Lesson 9: Measurement error (Part 2)

Goals

- Comprehend that percentage error is greatest when the actual values are smaller than the measured values for an object.
- Generalise (orally) a process for calculating the maximum percentage error for the area of a rectangle and volume of a cuboid.

Lesson Narrative

This lesson is optional. This is the second lesson exploring measurement error in more detail. While the previous lesson examined percentage error when measurements are added together, this lesson works with percentage error when measurements are multiplied. While the activities in this lesson can stand on their own, students will benefit from having done the activities in the previous lesson first.

In addition to examining accuracy of measurements carefully, students will work through examples and look for patterns in order to hypothesise, and eventually show, how percentage error behaves when measurements with error are multiplied to one another.

As with all lessons in this unit, all related topics have been addressed in prior units. This lesson provides an optional opportunity to go deeper and make connections between domains.

Building On

- Use proportional relationships to solve multistep ratio and percentage problems. Examples: simple interest, tax, markups and markdowns, gratuities and commissions, fees, percentage increase and decrease, percentage error.

Addressing

- Solve real-world and mathematical problems involving area, volume and surface area of two- and three-dimensional objects composed of triangles, quadrilaterals, polygons, cubes, and prisms.

Instructional Routines

- Clarify, Critique, Correct
- Discussion Supports
- Think Pair Share

Required Materials

Four-function calculators

Student Learning Goals

Let's check how accurate our calculations are.

9.1 Measurement Error for Area

Optional: 20 minutes

This activity examines how measurement errors behave when quantities are multiplied. In other words, if I have a measurement m with a maximum error of 5% and a measurement n with a maximum error of 5%, what percentage error can $m \times n$ have?

Monitor for students who use different methods to solve the problem, such as trying out sample numbers or using expressions with variables.

Instructional Routines

- Clarify, Critique, Correct
- Think Pair Share

Launch

Arrange students in groups of 2. Provide access to calculators.

If desired, suggest that students try out several different sample numbers for the length and width of the rectangle, calculate the maximum percentage error, and look for a pattern. Give students 4–5 minutes of quiet work time, followed by time to discuss their work with their partner and make revisions, followed by whole-class discussion.

Conversing, Representing: Clarify, Critique, Correct. Present an incorrect response that reflects a possible misunderstanding from the class such as: "Since both length and width have maximum errors of 5%, I multiplied 5% by 5% to get 25%, because area is length times width." Prompt pairs to clarify and then critique the incorrect response, and then write a correct version. This provides students with an opportunity to evaluate, and improve on, the written mathematical arguments of others.
Design Principle(s): Maximise meta-awareness; Support sense-making

Anticipated Misconceptions

Students may think that the maximum error possible for the area is 5% because both the length and width are within 5% of the actual values. Encourage these students to make calculations of the biggest and smallest possible length and the biggest and smallest possible width. Then have them make calculations for the biggest and smallest possible area.

Student Task Statement

Imagine that you measure the length and width of a rectangle and you know the measurements are accurate within 5% of the actual measurements. If you use your measurements to find the area, what is the maximum percentage error for the area of the rectangle?

Student Response

The maximum percentage error would be 10.25%. If x is the actual length and m is the measured length, then $0.95x < m$ and $m < 1.05x$ since $0.05x$ is 5% of x. If y is the actual width and n is the measured width, then the biggest possible error is $0.05y$ so $0.95y < n$ and $n < 1.05y$. If they are both maximum, the area would be $1.05^2xy = 1.1025xy$. If they are both minimum, the area would be $0.95^2xy = 0.9025xy$. So the maximum percentage error would be when they are both at the maximum possible error, and the percentage error would be 10.25%.

Activity Synthesis

Have students swap papers with a partner and check their work.

Invite students to share their solutions, especially those who looked for a pattern or used variables. Consider discussing questions like these:

- "Did you calculate the maximum percentage error for any specific sample measurements? What did you find?" (The maximum percentage error for the largest and smallest possible values were not the same: 9.75% and 10.25%.)
- "How do you know that this pattern is true for any possible length and width of the rectangle?" (I used variables to express the unknown measurements.)
- "How could you use variables to help solve this problem?" (I can use variables to represent the length and width of the rectangle and write expressions in terms of these variables to represent the largest and smallest possible areas.)

9.2 Measurement Error for Volume

Optional: 25 minutes

This challenging activity examines how measurement errors behave when 3 quantities are multiplied (versus 2 quantities in the previous activity). In other words, if I have measurements a, b, and c each with a maximum error of 5%, what percentage error can $a \times b \times c$ have? The arithmetic and algebraic demands of this task are high because students take a product of *three* quantities that each have a maximum percentage error of 5%.

Instructional Routines

- Discussion Supports

Launch

Arrange students in groups of 2. Provide access to calculators. Make sure students realise that the first question gives the measured values, not the actual values, for each dimension.

Give students 10 minutes to discuss with their partners, followed by whole-class discussion.

Conversing: Discussion Supports. Before students determine the maximum measurement error for the volume of a cuboid, invite pairs to make a prediction and justification. Use a sentence frame such as: "In a cuboid, if each dimension has a 5% maximum measurement error, we predict the volume's maximum percentage error is ___ because….". This will help students use the mathematical language of justifications to begin reasoning about the measurement error for volume.
Design Principle(s): Cultivate conversation; Support sense-making

Anticipated Misconceptions

Students may think that the maximum error possible for the volume is 1% because the length, width, and height are within 1% of the actual values. Encourage these students to make calculations of the biggest and smallest possible length, width, and height. Then ask them to make calculations for the biggest and smallest possible volumes.

Student Task Statement

1. The length, width, and height of a cuboid were measured to be 10 cm, 12 cm, and 25 cm. Assuming that these measurements are accurate to the nearest cm, what is the largest percentage error possible for:
   a. each of the dimensions?
   b. the volume of the cuboid?
2. If the length, width, and height of a cuboid have a maximum percentage error of 1%, what is the largest percentage error possible for the volume of the cuboid?

Student Response

1. The actual measurements could be as much as 0.5 cm over or under the measurements given.
   a. We know the largest percentage error occurs with the *smallest* measurement, so we will only check it for the minimum possible actual lengths.
      - $0.5 \div 9.5 \approx 0.05$ (or about 5%)
      - $0.5 \div 11.5 \approx 0.04$ (or about 4%)
      - $0.5 \div 24.5 \approx 0.02$ (or about 2%)

b. The measured volume is $3\,000$ cubic cm. The smallest the actual volume could be is $9.5 \times 11.5 \times 24.5 = 2\,676.625$. The percentage error in the case of the smallest one is $(3\,000 - 2\,676.625) \div 2\,676.625$ or about 12%. The biggest the actual volume could be is $10.5 \times 12.5 \times 25.5 = 3\,346.875$, which gives a percentage error of about 10%. Again, we find that the biggest possible error happens when the actual measurement is as small as possible.

2. If the actual dimensions are x, y, and z, then the minimum measured volume would be $(0.99)^3 xyz \approx 0.97xyz$ and the maximum measured volume would be $(1.01)^3 xyz \approx 1.03xyz$. So the largest percentage error in the volume is about 3%.

Activity Synthesis

Some discussion points include:

- The first problem gives measurements and errors (but no percentage error), while the second problem gives no measurements but does give the percentage error. This makes the calculations notably different for the two problems.
- In the first problem, we are given measurements and the possible size of error. We need to find the greatest percentage error and, as we have seen in other cases, this happens for the smallest possible value of the measurement. If we were to find the percentage error of each measurement, we would find that the error for the volume is a *larger* percentage error than for any of the individual measurements.
- In the second problem, we are given the maximum possible percentage error but no measurements, and we need to find the largest possible error for the volume, that is, for the product of the three unknown measurements. The greatest percentage error possible for the volume occurs when the measured value is as large as possible.
- A unifying feature in these two problems is that we notice that the largest percentage error occurs when the actual measurements are *smaller* than the measured values, as much smaller as possible.

Representation: Internalise Comprehension. Use colour and annotations to illustrate connections between representations. As students share their diagrams and reasoning, use colour and annotations to scribe their thinking on a display of each problem so that it is visible for all students.
Supports accessibility for: Visual-spatial processing; Conceptual processing