Mathematics Education Innovation

Lesson 12: Congruent polygons

Goals

- Comprehend that shapes with the same area and perimeter may or may not be congruent.
- Critique arguments (orally) that two shapes with congruent corresponding sides may be non-congruent shapes.
- Justify (orally and in writing) that two polygons on a grid are congruent using the definition of congruence in terms of transformations.

Learning Targets

- I can decide using transformations whether or not two shapes are congruent.

Lesson Narrative

In this lesson, students find transformations that show two shapes are congruent and make arguments for why two shapes are not congruent. They learn that, for many shapes, simply having corresponding side lengths that are equal will not guarantee the shapes are congruent.

In the previous lesson, students defined what it means for two shapes to be congruent and started to apply the definition to determine if a pair of shapes is congruent. In the first part of this lesson, students continue to determine whether or not pairs of shapes are congruent, but here they have the extra structure of a grid. For example:

- Instead of "translate down and to the left," students can say, "translate 3 units down and 2 units to the left"
- Instead of "reflect the shape," students can say, "reflect the shape in this vertical line."

In addition, students have to be careful how they name congruent polygons, making sure that corresponding vertices are listed in the proper order.

An optional part of the lesson begins to examine criteria to decide when two shapes are congruent. If two shapes are congruent, then their corresponding sides and angles are congruent. Is it true that having the same side lengths (or angles) is enough to determine whether or not two shapes are congruent? Students investigate this question for quadrilaterals in two different situations:

- 4 congruent side lengths.
- 2 pairs of congruent side lengths where the pairs are of different length.

Addressing

- Understand that a two-dimensional shape is congruent to another if the second can be obtained from the first by a sequence of rotations, reflections, and translations; given two congruent shapes, describe a sequence that exhibits the congruence between them.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Compare and Connect
- Discussion Supports
- Take Turns

Required Materials

Geometry toolkits

tracing paper, graph paper, coloured pencils, scissors, and an index card to use as a straightedge or to mark right angles, plus a ruler and protractor. Clear protractors with no holes and with radial lines printed on them are recommended.

Toothpicks, pencils, straws, or other objects

Required Preparation

If you choose to have students complete the optional activity, have sets of objects ready for students to build quadrilaterals. Each pair of students requires 12 objects (such as toothpicks, pencils, or straws) to be used as sides of quadrilaterals: 8 objects of one length and 4 objects of a different length.

Student Learning Goals

Let's decide if two shapes are congruent.

12.1 Translated Images

Warm Up: 5 minutes

This task helps students think strategically about what kinds of transformations they might use to show two shapes are congruent. Being able to recognise when two shapes have either a mirror orientation or rotational orientation is useful for planning out a sequence of transformations.

Launch

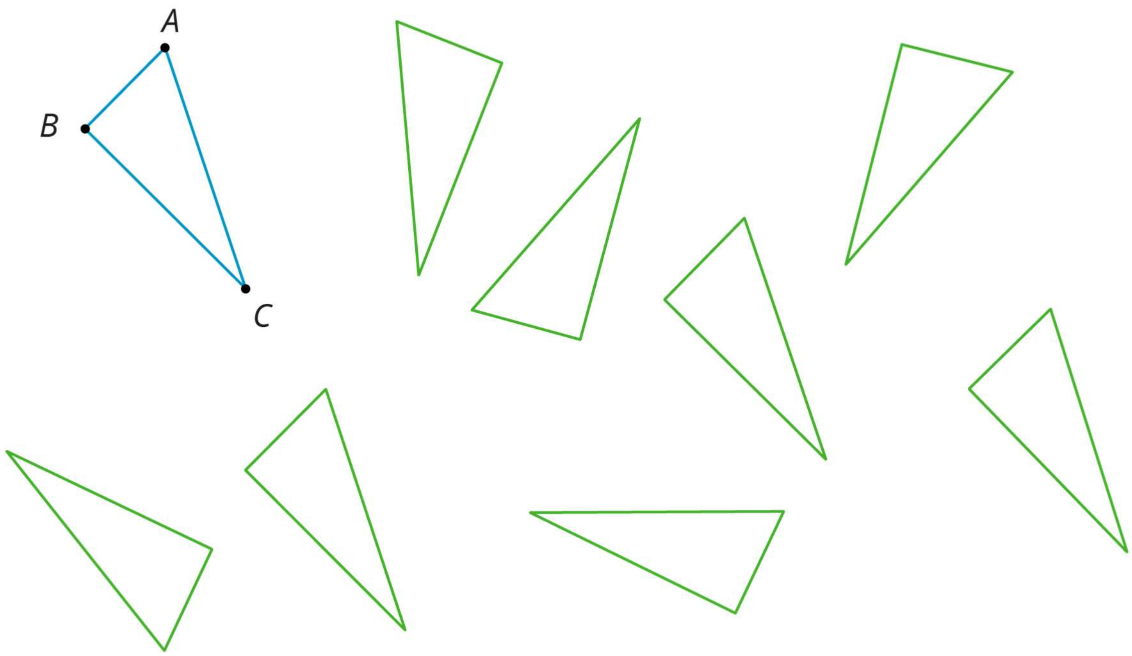
Provide access to geometry toolkits. Allow for 2 minutes of quiet work time followed by a whole-class discussion.

Anticipated Misconceptions

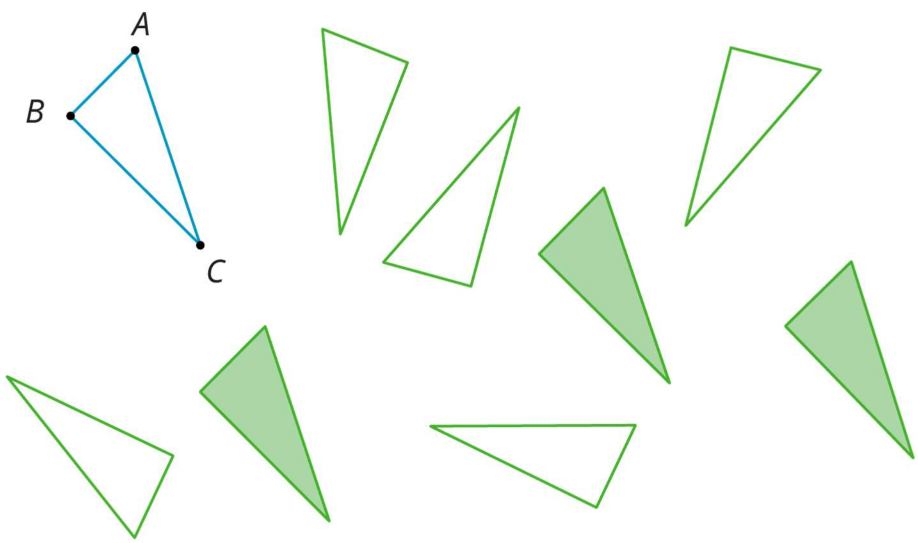
If any students assert that a triangle is a translation when it isn't really, ask them to use tracing paper to demonstrate how to translate the original triangle to land on it. Inevitably, they need to rotate or flip the paper. Remind them that a translation consists only of sliding the tracing paper around without turning it or flipping it.

Student Task Statement

All of these triangles are congruent. Sometimes we can take one shape to another with a translation. Shade the triangles that are images of triangle ABC under a translation.



Student Response



Activity Synthesis

Point out to students that if we just translate a shape, the image will end up pointed in the same direction. (More formally, the shape and its image have the same mirror and rotational orientation.) Rotations and reflections usually (but not always) change the orientation of a shape.

For a couple of the triangles that are not translations of the given shape, ask what sequence of transformations would show that they are congruent, and demonstrate any rotations or reflections required.

12.2 Congruent Pairs (Part 1)

15 minutes

In the previous lesson, students formulated a precise mathematical definition for congruence and began to apply this to determine whether or not pairs of shapes are congruent. This activity is a direct continuation of that work with the extra structure of a square grid. The square grid can be a helpful structure for describing the different transformations in a precise way. For example, with translations we can talk about translating up or down or to the left or right by a specified number of units. Similarly, we can readily reflect in horizontal and vertical lines and perform some simple rotations. Students may also wish to use tracing paper to help execute these transformations.

Students are given several pairs of shapes on grids and asked to determine if the shapes are congruent. The congruent shapes are deliberately chosen so that more than one transformation will likely be required to show the congruence. In these cases, students will likely find different ways to show the congruence. Monitor for different sequences of transformations that show congruence. For example, for the first pair of quadrilaterals, some different ways are:

- Translate $EFGH$ 1 unit to the right, and then rotate its image 180° about (0,0).
- Reflect $ABCD$ in the x-axis, then reflect its image in the y-axis, and then translate this image 1 unit to the left.

For the pairs of shapes that are *not* congruent, students need to identify a feature of one shape not shared by the other in order to argue that it is not possible to move one shape on top of another with translations, rotations or reflections. At this early stage, arguments can be informal. Monitor for these situations:

- The side lengths are different so it is not possible to make them match up.
- The angles are different so the two shapes cannot be made to match up.
- The areas of the shapes are different.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Compare and Connect

Launch

Provide access to geometry toolkits. Allow for 5–10 minutes of quiet work time followed by a whole-class discussion.

Engagement: Internalise Self-Regulation. Chunk this task into more manageable parts to differentiate the degree of difficulty or complexity. Invite students to choose and respond to 2 out of 4 questions. Once students have successfully completed them, invite them to share with a partner prior to the whole class discussion.
Supports accessibility for: Organisation; Attention

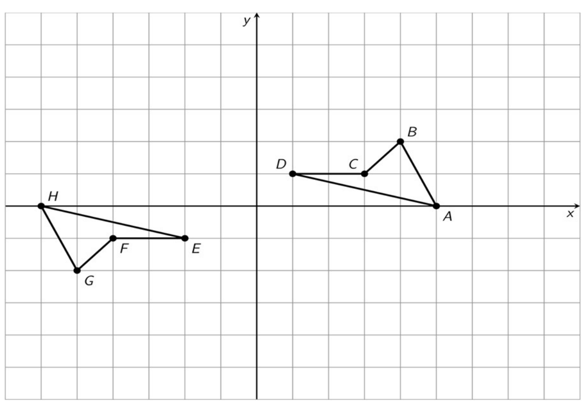
Anticipated Misconceptions

Students may want to visually determine congruence each time or explain congruence by saying, "They look the same." Encourage those students to explain congruence in terms of translations, rotations, reflections, and side lengths. For students who focus on features of the shapes such as side lengths and angles, ask them how they could show the side lengths or angles are the same or different using the grid or tracing paper.

Student Task Statement

For each of the following pairs of shapes, decide whether or not they are congruent. Explain your reasoning.

1.



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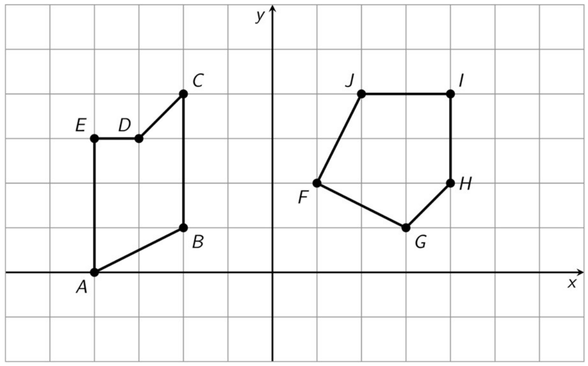

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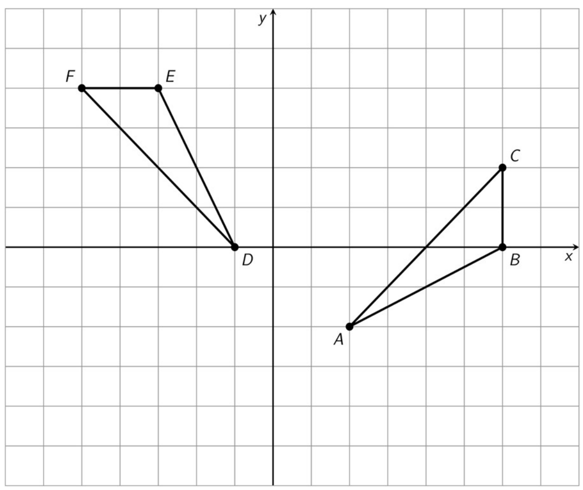

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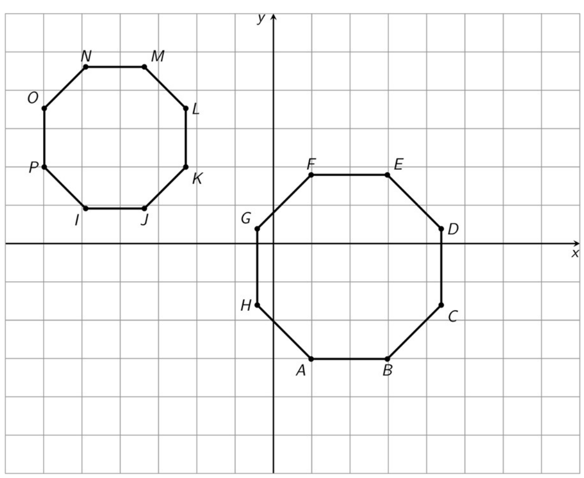


Student Response

1. These are congruent. Sample response: Rotate quadrilateral $ABCD$ around D by 180 degrees, and then translate left 3 units and down 2 units. It matches up perfectly with $HGFE$.

2. These are not congruent. Sample response 1: They are both pentagons, but $ABCDE$ has a pair of opposite parallel sides while $FGHIJ$ does not. Sample response 2: Angle D in $ABCDE$ measures more than 180 degrees, while all angles in $FGHIJ$ measure less than 180 degrees. Sample response 3: Side DE measures one unit in length, while all sides of $FGHIJ$ measure more than 1 unit in length.

3. These are congruent. Sample response: Rotate triangle ABC around (0,0) anti-clockwise by 90 degrees, and then translate it down 2 units and left 3 units. It matches up with triangle DEF perfectly.

4. These are not congruent. Sample response: Both are regular octagons, but $ABCDEFGH$ is larger than $IJKLMNOP$. This can be seen by comparing the images or by looking at sides AB and IJ. Side AB is 2 units in length, while side IJ is less than 2 units in length.

Activity Synthesis

Poll the class to identify which shapes are congruent (A and C) and which ones are not (B and D). For the congruent shapes, ask which motions (translations, rotations, or reflections) students used, and select previously identified students to show different methods. Sequence the methods from most steps to fewest steps when possible.

For the shapes that are *not* congruent, invite students to identify features that they used to show this and ask students if they tried to move one shape on top of the other. If so, what happened? It is important for students to connect the differences between identifying congruent vs non-congruent shapes.

The purpose of the discussion is to understand that when two shapes are congruent, there is a translation, rotation or reflection that matches one shape up perfectly with the other. Choosing the right sequence takes practice. Students should be encouraged to experiment, using technology and tracing paper when available. When two shapes are not congruent, there is no translation, rotation or reflection that matches one shape up perfectly with the other. It is not possible to perform every possible sequence of transformations in practice, so to show that one shape is *not* congruent to another, we identify a property of one shape that is not shared by the other. For the shapes in this problem set, students can focus on side lengths: for each pair of non congruent shapes, one shape has a side length not shared by the other. Since transformations do not change side lengths, this is enough to conclude that the two shapes are not congruent.

Representing, Conversing, Listening: Compare and Connect. As students prepare their work for discussion, look for approaches that focus on visually determining congruence and on approaches that focus on features of the shapes such as side lengths and angles. Encourage students to explain congruence in terms of translations, rotations, reflections, and side lengths and to show physical representations of congruence of side lengths and angles

using grid or tracing paper. Emphasise transformational language used to make sense of strategies to identify congruent and non-congruent shapes.
Design Principle(s): Maximise meta-awareness; Support sense-making

12.3 Congruent Pairs (Part 2)

15 minutes

Students take turns with a partner claiming that two given polygons are or are not congruent and explaining their reasoning. The partner's job is to listen for understanding and challenge their partner if their reasoning is incorrect or incomplete. This activity presents an opportunity for students to justify their reasoning and critique the reasoning of others.

This activity continues to investigate congruence of polygons on a grid. Unlike in the previous activity, the non-congruent pairs of polygons share the same side lengths.

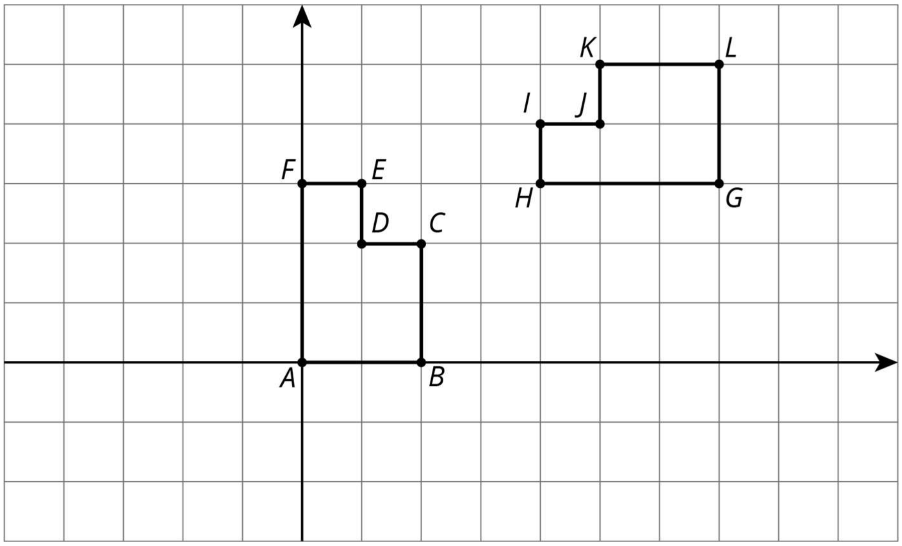
Instructional Routines

- Discussion Supports
- Take Turns

Launch

Arrange students in groups of 2, and provide access to geometry toolkits. Tell students that they will take turns on each question. For the first question, student A should claim whether the shapes are congruent or not. If student A claims they are congruent, they should describe a sequence of transformations to show congruence, while student B checks the claim by performing the transformations. If student A claims the shapes are not congruent, they should support this claim with an explanation to convince student B that they are not congruent. For each question, students exchange roles.

Ask for a student volunteer to help you demonstrate this process using the pair of shapes here.



Then, students work through this same process with their own partners on the questions in the activity.

Anticipated Misconceptions

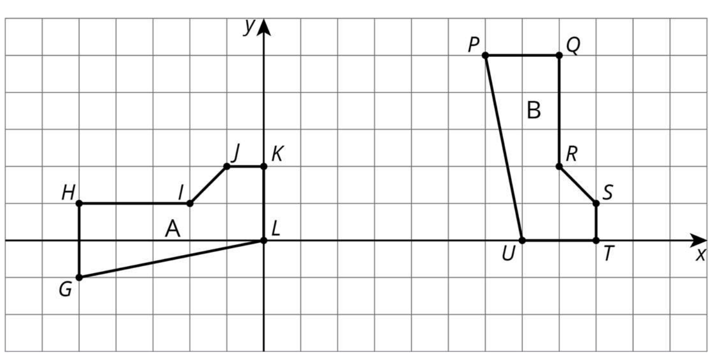
For D, students may be correct in saying the shapes are not congruent but for the wrong reason. They may say one is a 3-by-3 square and the other is a 2-by-2 square, counting the diagonal side lengths as one unit. If so, have them compare lengths by marking them on the edge of a card, or measuring them with a ruler.

In discussing congruence for problem 3, students may say that quadrilateral $GHIJ$ is congruent to quadrilateral $PQRS$, but this is not correct. After a set of transformations is applied to quadrilateral $GHIJ$, it corresponds to quadrilateral $QRSP$. The vertices *must* be listed in this order to accurately communicate the correspondence between the two congruent quadrilaterals.

Student Task Statement

For each pair of shapes, decide whether or not shape A is congruent to shape B. Explain how you know.

1.



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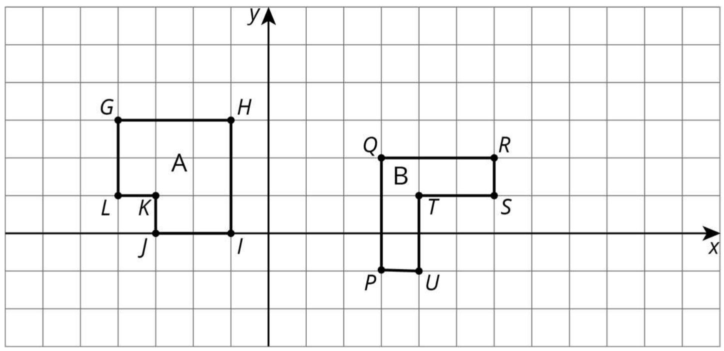

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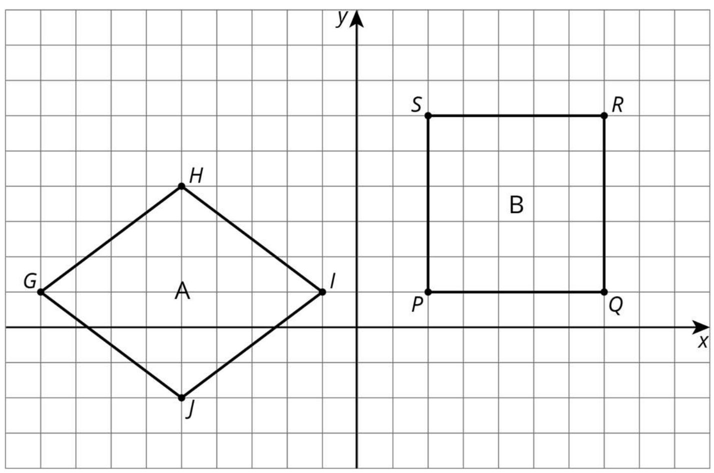

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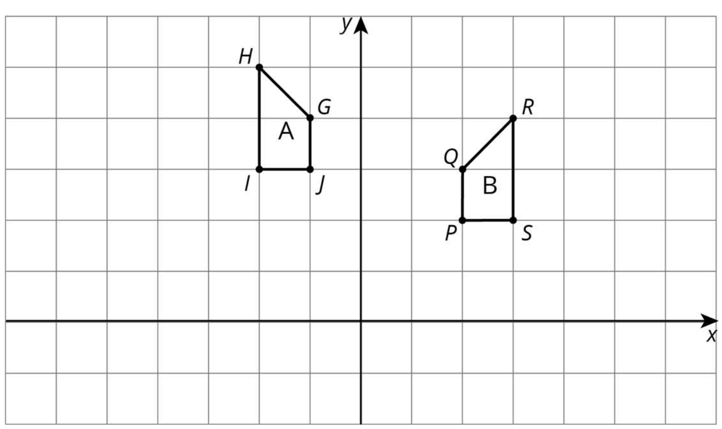

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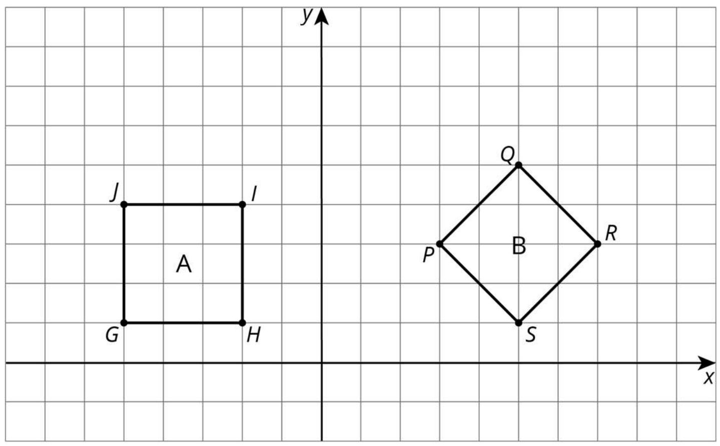


Student Response

1. Yes, they are congruent. Sample response: Rotate hexagon A 90 degrees clockwise with centre (0,0), and then translate it 7 units to the right. It matches up perfectly with hexagon B.

2. No, they are not congruent. Sample response: hexagon A has greater area (8 square units) than hexagon B (6 square units). They are not congruent because translations, rotations, and reflections do *not* change the area of a shape.

3. No, they are not congruent. Sample response: Both shapes are quadrilaterals, and the side lengths all appear to be 5 units in length. But the angles are not the same. Quadrilateral B is a square with 4 right angles. Quadrilateral A is a rhombus. Angles G and I are acute while angles H and J are obtuse. Since translations, rotations, and reflections do not change angles, there is no way to match up any of the angles of these quadrilaterals.

4. Yes, they are congruent. Sample response: Reflect quadrilateral A in the y-axis, and then translate one unit to the right and one unit down. It matches up perfectly with quadrilateral B.

5. No, they are not congruent. Sample response 1: Rotate quadrilateral B about S by 45 degrees anti-clockwise and then translate to the left by 7 units. Angle PSR matches up with angle GHI, but the sides of quadrilateral B are a little shorter than those of quadrilateral A, so the two shapes are not congruent. Sample response 2: The area of square quadrilateral A is 9 square units. The area of quadrilateral B (which is also a square) is 8 square units because it contains 4 whole unit squares and then 8 half unit squares that make 4 more unit squares. Congruent shapes have the same area so these two shapes are not congruent.

Are You Ready for More?

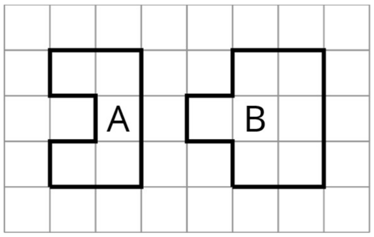
A polygon has 8 sides: five of length 1, two of length 2, and one of length 3. All sides lie on grid lines. (It may be helpful to use graph paper when working on this problem.)

1. Find a polygon with these properties.

2. Is there a second polygon, not congruent to your first, with these properties?

Student Response

Here are two non-congruent shapes that meet the conditions.



Activity Synthesis

To highlight student reasoning and language use, invite groups to respond to the following questions:

- "For which shapes was it easiest to give directions to your partner? Were some transformations harder to describe than others?"
- "For the pairs of shapes that were *not* congruent, how did you convince your partner? Did you use transformations or did you focus on some distinguishing features of the shapes?"
- "Did you use any measurements (length, area, angles) to help decide whether or not the pairs of shapes are congruent?"

For more practice articulating why two shapes are or are not congruent, select students with different methods to share how they showed congruence (or not). If the previous activity provided enough of an opportunity, this may not be necessary.

Listening, Representing: Discussion Supports. During the final discussion, revoice student ideas to demonstrate mathematical language use by restating a statement as a question in order to clarify, apply appropriate language, and involve more students.
Design Principle(s): Maximise meta-awareness

12.4 Building Quadrilaterals

Optional: 10 minutes

In previous activities, students saw that two congruent polygons have the same side lengths in the same order. They have also seen that congruent polygons have corresponding angles with the same sizes. In this activity, students build quadrilaterals that contain congruent sides and investigate whether or not they form congruent quadrilaterals.

In addition to building an intuition for how side lengths and angle sizes influence congruence, students also get an opportunity to revisit the taxonomy of quadrilaterals as they study which types of quadrilaterals they are able to build with specified side lengths.

This high level view of different types of quadrilaterals is a good example of seeing and understanding mathematical structure.

Watch for students who build both parallelograms and kites with the two pair of sides of different lengths. Invite them to share during the discussion.

Instructional Routines

- Discussion Supports

Launch

There are two sets of building materials. Each set contains 4 side lengths. Set A contains 4 side lengths of the same size. Set B contains 2 side lengths of one size and 2 side lengths of another size.

Divide the class into two groups. One group will be assigned to work with Set A, and the other with Set B. Within each group, students work in pairs. Each pair is given two of the same set of building materials. Each student uses the set of side lengths to build a quadrilateral at the same time. Each time a new set of quadrilaterals is created, the partners compare the two quadrilaterals created and determine whether or not they are congruent. Give students 5 minutes to work with their partner followed by a whole-class discussion.

Representation: Develop Language and Symbols. Display or provide charts with symbols and meanings. For example, display a chart of the taxonomy of quadrilaterals to provide access to precise language of the different types of quadrilaterals they are building.
Supports accessibility for: Conceptual processing; Memory

Anticipated Misconceptions

Students may assume when you are building quadrilaterals with a set of objects of the same length, the resulting shapes are congruent. They may think that two shapes are congruent because they can physically manipulate them to make them congruent. Ask them to first build their quadrilateral and then compare it with their partner's. The goal is not to ensure the two are congruent but to decide whether they *have to be* congruent.

Student Task Statement

Your teacher will give you a set of four objects.

1. Make a quadrilateral with your four objects and record what you have made.

2. Compare your quadrilateral with your partner’s. Are they congruent? Explain how you know.

3. Repeat steps 1 and 2, forming different quadrilaterals. If your first quadrilaterals were not congruent, can you build a pair that is? If your first quadrilaterals were congruent, can you build a pair that is not? Explain.

Student Response

There should be a variety of rhombuses and squares from Set A and parallelograms and kites from Set B. It is possible to build both congruent and non-congruent polygons from both sets of objects.

Activity Synthesis

To start the discussion, ask:

- "Were the quadrilaterals that you built always congruent? How did you check?"
- "Was it *possible* to build congruent quadrilaterals? What parts were important to be careful about when building them?"

Students should recognise that there are three important concerns when creating congruent polygons: congruent sides, congruent angles, and the order in which they are assembled.

Tell students that it is actually enough to guarantee congruence between two *polygons* if all three of those criteria are met. That is, "Two polygons are congruent if they have corresponding sides that are congruent *and* corresponding angles that are congruent."

Also highlight the fact that with two pairs of different congruent sides, there are two different types of quadrilaterals that can be built: kites (the pairs of congruent sides are adjacent) and parallelograms (the pairs of congruent sides are opposite one another). When all 4 sides are congruent, the quadrilaterals that can be built are all rhombuses.

Speaking: Discussion Supports. Give students extra time to make sure that every pair can explain or justify each step or part of the problem. Revoice student ideas to model uses of disciplinary language as you press for details. Make sure to vary who is called on to represent the work during the discussion so students get accustomed to preparing each other to fill that role. This will prepare students for the role of group representative and to support each other to take on that role.
Design Principle(s): Support sense-making

Lesson Synthesis

The main points to highlight at the conclusion of the lesson are:

- Two shapes are congruent when there is a sequence of translations, rotations, and reflections that match one shape up perfectly with the other (this is from the previous lesson but it is vital to thinking in this lesson as well.).
- When showing that two shapes are congruent on a grid, we use the structure of the grid to describe each translation, rotation or reflection. For example, translations can be described by indicating how many grid units to move left or right and how many grid units to move up or down. Reflections can be described by indicating the line of reflection (an axis or a particular grid line are readily available).

- Two shapes are *not* congruent if they have different side lengths, different angles, or different areas.

- Even if two shapes have the same side lengths, they may not be congruent. With four sides of the same length, for example, we can make many different rhombuses that are not congruent to one another because the angles are different.

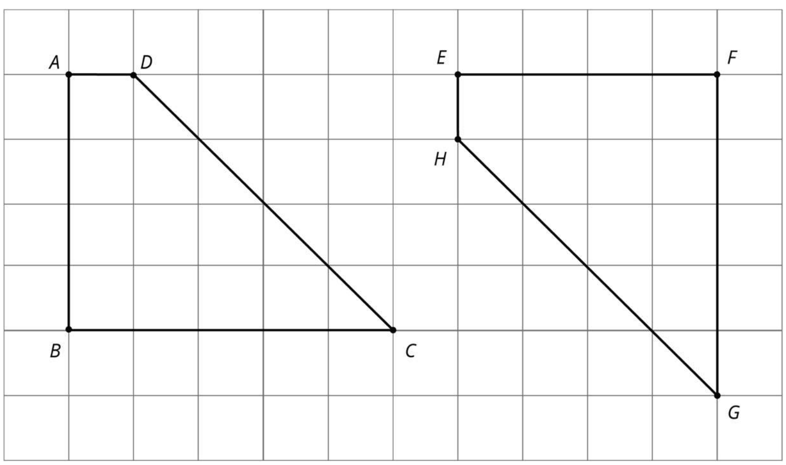
12.5 Moving to Congruence

Cool Down: 5 minutes

Students describe explicit transformations that take one polygon to another.
Several solutions are possible. Though students may use tracing paper to help visualise the different transformations, at this point they should be able to articulate the move abstractly using the language of translations, rotations, and reflection.

Student Task Statement

Describe a sequence of reflections, rotations, and translations that shows that quadrilateral $ABCD$ is congruent to quadrilateral $EFGH$.



Student Response

Answers vary. Sample response: Translate $ABCD$ down 1 and 5 to the right. Then reflect in line GH.

Student Lesson Summary

How do we know if two shapes are congruent?

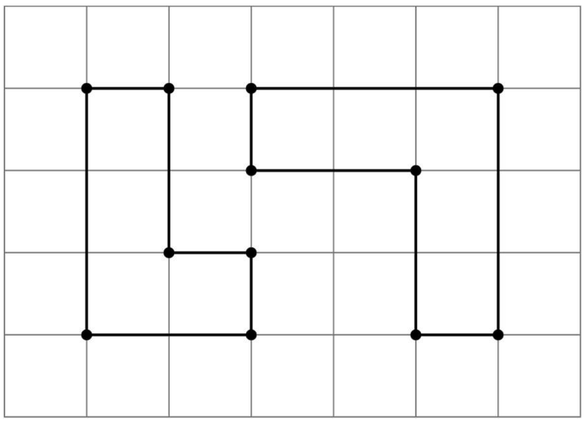
- If we copy one shape on tracing paper and move the paper so the copy covers the other shape exactly, then that suggests they are congruent.

- We can prove that two shapes are congruent by describing a sequence of translations, rotations, and reflections that move one shape onto the other so they match up exactly.

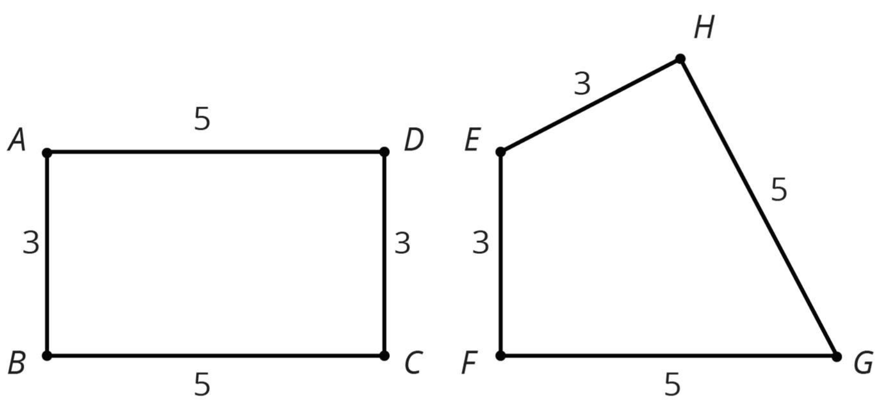
How do we know that two shapes are *not* congruent?

- If there is no correspondence between the shapes where the parts have equal size, that proves that the two shapes are *not* congruent. In particular,

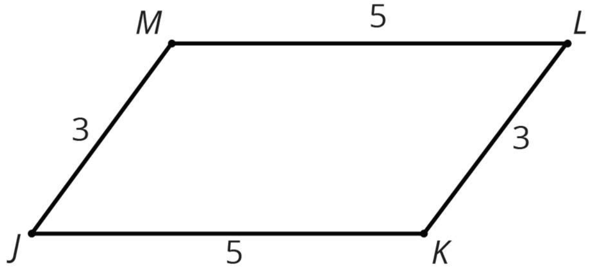
    – If two polygons have different sets of side lengths, they can't be congruent. For example, the shape on the left has side lengths 3, 2, 1, 1, 2, 1. The shape on the right has side lengths 3, 3, 1, 2, 2, 1. There is no way to make a correspondence between them where all corresponding sides have the same length.



    – If two polygons have the same side lengths, but their orders can't be matched as you go around each polygon, the polygons can't be congruent. For example, rectangle $ABCD$ can't be congruent to quadrilateral $EFGH$. Even though they both have two sides of length 3 and two sides of length 5, they don't correspond in the same order. In $ABCD$, the order is 3, 5, 3, 5 or 5, 3, 5, 3; in $EFGH$, the order is 3, 3, 5, 5 or 3, 5, 5, 3 or 5, 5, 3, 3.



    – If two polygons have the same side lengths, in the same order, but different corresponding angles, the polygons can't be congruent. For example, parallelogram $JKLM$ can't be congruent to rectangle $ABCD$. Even though they have the same side lengths in the same order, the angles are different. All angles in $ABCD$ are **right angles**. In $JKLM$, angles J and L are less than 90 degrees and angles K and M are more than 90 degrees.



Glossary

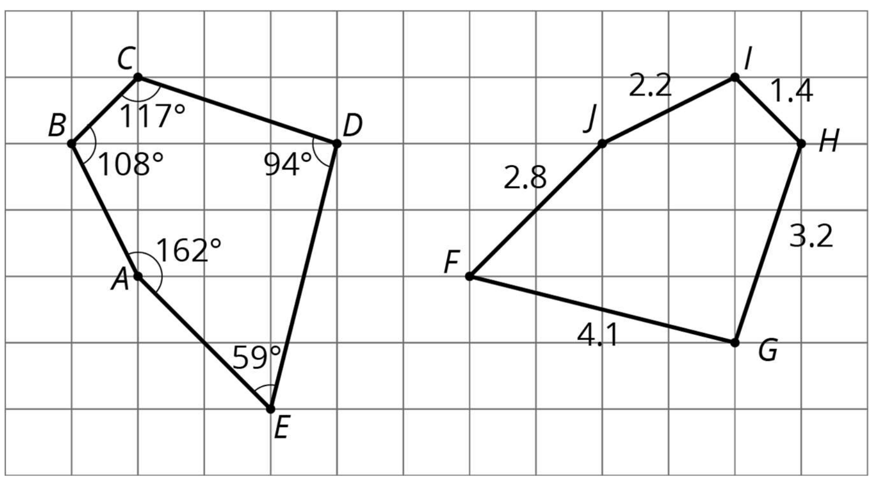
- right angle

Lesson 12 Practice Problems

1. **Problem 1 Statement**

    a. Show that the two pentagons are congruent.

    b. Find the side lengths of $ABCDE$ and the angles of $FGHIJ$.



Solution

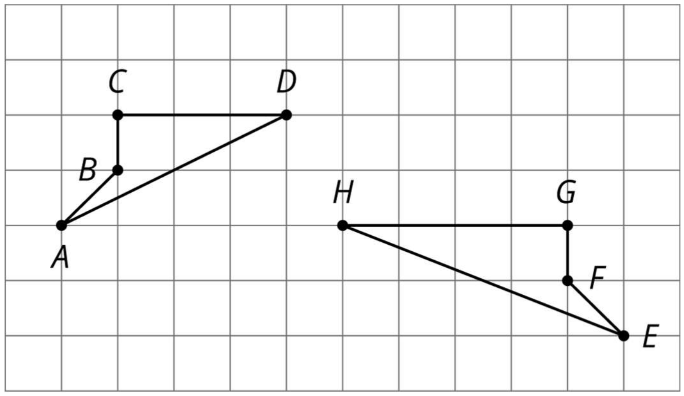
a. After performing a 90-degree clockwise rotation with centre D, then translating 3 units down and 6 units to the right, $ABCDE$ matches up perfectly with $JIHGF$. The rotation and translation do not change side lengths or angles.

b. $AB = 2.2$, $BC = 1.4$, $CD = 3.2$, $DE = 4.1$, and $EA = 2.8$. $\angle F = 59°$, $\angle G = 94°$, $\angle H = 117°$, $\angle I = 108°$, and $\angle J = 162°$.

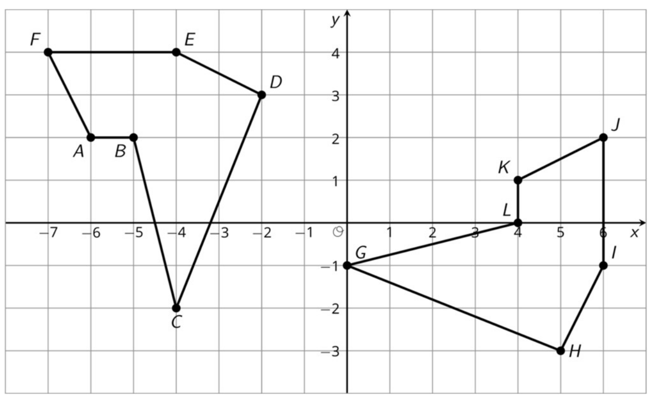
2. **Problem 2 Statement**

For each pair of shapes, decide whether or not the two shapes are congruent. Explain your reasoning.

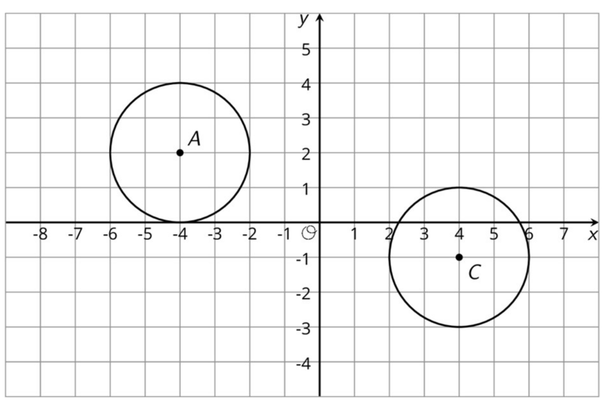
a.



b.



c.



Solution

a. Not congruent. Segment EH in polygon $EFGH$ is longer than any of the sides in polygon $ABCD$. A, B, and C can be matched up with vertices E, F, and G, but H does not match up with D.

b. Congruent. If $ABCDEF$ is rotated 90 degrees clockwise about C and then moved 4 units to the right and 1 unit up, it matches up perfectly with $GHIJKL$.

c. Congruent. If the circle on the top left is translated to the right by 8 units and down 3 units, it lands on top of the other circle.

3. **Problem 3 Statement**

   a. Draw line segment PQ.

   b. When PQ is rotated 180° around point R, the resulting line segment is the same as PQ. Where could point R be located?

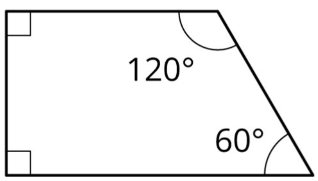
   Solution

   a. Answers vary.

   b. R must be the midpoint of PQ.

4. **Problem 4 Statement**

   Here is trapezium $ABCD$.



   Using translations, rotations or reflections on the trapezium, build a pattern. Describe some of the transformations you used.

   Solution

   Answers vary. Sample response: clockwise rotations, centred at the vertex of the 60° angle, of 60°, 120°, 180°, 240°, and 300° make a "windmill" type shape with copies of the trapezium.