Lesson 12: Using graphs to compare relationships

Goals

- Create and interpret graphs that show two different proportional relationships on the same axes.
- Generalise (orally and in writing) that when two different proportional relationships are graphed on the same axes, the steeper line has the greater constant of proportionality.

Learning Targets

- I can compare two, related proportional relationships based on their graphs.
- I know that the steeper graph of two proportional relationships has a larger constant of proportionality.

Lesson Narrative

In this lesson students continue their work with interpreting graphs of proportional relationships. An important goal of the lesson is for students to start to interpret the steepness of the graph in terms of the context. They use distance-versus-time graphs to decide which person from a group is going the fastest. They also work with graphs where the scale is not specified on each axis, and realise that they can still use graphs to compare rates.

Building On

- Apply and extend previous understandings of multiplication and division to multiply and divide fractions.

Addressing

- Recognise and represent proportional relationships between quantities.

Building Towards

- Compute unit rates associated with ratios of fractions, including ratios of lengths, areas and other quantities measured in like or different units. For example, if a person walks $\frac{1}{2}$ mile in each $\frac{1}{4}$ hour, compute the unit rate as the complex fraction $\frac{1/2}{1/4}$ miles per hour, equivalently 2 miles per hour.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Discussion Supports
- Number Talk

Required Materials

Coloured pencils
Rulers

Required Preparation

Have available the information from the activity "Tyler's Walk" from the previous lesson.

Student Learning Goals

Let's graph more than one relationship on the same grid.

12.1 Number Talk: Fraction Multiplication and Division

Warm Up: 5 minutes

The purpose of this Number Talk is to elicit strategies and understandings students have for multiplying and dividing fractions. These understandings help students develop fluency and will be helpful throughout this unit when students find constants of proportionality from graphs, tables, and equations. While four problems are given, it may not be possible to share every strategy. Consider gathering only two or three different strategies per problem, saving most of the time for the final question.

Instructional Routines

- Discussion Supports
- Number Talk

Launch

Display one problem at a time. Give students 30 seconds of quiet think time for each problem and ask them to give a signal when they have an answer and a strategy. Keep all problems displayed throughout the talk. Follow with a whole-class discussion.

Representation: Internalise Comprehension. To support working memory, provide students with sticky notes or mini whiteboards.
Supports accessibility for: Memory; Organisation

Student Task Statement

Find each product or quotient mentally.

$\frac{2}{3} \times \frac{1}{2}$

$\frac{4}{3} \times \frac{1}{4}$

$4 \div \frac{1}{5}$

$\frac{9}{6} \div \frac{1}{2}$

Student Response

- $\frac{1}{3}$. Explanations vary. Sample response: Half of two-thirds is one third.
- $\frac{1}{3}$. Explanations vary. Sample response: One fourth of four-thirds is one third.
- 20. Explanations vary. Sample response: There are 5 fifths in 1, so there are 20 fifths in 4.
- 3. Explanations vary. Sample response: $\frac{9}{6}$ is $\frac{3}{2}$. Dividing by $\frac{1}{2}$ is the same as multiplying by 2. Twice $\frac{3}{2}$ is 3.

Activity Synthesis

Ask students to share their strategies for each problem. Record and display their answers and explanations for all to see.

To involve more students in the conversation, consider asking:

- "Who can restate ___'s reasoning in a different way?"
- "Did anyone solve the problem the same way but would explain it differently?"
- "Did anyone solve the problem in a different way?"
- "Does anyone want to add on to _____'s strategy?"
- "Do you agree or disagree? Why?"

If time permits, ask students if they notice any connections between the problems. Have them share any relationships they notice.

Speaking: Discussion Supports.: Display sentence frames to support students when they explain their strategy. For example, "First, I _____ because . . ." or "I noticed _____ so I . . . ." Some students may benefit from the opportunity to rehearse what they will say with a partner before they share with the whole class.
Design Principle(s): Optimise output (for explanation)

12.2 Race to the Bumper Cars

15 minutes (there is a digital version of this activity)

In this activity, students graph three time-distance relationships along with the one from the previous lesson, “Tyler’s Walk.” One of these is not a proportional relationship, so students must pay close attention to the quantities represented. The purpose of this activity is to give students many opportunities to connect the different features of a graph with parts of the situation it represents. In particular, they attach meaning to any point that is on a graph, and they interpret the meaning of the distance when the time is 1 second as both the constant of proportionality of the relationship and the person’s speed in the context in metres per second. Comparing different but related situations and their graphs supports students as they make sense of the situation.

Launch

Arrange students in groups of 2–3. Provide access to coloured pencils and rulers.

Tell students that this activity is tied to the activity titled "Tyler's Walk" from the previous lesson. All references to Tyler going to the bumper cars come from the statements in that activity.

The digital version has an applet with options to change line colours and hide points. You may want to demonstrate the applet before students use it, perhaps graphing Tyler's data from the previous activity together. Note: the applet can graph lines, rays, or segments. Your class can decide how to represent the data.

Representation: Internalise Comprehension. Demonstrate and encourage students to use colour coding and annotations to highlight connections between representations in a problem. For example, use a different colour for each person to highlight the connection between the table, graph, and constant of proportionality.
Supports accessibility for: Visual-spatial processing

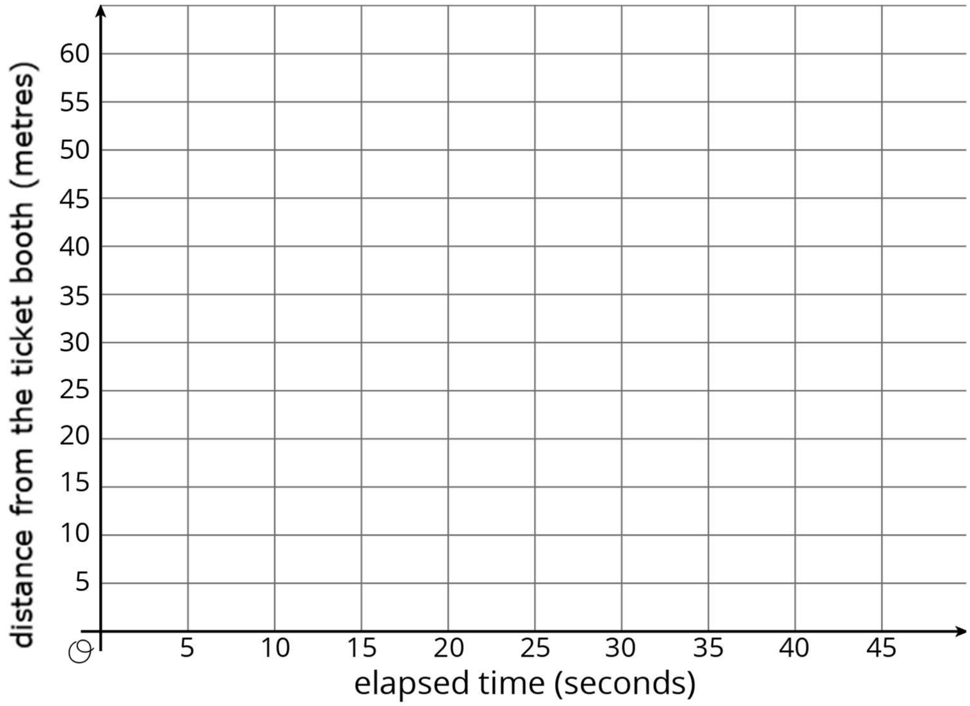
Student Task Statement

Diego, Lin, and Mai went from the ticket booth to the bumper cars.

1. Use each description to complete the table representing that person’s journey.
    a. Diego left the ticket booth at the same time as Tyler. Diego jogged ahead at a steady pace and reached the bumper cars in 30 seconds.
    b. Lin left the ticket booth at the same time as Tyler. She ran at a steady pace and arrived at the bumper cars in 20 seconds.
    c. Mai left the booth 10 seconds later than Tyler. Her steady jog enabled her to catch up with Tyler just as he arrived at the bumper cars.

Diego's time (seconds)	Diego's distance (metres)
0	
15	
30	50
1	
Lin's time (seconds)	Lin's distance (metres)
	0
	25
20	50
1	
Mai's time (seconds)	Mai's distance (metres)
	0
	25
40	50
1	

2. Using a different colour for each person, draw a graph of all four people's journeys (including Tyler's from the other day).



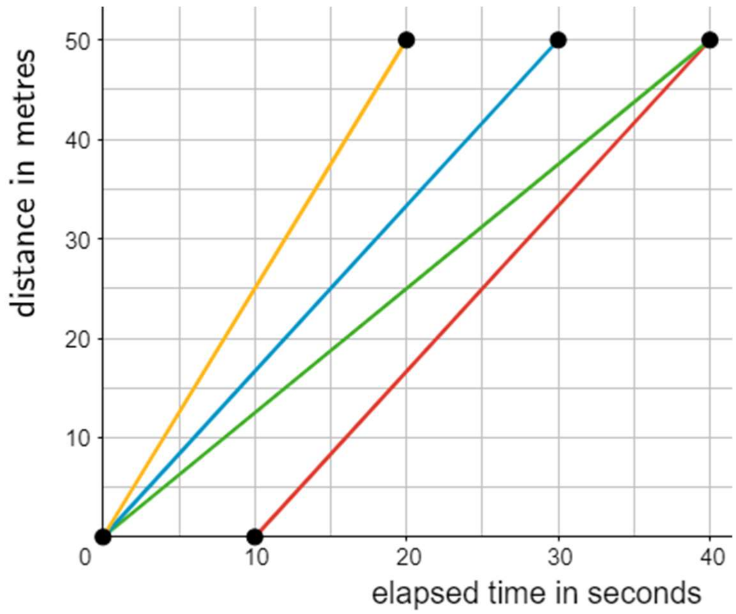
3. Which person is moving the most quickly? How is that reflected in the graph?

Student Response

1. Tables:

Diego's time (seconds)	Diego's distance (metres)
0	0
15	25
30	50
1	$\frac{5}{3}$
Lin's time (seconds)	Lin's distance (metres)
0	0
10	25
20	50
1	2.5
Mai's time (seconds)	Mai's distance (metres)
10	0
25	25
40	50
1	0

2.



3. Lin left the booth at the same time as Tyler but reached the bumper cars in 20 seconds; her time of arrival at the bumper cars is shown by the point (20,50). Diego left at the same time as the first three, but it took him 30 seconds to reach the bumper cars. His arrival at the bumper cars is shown by the point (30,50). Tyler's arrival at the bumper cars is shown by the point (40,50).

   We don't know anything about Mai's distance from the ticket booth before she leaves 10 seconds after Tyler, so her graph does not include any points to the left of $x = 10$. The point (10,0) is on her graph. The pair (40,50) in her table shows that she and Tyler arrived at the bumper cars at the same time, 40 seconds after Tyler started walking, and that the point (40,50) is on her graph. Mai's graph does not represent a proportional relationship: The distance she travels is not proportional to time elapsed since Tyler left the ticket booth. (It is, however, proportional to time elapsed since *Mai* left the booth.)

   – Lin is moving most quickly. Explanations vary for how you can see this on the graph. Sample responses:

   – At any given time between 0 and 20 seconds, she has travelled the farthest.

   – For any given distance between 0 and 50 metres, it takes her the least amount of time to get there.

– She is travelling at $2\frac{1}{2}$ metres per second, while Diego is travelling at $1\frac{2}{3}$ metres per second and Tyler at $1\frac{1}{4}$ metres per second. You can see this on the graph by looking at the points with x-coordinate 1.

Are You Ready for More?

Write equations to represent each person's relationship between time and distance.

Student Response

Let t represent elapsed time in seconds and d represent distance from the ticket booth in metres. Lin: $d = 2.5t$ or equivalent. Diego: $d = \frac{5}{3}t$ or equivalent. Tyler: $d = 1.25t$ or equivalent. Mai: $d = \frac{5}{3}(t - 10)$ or equivalent.

Activity Synthesis

Students may expect the graphs to intersect because everyone arrives at the same location. However, they did not arrive there at the same time (with the exception of Tyler and Mai). Because all characters travelled the same distance from the ticket booth and no further, the endpoints of their graphs lie on the same horizontal line $y = 50$, that is, they have the same y-coordinate. The points will vary in position from right to left depending on the number of seconds after Tyler left the ticket booth it took each person to arrive at the bumper cars. Note these features in a whole-class discussion.

The most important goals of the discussion are to attach meaning to any point that is on a graph, and to interpret the meaning of the distance when the time is 1 second as both the constant of proportionality of the relationship and the person's speed in the context in metres per second.

These questions may be used to facilitate the class discussion:

- "For each graph that shows a proportional relationship, what is the constant of proportionality?" (Tyler's was 1.25, Diego's was $1\frac{2}{3}$, Lin's was 2.5.)
- "How did you find them?" (Answers will vary, but students could have divided a y-coordinate by its associated x-coordinate.)
- "Where do constants of proportionality occur in the tables, and where do they occur on the graphs?" (They occur in the tables as the values in the second column that correspond to the value of 1 in the first column; on the graphs as the y-coordinate of points where the x-coordinate is 1.)
- "Which is the only graph that does not represent a proportional relationship?"

- "A classmate argues that Mai's graph must represent a proportional relationship, because she jogged at a steady rate. How do you answer?" (Mai's graph does not pass through the origin, so it does not represent a proportional relationship. That is, the distance she travelled is not proportional to the time elapsed. What does "time elapsed" mean? It is time elapsed since Tyler left the ticket booth. However, distance vs. time elapsed since *Mai* left the ticket booth until she arrived at the bumper cars is a proportional relationship.)

12.3 Space Rocks and the Price of Rope

10 minutes (there is a digital version of this activity)

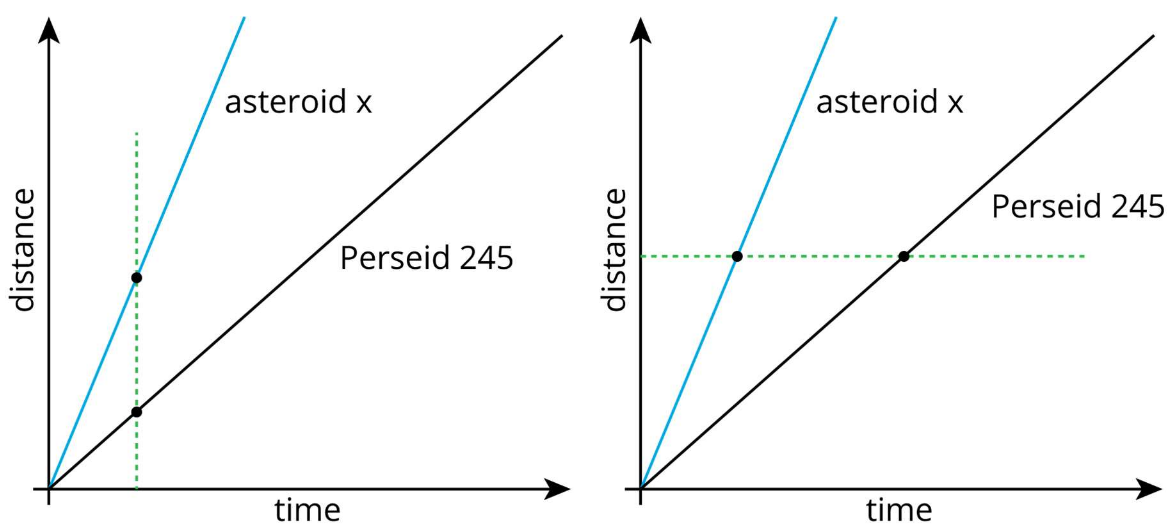
This activity is intended to help students interpret features of graphs in terms of the proportional relationships they represent. This task highlights two fundamental ideas:

- The steeper the graph, the greater the constant of proportionality.
- If the graph represents distance of an object vs. time, the constant of proportionality is the speed of the object.

Students can reason abstractly by picking an arbitrary time and comparing the corresponding distances, or picking an arbitrary distance and comparing the corresponding times. Ideally, both of these ways of reasoning are shared in a whole-class discussion of the task because they will be needed in future work.

While students work, monitor for these approaches:

- At the same time, Asteroid x has travelled a greater distance (as in the graph with the vertical dashed line). We can't tell exactly where 1 unit of time is on the graph, but wherever it is, we can tell that Asteroid x has covered a greater distance.



- Asteroid x takes less time than Perseid 245 to cover the same distance. See the graph with the horizontal dashed line.
- When distance is proportional to time and distance is graphed against time, the constant of proportionality represents the magnitude of the speed (unit of distance travelled per unit of time). It was shown in the previous activity that a steeper line has a greater constant of proportionality. Therefore, a line steeper than Perseid 245's line represents a greater speed.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Discussion Supports

Launch

Keep students in the same groups of 2–3.

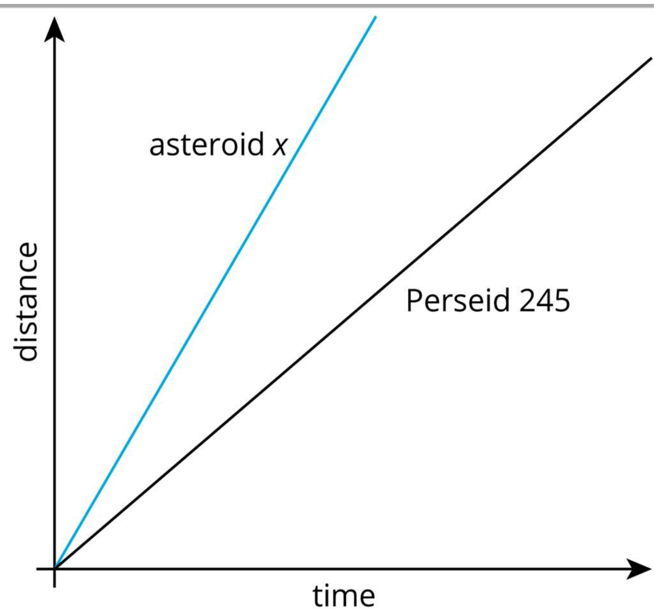
If using the digital activity, have students explore the applet to develop their reasoning around the following two concepts:

- The steeper the graph, the greater the constant of proportionality.
- If the graph represents distance vs. time, the constant of proportionality is the speed.

Engagement: Develop Effort and Persistence. Encourage and support opportunities for peer interactions. Invite students to talk about their ideas with a partner before writing them down. Display sentence frames to support students when they explain their strategy. For example: "First, I _____ because _____. Then, I….," "I noticed _____ so I….," and "I tried _____ and what happened was.…"
Supports accessibility for: Language; Social-emotional skills

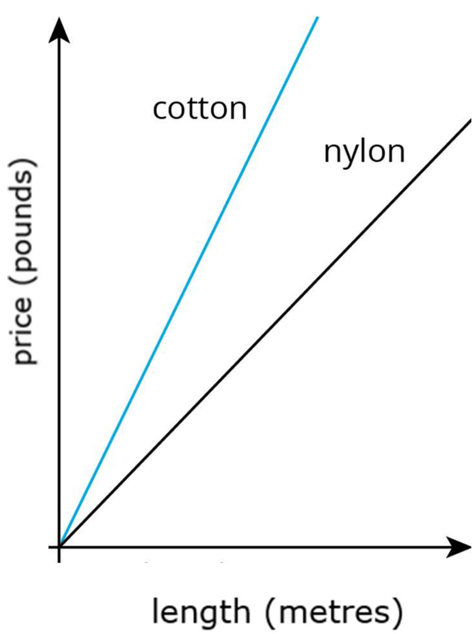
Student Task Statement

1. Meteoroid Perseid 245 and Asteroid x travel through the solar system. The graph shows the distance each travelled after a given point in time.



Is Asteroid x travelling faster or slower than Perseid 245? Explain how you know.

2. The graph shows the price of different lengths of two types of rope.



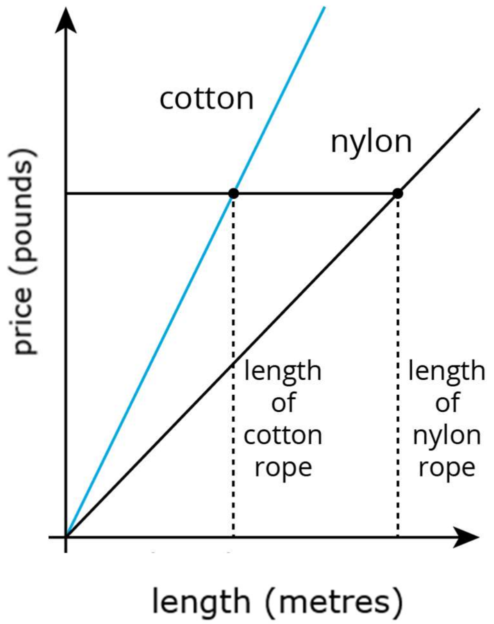
If you buy £1.00 of each kind of rope, which one will be longer? Explain how you know.

Student Response

1. Asteroid x is travelling faster than Perseid 245. Explanations vary. Students might consider the same time on each graph and compare the distance travelled, they might

consider the same distance travelled on each graph and compare the time it took, or they might reason about each object's speed in distance units per time unit.

2. The nylon rope would be longer. The graph shows that a greater length of nylon rope can be purchased for the same price as a shorter length of cotton rope.



Activity Synthesis

It is important that students not assume "steeper always means faster," but that they understand why it is in this case by reasoning abstractly and attending to the references for points on the graphs. If the same relationships were graphed with distance on the horizontal axis and time on the vertical axis, a steeper line would indicate a slower speed. If the same relationships were graphed on separate axes, their scales could be different. Because the graphs share the same axes, it is implicit that comparisons between them occur relative to the same units.

Try to find students who took each approach, and invite each to share reasoning with the class. Important points to highlight are:

- A steeper graph has a larger constant of proportionality and a larger constant of proportionality will have a steeper graph. ("When you look at two graphs of a proportional relationship, how can you tell which one has a greater constant of proportionality?")
- In a distance vs. time graph, a steeper graph indicates a greater speed. ("When you look at two distance vs. time graphs, how can you tell which represents an object travelling at a greater speed?")

- The two graphs in question 1 can be considered from two perspectives:
  - Same point in time, noting that one object has covered a greater distance. ("Which student focused on the two objects at the same moment in time? How did they know that Asteroid x was travelling faster?")
  - Same distance, noting that one object took less time. ("The other student did not focus on the same moment in time. What did they focus on in their explanation?")

Speaking: Discussion Supports. As students describe their explanations for which asteroid moved faster or slower, revoice student ideas to demonstrate mathematical language use. Press for details in students' explanations by requesting that students challenge an idea, elaborate on an idea, or give an example. Show central concepts multi-modally by using different types of sensory inputs: acting out scenarios or inviting students to do so, using gestures, and talking about the context of moving objects. This will help students to produce and make sense of the language needed to communicate their own ideas.
Design Principle(s): Support sense-making, Optimise output (for explanation)

Lesson Synthesis

It would be helpful to display the completed graph from the "Race to the Bumper Car" activity during discussion; the one that shows the distance time relationship for Tyler, Diego, Lin, and Mai's trip from the ticket booth to the bumper cars. If the right technology is available, display this graph to facilitate discussion:
https://www.desmos.com/calculator/dpthqt3sld.

Lines and points can be shown and hidden by clicking the folder icons along the left side of the window. The graphed points, once turned on, can be dragged along the lines. Turn the coordinates on and off by clicking on a point. The most useful aspect of using this dynamic graph for this discussion is that the graph can be zoomed in to easily see a point when its x-coordinate is 1.

Revisit the connections made in the this activity.

- "How can we tell from the graph who had gone the farthest after 10 seconds?" (Find points of the graph with first coordinate 10 and compare second coordinate.)
- "How can we tell from the graph how long it took everybody to get to the bumper cars?" (Find the points on the graph when the second coordinate is 50.)
- "How can we tell from the graph who was moving the fastest?" (Find the constant of proportionality k by locating the point $(1, k)$, or in this case, see which graph is the steepest.)

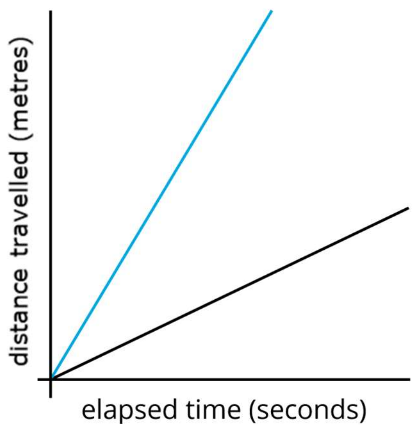
12.4 Revisiting the Amusement Park

Cool Down: 5 minutes

Student Task Statement

Noah and Diego left the amusement park's ticket booth at the same time. Each moved at a constant speed toward his favourite ride. After 8 seconds, Noah was 17 metres from the ticket booth, and Diego was 43 metres away from the ticket booth.

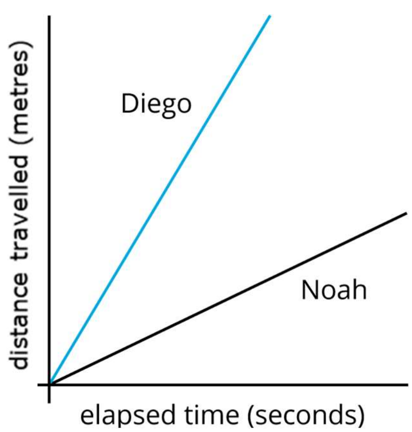
1. Which line represents the distance travelled by Noah, and which line represents the distance travelled by Diego? Label each line with one name.



2. Explain how you decided which line represents which person's travel.

Student Response

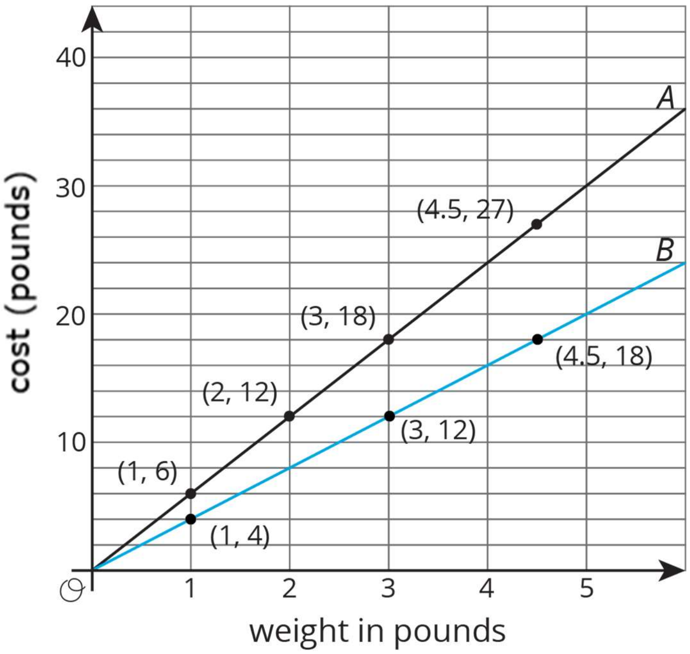
1. The steeper line represents the distance travelled by Diego.



2. Answers vary. Sample response: Diego had gone farther after 8 seconds. If you pick a time and look at which line represents a person who has gone farther, that is the steeper graph. So that must be Diego's line.

Student Lesson Summary

Here is a graph that shows the price of blueberries at two different stores. Which store has a better price?



We can compare points that have the same x value or the same y value. For example, the points (2,12) and (3,12) tell us that at store B you can get more pounds of blueberries for the same price.

The points (3,12) and (3,18) tell us that at store A you have to pay more for the same quantity of blueberries. This means store B has the better price.

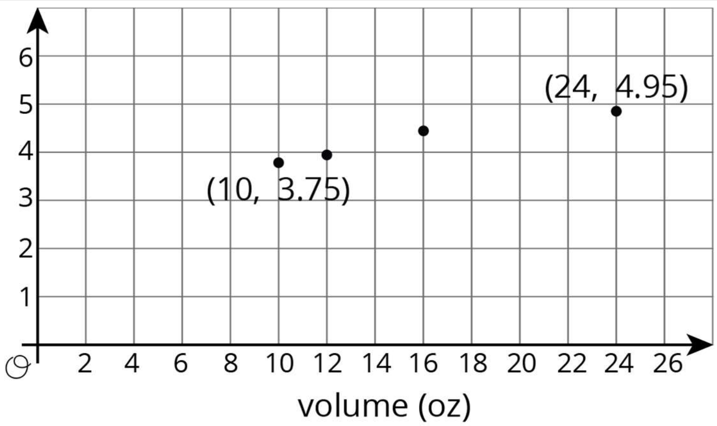
We can also use the graphs to compare the constants of proportionality. The line representing store B goes through the point (1,4), so the constant of proportionality is 4. This tells us that at store B the blueberries cost £4 per pound. This is cheaper than the £6 per pound unit price at store A.

Lesson 12 Practice Problems

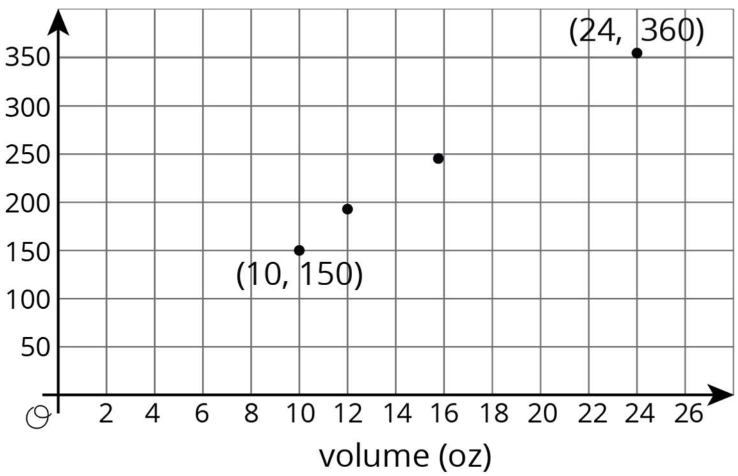
1. **Problem 1 Statement**

   The graphs below show some data from a coffee shop menu. One of the graphs shows cost (in pounds) vs. drink volume (in ounces), and one of the graphs shows calories vs. drink volume (in ounces).

   ________________ vs volume



__________________ vs volume



a. Which graph is which? Give them the correct titles.

b. Which quantities appear to be in a proportional relationship? Explain how you know.

c. For the proportional relationship, find the constant of proportionality. What does that number mean?

Solution

a. The first graph is cost vs volume, and the second graph is calories vs volume. You can tell because the y-values are appropriate for cost in pounds on the first graph, and the y-values are appropriate for calories on the second.

b. It appears there is a proportional relationship between calories and volume. The points appear to lie on a line that would pass through the origin. Also, it makes sense that every one ounce would contain the same number of calories. Regarding the cost relationship, the points do not appear to lie on a precise line, and the line definitely would not pass through the origin. This makes sense because there is more to the cost of a cup of coffee than the amount of coffee.

c. The constant of proportionality is 15 calories per ounce, which can be found using $\frac{150}{10}$ or $\frac{360}{24}$. It means the coffee drink contains 15 calories in 1 ounce.

2. **Problem 2 Statement**

Lin and Andre biked home from school at a steady pace. Lin biked 1.5 km and it took her 5 minutes. Andre biked 2 km and it took him 8 minutes.

a. Draw a graph with two lines that represent the bike rides of Lin and Andre.

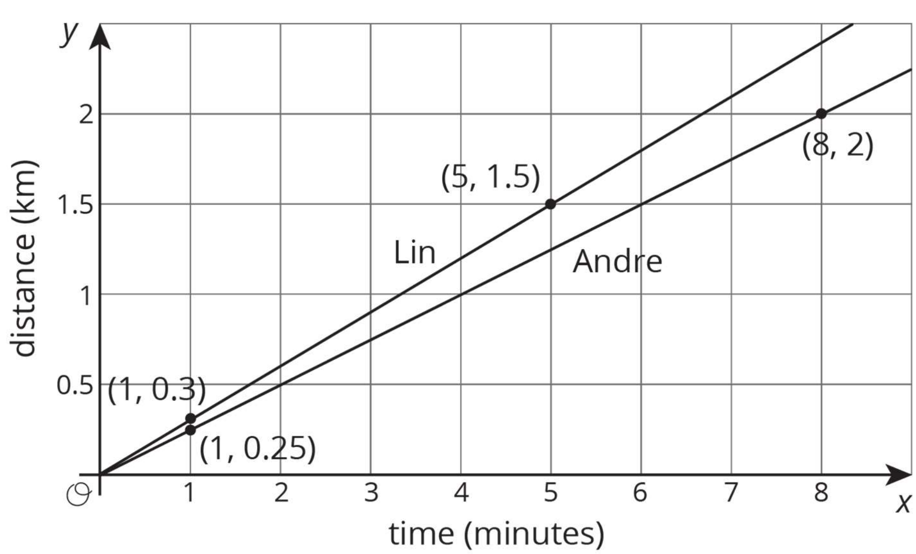
b. For each line, highlight the point with coordinates $(1, k)$ and find k.

c. Who was biking faster?

Solution

Lin and Andre biked home from school at a steady pace. Lin biked 1.5 km and it took her 5 minutes. Andre biked 2 km and it took him 8 minutes.

a.



b. For Lin's graph, $k = 0.3$. For Andre's graph, $k = \frac{2}{8}$.

c. Lin is going slightly faster at 0.3 km per minutes. Andre is going $\frac{2}{8}$ or 0.25 km per minute.

3. **Problem 3 Statement**

Match each equation to its graph.

a. $y = 2x$

b. $y = \frac{4}{5}x$

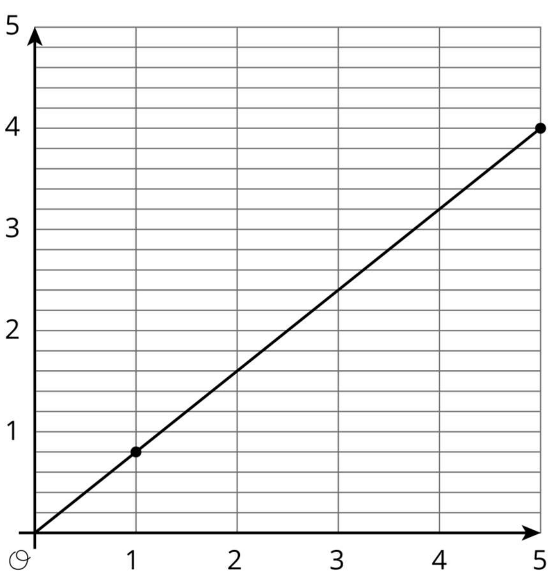
c. $y = \frac{1}{4}x$

d. $y = \frac{2}{3}x$

e. $y = \frac{4}{3}x$

f. $y = \frac{3}{2}x$

1



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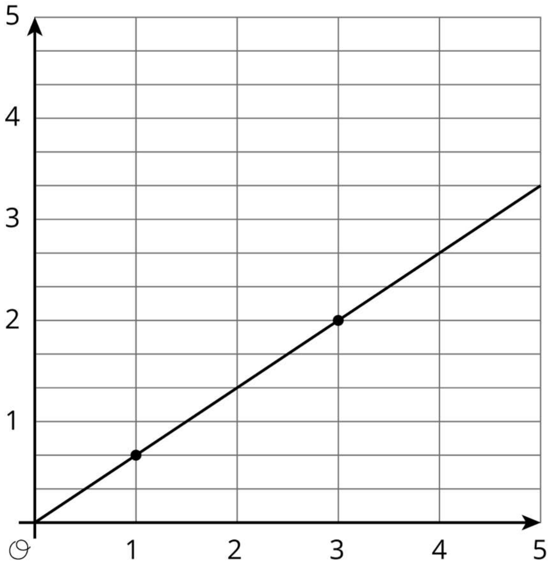

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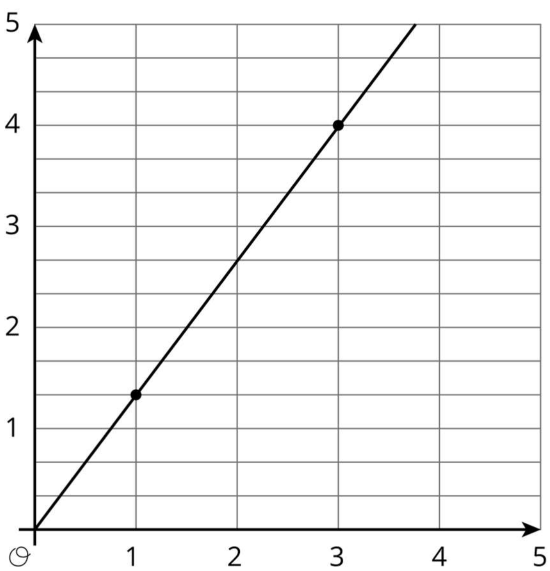

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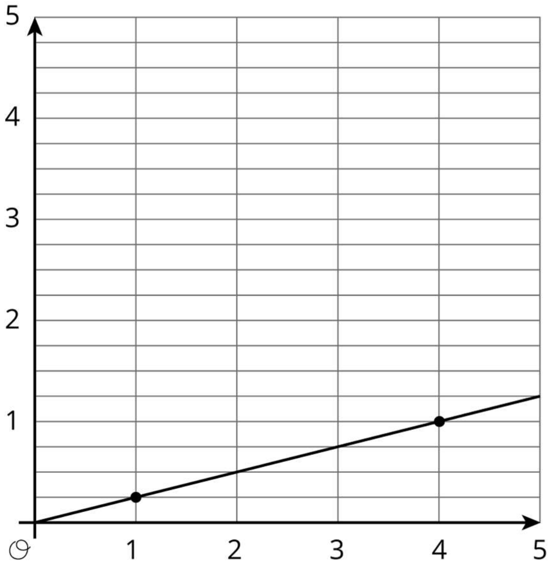

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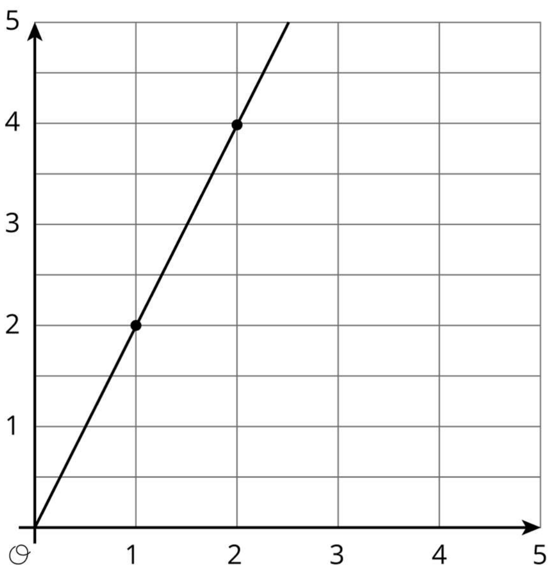

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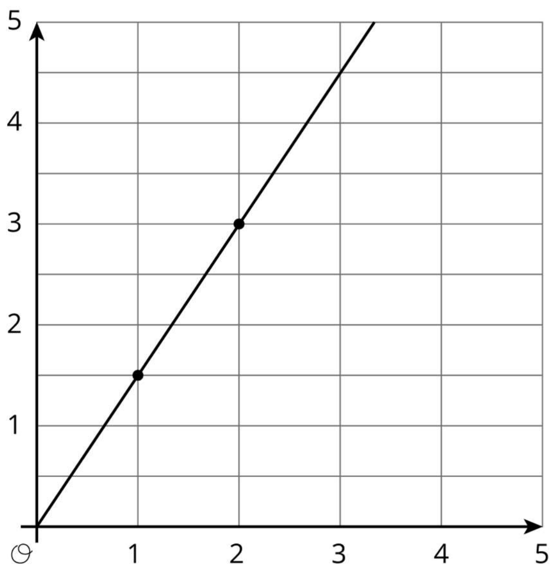

Solution

a. 5

b. 1

c. 4

d. 2

e. 3

f. 6