Lesson 4: Estimating probabilities through repeated experiments

Goals

- Describe (orally and in writing) patterns observed on a table or graph that shows the relative frequency for a repeated experiment.
- Generalise (orally) that the cumulative relative frequency approaches the probability of the event as an experiment is repeated many times.
- Generate possible results that would or would not be surprising for a repeated experiment, and justify (orally) the reasoning.

Learning Targets

- I can estimate the probability of an event based on the results from repeating an experiment.
- I can explain whether certain results from repeated experiments would be surprising or not.

Lesson Narrative

In this lesson students roll a dice many times and calculate the cumulative fraction of the time that an event occurs to see that in the long run this relative frequency approaches the probability of the chance event. They also see that the relative frequency of a chance event will not usually exactly match the actual probability. For example, when flipping a coin 100 times, the coin may land showing a head 46 times instead of exactly 50 times and not be considered unreasonable.

In future lessons students will be asked to design and use simulations. Each lesson leading up to that helps prepare students by giving them hands-on experience with different types of chance experiments they could choose to use in their simulations. In this lesson students work with rolling a dice and tossing a coin.

Building On

- Read, write, and compare decimals to thousandths.
- Convert a rational number to a decimal using long division; know that the decimal form of a rational number terminates in 0s or eventually repeats.

Addressing

- Analyse proportional relationships and use them to solve real-world and mathematical problems.
- Understand that the probability of a chance event is a number between 0 and 1 that expresses the likelihood of the event occurring. Larger numbers indicate greater likelihood. A probability near 0 indicates an unlikely event, a probability around $\frac{1}{2}$

indicates an event that is neither unlikely nor likely, and a probability near 1 indicates a likely event.

- Approximate the probability of a chance event by collecting data on the chance process that produces it and observing its long-run relative frequency, and predict the approximate relative frequency given the probability. For example, when rolling a dice 600 times, predict that a 3 or 6 would be rolled roughly 200 times, but probably not exactly 200 times.
- Develop a probability model and use it to find probabilities of events. Compare probabilities from a model to observed frequencies; if the agreement is not good, explain possible sources of the discrepancy.
- Develop a probability model (which may not be uniform) by observing frequencies in data generated from a chance process. For example, find the approximate probability that a spinning penny will land heads up or that a tossed paper cup will land open-end down. Do the outcomes for the spinning penny appear to be equally likely based on the observed frequencies?

Instructional Routines

- Collect and Display
- Discussion Supports
- Think Pair Share

Required Materials

Graph paper
Dice
cubes with sides numbered from 1 to 6

Required Preparation

The In the Long Run activity requires 1 dice for every 3 students. Access to graph paper may be useful, but is not required.

Student Learning Goals

Let’s do some experimenting.

4.1 Decimals on the Number Line

Warm Up: 5 minutes

The purpose of this warm-up is for students to practice placing numbers represented with decimals on a number line and thinking about probabilities of events that involve the values of the numbers. In the following activity, students are asked to graph points involving probabilities that are represented by numbers similar to the ones in this activity.

Instructional Routines

- Think Pair Share

Launch

Arrange students in groups of 2. Give students 2 minutes of quiet work time followed by time to share their responses with a partner. Follow with a whole-class discussion.

Student Task Statement

1. Locate and label these numbers on the number line.

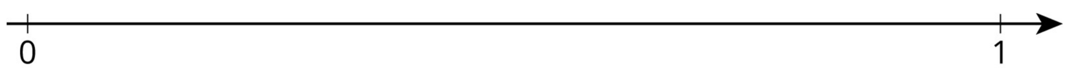
   a. 0.5

   b. 0.75

   c. 0.33

   d. 0.67

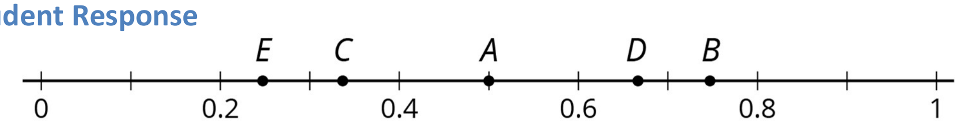
   e. 0.25



2. Choose one of the numbers from the previous question. Describe a game in which that number represents your probability of winning.

Student Response

1.



2. Answers vary. Sample response: Get blue when randomly selecting a colour from the primary colours: red, yellow, and blue. This has a probability of 0.33.

Activity Synthesis

Select some partners to share their responses and methods for positioning the points on the number line. If time allows, select students to share a chance event for each of the values listed.

4.2 In the Long Run

20 minutes (there is a digital version of this activity)

This activity begins to answer the question brought up in the previous lesson about finding the probability when the sample space is not available. Students have the opportunity to use this experiment for which the sample space is available to check its agreement and estimate based on repeating the experiment many times.

Students make the connection between probability and the fraction of outcomes for which the event occurs in the long-run. This activity highlights that a probability describes what happens in the the long run and that it does not guarantee that the event will occur a specific number of times after any specific number of trials. For example, an event that has probability 0.6 means that the event will occur about 60% of the time in the long run, but it does not mean that it will occur exactly 60 times when the experiment is performed 100 times.

Instructional Routines

- Discussion Supports

Launch

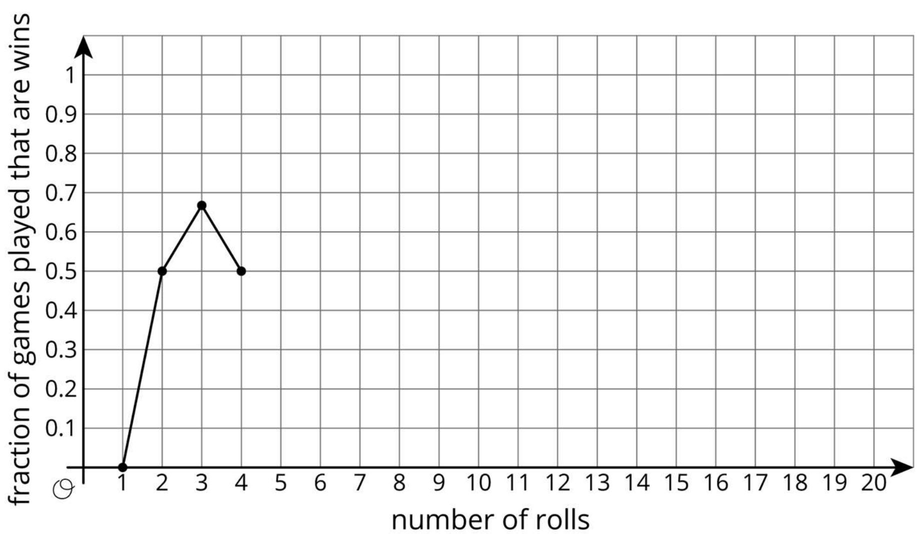
Arrange students in groups of 3. Provide one standard dice for each group. Following the teacher demonstration, allow 10 minutes for group work, followed by a whole-class discussion.

Demonstrate how to calculate and plot the current fraction of the times an event occurs.

Classes using the digital version have an applet available that automates the calculation and the graphing, allowing students to focus on the probabilities.

Display the table and graph for all to see as an example of how to fill in the table and graph the results.

roll	number rolled	total number of wins for Mai	fraction of games that are wins
1	5	0	0
2	1	1	$\frac{1}{2} = 0.50$
3	2	2	$\frac{2}{3} \approx 0.67$
4	4	2	$\frac{2}{4} = 0.50$



To help students understand the graph, consider asking these questions. Ask students why the y-axis only shows 0 to 1. Ask students what the point at (3,0.66) represents.

Anticipated Misconceptions

Students may not notice a pattern in the graph. Ask if they can see a pattern with the decimal values for the fraction of wins in their table. If their data does not fit the expected pattern, tell them that this is not typical and ask them to look at another group's results.

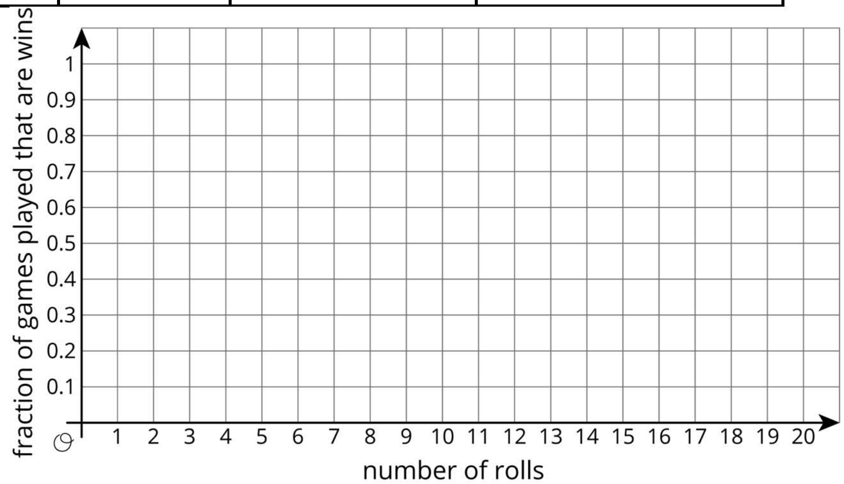
Student Task Statement

Mai plays a game in which she only wins if she rolls a 1 or a 2 with a standard dice.

1. List the outcomes in the sample space for rolling the dice.
2. What is the probability Mai will win the game? Explain your reasoning.
3. If Mai is given the option to flip a coin and win if it comes up heads, is that a better option for her to win?
4. With your group, follow these instructions 10 times to create the graph.
   - One person rolls the dice. Everyone records the outcome.
   - Calculate the fraction of rolls that are a win for Mai so far. Approximate the fraction with a decimal value rounded to the hundredths place. Record both the fraction and the decimal in the last column of the table.
   - On the graph, plot the number of rolls and the fraction that were wins.
   - Pass the dice to the next person in the group.

roll	outcome	total number of wins for Mai	fraction of games played that are wins
1			
2			
3			
4			
5			
6			
7			
8			

9			
10			



5. What appears to be happening with the points on the graph?

   a. After 10 rolls, what fraction of the total rolls were a win?

   b. How close is this fraction to the probability that Mai will win?

6. Roll the dice 10 more times. Record your results in this table and on the graph from earlier.

roll	outcome	total number of wins for Mai	fraction of games played that are wins
11			
12			
13			
14			
15			
16			
17			
18			
19			
20			

a. After 20 rolls, what fraction of the total rolls were a win?

b. How close is this fraction to the probability that Mai will win?

Student Response

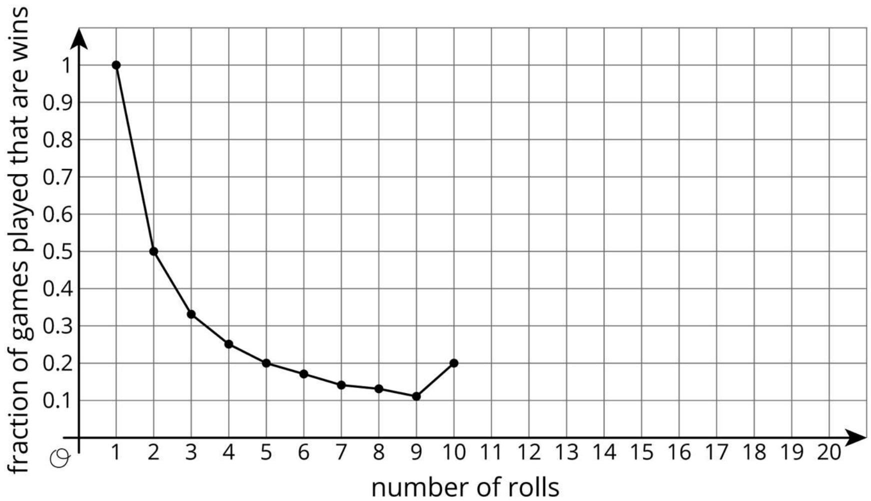
1. Sample space: 1, 2, 3, 4, 5, 6.

2. Mai should win with probability $\frac{2}{6} = \frac{1}{3}$, since 2 out of the 6 numbers win.

3. Flipping the coin gives Mai a better chance of winning, since the probability of getting heads is $\frac{1}{2}$. That is greater than $\frac{1}{3}$ for the dice.

4. Answers vary. Sample response:

roll	outcome	total number of wins for Mai	fraction of games played that are wins
1	2	1	$\frac{1}{1} = 1$
2	3	1	$\frac{1}{2} = 0.5$
3	6	1	$\frac{1}{3} \approx 0.33$
4	4	1	$\frac{1}{4} = 0.25$
5	5	1	$\frac{1}{5} = 0.2$
6	3	1	$\frac{1}{6} \approx 0.17$
7	5	1	$\frac{1}{7} \approx 0.14$
8	4	1	$\frac{1}{8} \approx 0.13$
9	5	1	$\frac{1}{9} \approx 0.11$
10	2	2	$\frac{2}{10} = 0.2$



5. The points seem to be jumping less wildly up and down.

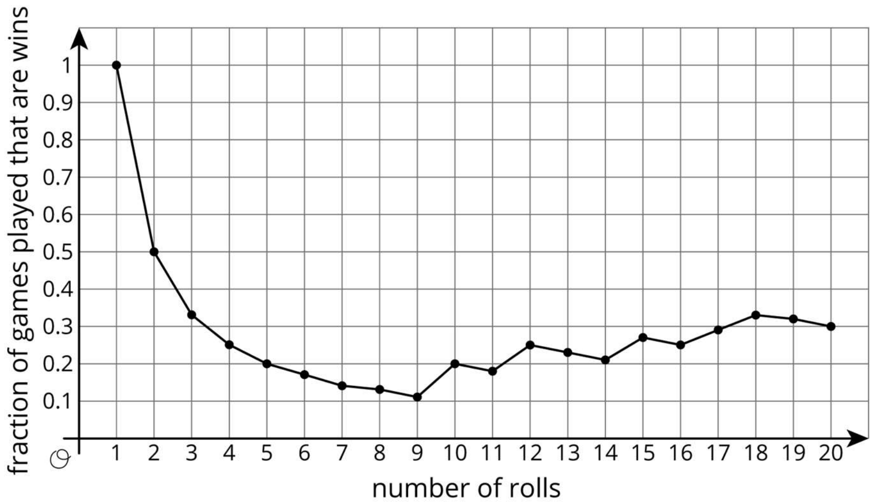
6. Answers vary. Sample response:

   a. $\frac{2}{10}$

   b. $\frac{2}{15} \approx 0.13$ below the expected probability since $\frac{1}{3} - \frac{2}{10} = \frac{2}{15}$

7. Answers vary. Sample response:

roll	outcome	total number of wins for Mai	fraction of games played that are wins
11	6	2	$\frac{2}{11} \approx 0.18$
12	1	3	$\frac{3}{12} = 0.25$
13	6	3	$\frac{3}{13} \approx 0.23$
14	5	3	$\frac{3}{14} \approx 0.21$
15	1	4	$\frac{4}{15} \approx 0.27$
16	6	4	$\frac{4}{16} = 0.25$
17	1	5	$\frac{5}{17} \approx 0.29$
18	2	6	$\frac{6}{18} \approx 0.33$
19	5	6	$\frac{6}{19} \approx 0.32$
20	4	6	$\frac{6}{20} = 0.3$



8. Answers vary. Sample response:

   a. $\frac{6}{20}$.

   b. My current fraction of rolls that are wins is $\frac{6}{20}$, but I expect the probability to be $\frac{2}{6}$, so they are about 0.03 apart.

Activity Synthesis

The purpose of this discussion is for students to understand that calculating the fraction of the time an event occurs can be used to estimate the probability of the event and that more repetitions should make the estimation more accurate.

Select some students to share their answer and reasoning for the second question. If it is not mentioned by students, tell them that when there is more than one outcome that is in the desired event, then the probability of that event is the number of outcomes in the desired event divided by the number of outcomes in the sample space. In this example, there are 2 outcomes that win (a roll of 1 or 2) and 6 outcomes in the sample space, so the probability of winning is $\frac{2}{6}$ which is equivalent to $\frac{1}{3}$.

Collect the number of 1s and 2s for each group and compute the fraction for the whole class with all the data. The value should be very close to $\frac{1}{3}$.

Select students to share their thoughts on what appears to be happening with the points on their graph. (They are levelling out at 0.33.) If students struggle with noticing that the points are levelling out at a y value of 0.33, ask them to draw a horizontal line on their graphs at their answer for the probability they got in the second question.

Ask the class how many times the entire class rolled dice. Then ask, "Based on the probability predicted in the second question, how many times do we expect the class to have simulated a win for Mai? How does this compare to the actual number of wins the class rolled."

A probability tells you how likely an event is to occur. While it is not guaranteed to be an exact match, if the chance experiment is repeated many times, we expect the fraction of times that an event occurs to be fairly close to the calculated probability.

Representation: Internalise Comprehension. Activate or supply background knowledge about finding patterns with decimal values for the fraction of wins in the last statement. Some students may benefit from a demonstration of how to approximate fractions with decimal values to graph. Invite students to engage in the process by offering suggested directions as you demonstrate.
Supports accessibility for: Visual-spatial processing; Organisation Speaking: Discussion Supports. Use this routine to support whole-class discussion. For each observation that is shared, ask students to restate and/or revoice what they heard using precise mathematical language. Consider providing students time to restate what they hear to a partner, before selecting one or two students to share with the class. Ask the original speaker if their peer was accurately able to restate their thinking. Call students' attention to any words or phrases that helped to clarify the original statement. This will provide more students with an opportunity to produce language as they interpret the reasoning of others.
Design Principle(s): Support sense-making

4.3 Due For a Win

10 minutes

This activity gives students the opportunity to see that an estimate of the probability for an event should be close to what is expected from the exact probability in the long-run; however, the outcome for a chance event is not guaranteed and estimates of the probability for an event using short-term results will not usually match the actual probability exactly.

Instructional Routines

- Collect and Display

Launch

Tell students that the probability of a coin landing heads up after a flip is $\frac{1}{2}$.

Give students 5 minutes of quiet work time followed by a whole-class discussion.

Student Task Statement

1. For each situation, do you think the result is surprising or not? Is it possible? Be prepared to explain your reasoning.

    a. You flip the coin once, and it lands heads up.

    b. You flip the coin twice, and it lands heads up both times.

    c. You flip the coin 100 times, and it lands heads up all 100 times.

2. If you flip the coin 100 times, how many times would you expect the coin to land heads up? Explain your reasoning.

3. If you flip the coin 100 times, what are some other results that would not be surprising?

4. You've flipped the coin 3 times, and it has come up heads once. The cumulative fraction of heads is currently $\frac{1}{3}$. If you flip the coin one more time, will it land heads up to make the cumulative fraction $\frac{2}{4}$?

Student Response

1.
   a. It is not surprising and is possible.
   b. It is a little more rare than the first one, but not very surprising. It is possible.
   c. It is very surprising and we may suspect the coin is not fair. It is possible, though.

2. It should be heads up about 50 times out of the 100. Since the probability is $\frac{1}{2}$, there should be about the same number of heads and tails.

3. Answers vary. Possible answers should range from approximately 40 to 60.

4. Not necessarily. There's still a 50% chance it will come up tails.

Activity Synthesis

The purpose of the discussion is for students to recognise that the actual results from repeating an experiment should be close to the expected probability, but may not match exactly.

For the first problem, ask students to indicate whether or not they think each result seems surprising. For the second and third questions, select several students to provide answers and display for all to see, then create a range of values that might not be surprising based on student responses. Ask the class if they agree with this range or to provide a reason the range is too large. It is not important for the class to get exact values, but a general agreement should arise that *some* range of values makes sense so that there does not need to be exactly 50 heads from the 100 flips.

An interesting problem in statistics is trying to define when things get "surprising." Flipping a fair coin 100 times and getting 55 heads should not be surprising, but getting either 5 or 95 heads probably is. (Although there is not a definite answer for this, a deeper study of statistics using additional concepts in secondary school or college can provide more information to help choose a good range of values (hypothesis testing).)

Explain that a probability represents the expected likelihood of an event occurring for a single trial of an experiment. Regardless of what has come before, each coin flip should still be equally likely to lands heads up as tails up.

As another example: A netball player who tends to make 75% of her shots will probably make about $\frac{3}{4}$ of the shots she attempts, but there is no guarantee she will make any individual shot even if she has missed a few in a row.

Conversing, Reading: Collect and Display. As students share whether each result is surprising or not, write down the words and phrases students use to explain their reasoning. Listen for students who state that the actual results from repeating an experiment should be close to the expected probability. As students review the language collected in the visual display, encourage students to revise and improve how ideas are communicated. For example, a phrase such as: “If you flip a coin 100 times, it is impossible for the coin to land heads up all 100 times” can be improved with the phrase “If you flip a coin 100 times, it is very unlikely for the coin to land heads up all 100 times, but it is possible.” This routine will provide feedback to students in a way that supports sense-making while simultaneously increasing meta-awareness of language.
Design Principle(s): Support sense-making; Maximise meta-awareness

Lesson Synthesis

Consider asking these questions:

- "You conduct a chance experiment many times and record the outcomes. How are these outcomes related to the probability of a certain event occurring?" (The fraction of times the event occurs after many repetitions should be fairly close to the expected probability of the event.)
- "What is the probability of rolling a 2, 3, or 4 on a standard dice? If you roll 3 times and none of them result in a 2, 3, or 4, does the probability of getting one of those values change with the next roll?" (The probability is 0.5 since 3 outcomes out of 6 possible are in the event. The probability should not change after 3 times. If a 2, 3, or 4 does not appear after a lot of rolls—say, 100—then we might suspect the dice of being non-standard.)
- "The probability of getting the flu during flu season is $\frac{1}{8}$. If a family has 8 people living in the same house, is it guaranteed that one of them will get the flu? If a country has 8 million people, about how many do you expect will get the flu? Does this number have to be exact?" (No, it is very possible that none of the people in the family will get the flu and also possible that more than 1 person will get the flu. We might expect about 1 million people in the country to get the flu, but this is probably not exact.)

4.4 Fiction or Non-fiction?

Cool Down: 5 minutes

Student Task Statement

A librarian is curious about the habits of the library's customers. He records the type of item that the first 10 customers check out from the library.

customer	item type
1	fiction book
2	non-fiction book
3	fiction book
4	fiction book
5	audiobook
6	non-fiction book
7	DVD
8	non-fiction book
9	fiction book
10	DVD

Based on the information from these customers . . .

1. Estimate the probability that the next customer will check out a fiction book. Explain your reasoning.
2. Estimate the number of DVDs that will be checked out for every 100 customers. Explain your reasoning.

Student Response

1. $\frac{4}{10}$ or equivalent since 4 of the 10 customers in the list checked out fiction books.
2. 20 DVDs. Since 2 of the 10 customers in the list checked out DVDs and this is $\frac{1}{5}$ of the customers, we can expect $\frac{1}{5}$ of every 100 customers to check out DVDs. $\frac{1}{5}$ of 100 is 20.

Student Lesson Summary

A probability for an event represents the proportion of the time we expect that event to occur in the long run. For example, the probability of a coin landing heads up after a flip is $\frac{1}{2}$, which means that if we flip a coin many times, we expect that it will land heads up about half of the time.

Even though the probability tells us what we should expect if we flip a coin many times, that doesn't mean we are more likely to get heads if we just got three tails in a row. The chances of getting heads are the same every time we flip the coin, no matter what the outcome was for past flips.

Lesson 4 Practice Problems

Problem 1 Statement

A carnival game has 160 rubber ducks floating in a pool. The person playing the game takes out one duck and looks at it.

- If there's a red mark on the bottom of the duck, the person wins a small prize.
- If there's a blue mark on the bottom of the duck, the person wins a large prize.
- Many ducks do not have a mark.

After 50 people have played the game, only 3 of them have won a small prize, and none of them have won a large prize.

Estimate the number of the 160 ducks that you think have red marks on the bottom. Then estimate the number of ducks you think have blue marks. Explain your reasoning.

Solution

Answers vary. Sample response: There are about 10 ducks with red marks on the bottom and 3 or fewer ducks with blue marks on the bottom.

- If $\frac{3}{50}$ of the people won a small prize, then the probability of getting a duck with a red mark appears to be around 0.06. Since $0.06 \times 160 = 9.6$, there are probably 9 or 10 ducks that have red marks out of the 160. If 9 of the ducks have a red mark, then the probability would be $\frac{9}{160} = 0.05625$. If 10 of the ducks have a red mark, then the probability would be $\frac{10}{160} = 0.0625$.
- The probability of getting a duck with a blue mark appears to be less than $\frac{1}{50}$, or 0.02. Since $0.02 \times 160 = 3.2$, there are probably 3 or fewer ducks that have a blue mark out of the 160. If 3 ducks have a blue mark, then the probability would be $\frac{3}{160} = 0.01875$. If 1 or 2 ducks have a blue mark, then the probability would be lower but still positive.

Problem 2 Statement

Lin wants to know if flipping a coin really does have a probability of $\frac{1}{2}$ of landing heads up, so she flips a coin 10 times. It lands heads up 3 times and tails up 7 times. Has she proven that the probability is not $\frac{1}{2}$? Explain your reasoning.

Solution

No. The actual results from experiments may only get close to the expected probability if they are done many, many times. Ten flips may not be enough to get close to the expected $\frac{1}{2}$ probability.

Problem 3 Statement

A spinner has five equal sections, with one letter from the word "MATHS" in each section.

a. You spin the spinner 20 times. About how many times do you expect it will land on A?

b. You spin the spinner 80 times. About how many times do you expect it will land on something other than A? Explain your reasoning.

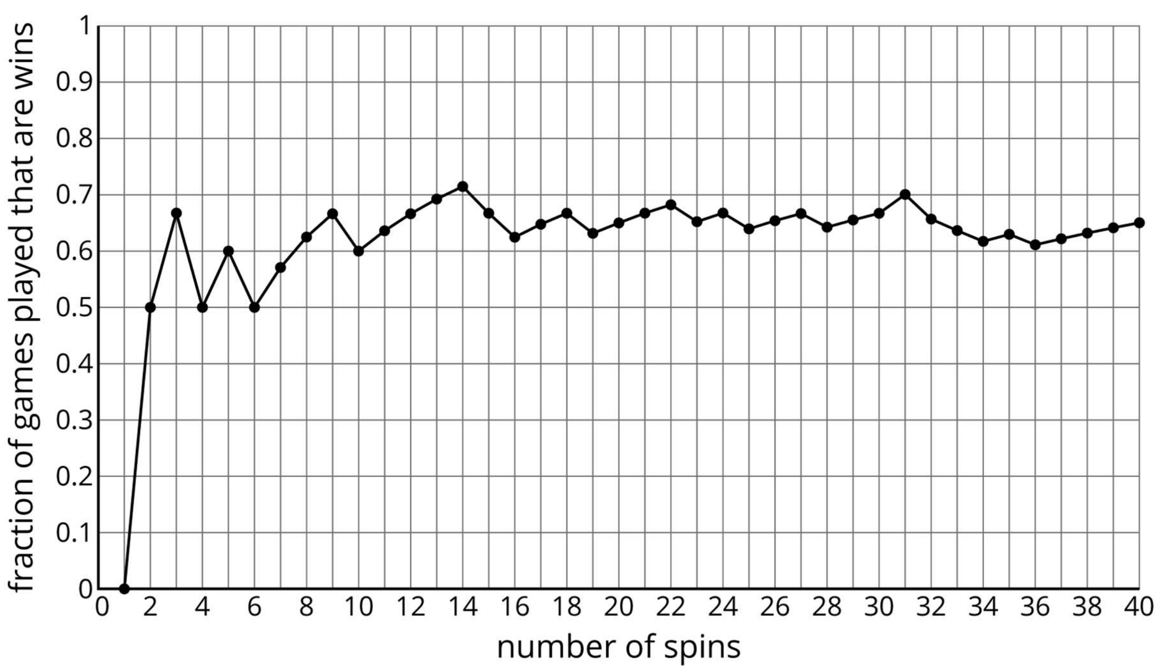
Solution

a. About 4 times, because $\frac{1}{5} \times 20 = 4$.

b. About 64 times, because $\frac{4}{5} \times 80 = 64$.

Problem 4 Statement

A spinner is spun 40 times for a game. Here is a graph showing the fraction of games that are wins under some conditions.



Estimate the probability of a spin winning this game based on the graph.

Solution

0.65

Problem 5 Statement

Which event is more likely: rolling a standard dice and getting an even number, or flipping a coin and having it land heads up?

Solution

Both events are equally likely. Sample explanations:

- Each event has a 50% chance of occurring.
- Both events are as likely to happen as to not happen.

Problem 6 Statement

Noah will select a letter at random from the word "FLUTE." Lin will select a letter at random from the word "CLARINET."

Which person is more likely to pick the letter "E?" Explain your reasoning.

Solution

Noah. Explanations vary. Sample response: Getting the letter "E" is more likely when selecting from the word "FLUTE" because there are fewer possible outcomes in the sample space, and each outcome is equally likely.