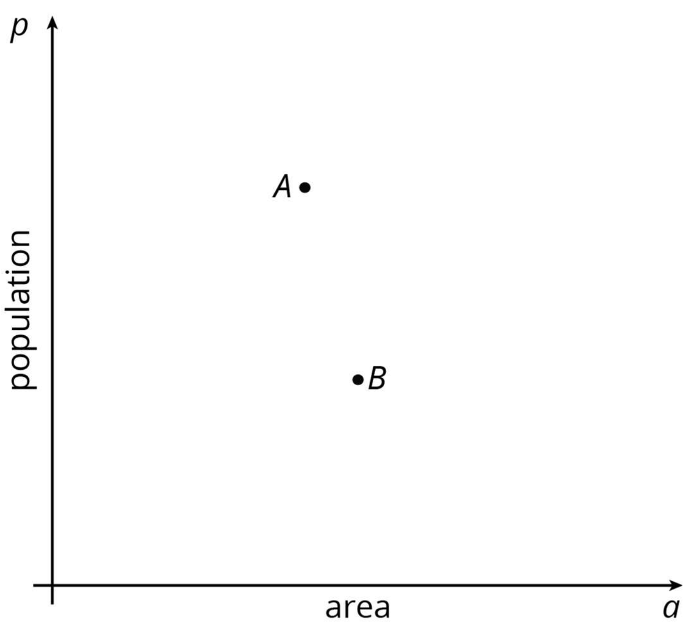
- Thinking that the numbers on the p-axis represent people, rather than thousands of people.
- Trying to find the exact values given, rather than approximating or rounding.
- Trouble adapting concepts or skills from graphing proportional relationships to this more complex situation.

If students have difficulty understanding numbers that are expressed in units of 1 000, they may need either some scaffolding or adequate time to talk about what these numbers mean. Alternatively, the numbers can be given in terms of the more familiar millions, in which case the population densities will require some extra effort to understand.

Student Task Statement

- New York City has a population of 8 406 thousand people and covers an area of 1 214 square kilometres.
- Los Angeles has a population of 3 884 thousand people and covers an area of 1 302 square kilometres.

1. The points labelled A and B each correspond to one of the two cities. Which is which? Label them on the graph.



2. Write an equation for the line that passes through $(0,0)$ and A. What is the constant of proportionality?

3. Write an equation for the line that passes through $(0,0)$ and B. What is the constant of proportionality?

4. What do the constants of proportionality tell you about the crowdedness of these two cities?

Student Response

1. New York City has a greater population and a smaller area, so it must correspond to point A.

2. An equation for the line through point A is $p = 6.9a$; the constant of proportionality is about 6.9.

3. An equation for the line through point B is $p = 3.0a$; the constant of proportionality is about 3.0.

4. The constants of proportionality tell us that in New York City there are about 6.9 thousand people per square kilometre, or 6 900 people per square kilometre; and in LA there are about 3 thousand people per square kilometre, or 3 000 people per square kilometre.

Are You Ready for More?

1. Predict where these types of regions would be shown on the graph:

   a. a suburban region where houses are far apart, with big yards

   b. a neighbourhood in an urban area with many high-rise apartment buildings

   c. a rural state with lots of open land and not many people

2. Next, use this data to check your predictions:

place	description	population	area (km^2)
Chalco	a suburb of Omaha, Nebraska	10 994	7.5
Anoka County	a county in Minnesota, near Minneapolis/St. Paul	339 534	1 155
Guttenberg	a city in New Jersey	11 176	0.49
New York	a state	19 746 227	141 300
Rhode Island	a state	1 055 173	3 140
Alaska	a state	736 732	1 717 856
Tok	a community in Alaska	1 258	342.7

Student Response

Note that it's not really possible to see all the points on the same graph: the populations of Los Angeles and New York are so large, and the population of Tok so small, that if you could distinguish the point for Tok, LA and NY would be far off the paper or screen. And on the graph above showing LA and NY, Tok's population would be so small that it could not be distinguished from (0,0).

A computer graphing program can help students understand this as it will take many steps of zooming in or out to switch between very small cities and very large cities.

Activity Synthesis

Invite some students to display their graphs and equations for all to see. Ask all students if they agree or disagree and why. Once students agree, focus on the meaning of the constants of proportionality and what they tell us about the crowdedness in the two cities.

Consider asking the following questions:

- "Why did you choose point A to represent New York City?" (New York City has more people per square kilometre than Los Angeles, so New York is more crowded than Los Angeles.)
- "What does the constant of proportionality tell us about the crowdedness in the two cities?" (Even though people aren't distributed uniformly throughout the cities, we might say that if we were to distribute everyone who has an address in the city uniformly throughout the city, then there would be about 6 900 people in every square kilometre in NYC and about 3 000 people in every square kilometre in LA.)

Representing: Compare and Connect. Use this routine to help students discuss what the constant of proportionality tells them about the crowdedness of the two cities. Ask students to compare the city's equations and graphs to look for "What is the same and what is different?" between the representations. Ask students to look for where each city's constant of proportionality is represented in the equations and the graphs.
Design Principle(s): Optimise output (for comparison); Maximise meta-awareness