MEI Mathematics Education Innovation

Lesson 8: Linear functions

Goals

- Comprehend that any linear function can be represented by an equation in the form $y = mx + c$, where m and c are rate of change and initial value of the function, respectively.
- Coordinate (orally and in writing) the graph of a linear function and its rate of change and initial value.

Learning Targets

- I can determine whether a function is increasing or decreasing based on whether its rate of change is positive or negative.
- I can explain in my own words how the graph of a linear function relates to its rate of change and initial value.

Lesson Narrative

This is the first of three lessons about linear functions. Students are already familiar with linear equations and their graphs from previous units.

In the first activity, students see that a proportional relationship between two quantities can be viewed as a function. They see that either quantity can be chosen as the independent variable and that the only difference in the equation and the graph is the constant of proportionality, which is visible on the graph as the gradient of the line through the origin.

In the next activities, students investigate and make connections between linear functions as represented by graphs, descriptions, and by the equation $y = mx + c$. They interpret the gradient of the line as the rate of change m of the dependent variable with respect to the independent variable and the vertical intercept of the line as the initial value c of the function. Students also compare properties of linear functions represented in different ways to determine, for example, which function has the greater rate of change. Consider using the optional activity if students need more practice comparing linear functions represented in different ways.

Addressing

- Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). For example, given a linear function represented by a table of values and a linear function represented by an algebraic expression, determine which function has the greater rate of change.
- Interpret the equation $y = mx + c$ as defining a linear function, whose graph is a straight line; give examples of functions that are not linear. For example, the function

$A = s^2$ giving the area of a square as a function of its side length is not linear because its graph contains the points (1,1), (2,4) and (3,9), which are not on a straight line.

- Construct a function to model a linear relationship between two quantities. Determine the rate of change and initial value of the function from a description of a relationship or from two (x, y) values, including reading these from a table or from a graph. Interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of its graph or a table of values.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Stronger and Clearer Each Time
- Co-Craft Questions
- Discussion Supports

Student Learning Goals

Let's investigate linear functions.

8.1 Bigger and Smaller

Warm Up: 5 minutes

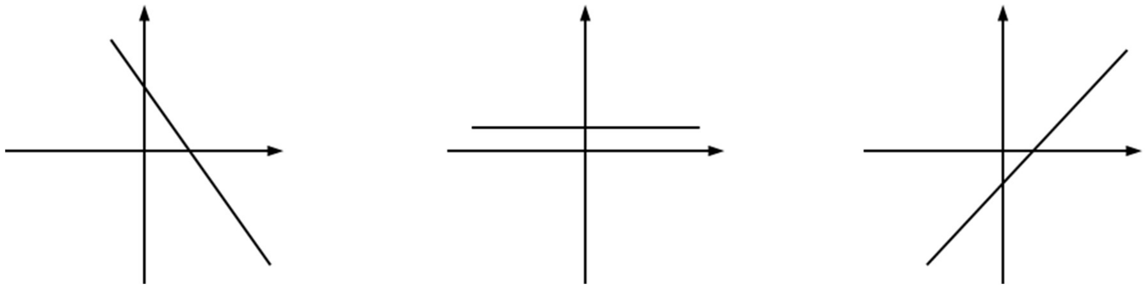
The purpose of this warm-up is for students to reason about the values we can assign graphs based on which feature of the graph, such as gradient and y-intercept, the viewer focuses on. Since there are no numbers on the graph, it is important for students to explain how they know the sign of the gradient and y-intercept based on the position of the graph.

Launch

Display the three graphs for all to see. Tell students that all three graphs have the same scale. Give students 1–2 minutes of quiet work time, followed by a whole-class discussion.

Student Task Statement

Diego said that these graphs are ordered from smallest to largest. Mai said they are ordered from largest to smallest. But these are graphs, not numbers! What do you think Diego and Mai are thinking?



Student Response

In the first graph, the line decreases when we read left to right. That means it has a negative gradient. The second line stays horizontal the entire time, so it must have a gradient of zero. The third graph increases as we read left to right, so it has a positive gradient. That means the gradients are ordered from least to greatest.

On the other hand, the y-intercept of the graph on the left is positive and higher than the second graph. The y-intercept of the last graph is negative, so the y-intercepts are ordered from greatest to least.

Alternatively, Mai may just be looking at the left side of the graphs where they "start" while Diego is looking at the right side of the graph where they "end up."

Activity Synthesis

Display the three graphs for all to see. Invite students to share what they think Diego and Mai are thinking. Encourage students to reference the graphs in their explanation. Record and display their responses for all to see. Emphasise that even though there are no numbers shown, we can tell the sign of the gradient and the sign of the y-intercept based on the position of the line.

8.2 Proportional Relationships Define Linear Functions

15 minutes

This activity begins connecting proportional relationships, which students learned in previous years, to functions. Students use function language with proportional relationships and make connections between what they know about functions and what they know about proportional relationships. Students use similar contexts from previous years to learn that proportional relationships are linear functions. They are also asked to determine the independent and dependent variables and the equation of the function.

Monitor for students who made opposite decisions assigning the independent and dependent variables. For the first problem, some may have written $M = 7t$ and some may have written $t = \frac{1}{7}M$. For the second problem, some may have written $f = 3y$, and some may have written $y = \frac{1}{3}f$.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Discussion Supports

Launch

Arrange students in groups of 2. Give students 3–5 minutes of quiet work time and then time to share responses with their partner. Follow with a whole-class discussion.

Representation: Internalise Comprehension. Demonstrate and encourage students to use colour coding and annotations to highlight connections between representations in a problem. For example, ask students to use the two different colours to represent the independent and dependent variables. Encourage students to use each respective colour to label the graph, plot points, and represent variables in equations.
Supports accessibility for: Visual-spatial processing Speaking: Discussion Supports. Display sentence frames to support students as they justify the independent and dependent variables they selected.
For example, "______ is the independent variable, because_____." or "______ depends on _____, because ______."
Design Principle(s): Support sense-making, Optimise output (for justification)

Anticipated Misconceptions

Students may use different scales on the axes and then try to compare rates. Point out that in order to compare the constant rate of change visually, the scales and labels on the axes must be the same. For example, students who use the representation $M = 7t$ versus the representation $t = \frac{M}{7}$ may have graphs that look either very similar or very different depending on how they scaled the axes, and students who write the same equation but use different scales may have graphs that look different from each other.

If students appear to be stuck on this misconception, consider selecting two graphs of the same function that use different scales and adding a point to the discussion about using scale to compare representations of the graph. Consider asking, "What could we do to the scale on the axes to see the constant rate of change on each graph and accurately compare them?" (Use the same scale on each axis, or graph both axes using the same length to represent 1 unit.)

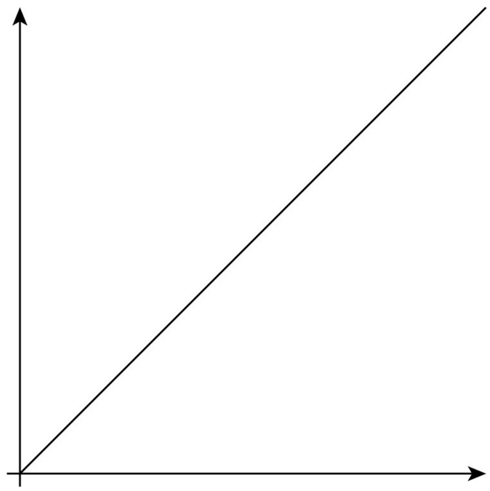
Student Task Statement

1. Jada earns £7 per hour mowing her neighbours' lawns.

    a. Name the two quantities in this situation that are in a functional relationship. Which did you choose to be the independent variable? What is the variable that depends on it?

    b. Write an equation that represents the function.

c. Here is a graph of the function. Label the axes. Label at least two points with input-output pairs.



2. To convert feet to yards, you multiply the number of feet by $\frac{1}{3}$.

a. Name the two quantities in this situation that are in a functional relationship. Which did you choose to be the independent variable? What is the variable that depends on it?

b. Write an equation that represents the function.

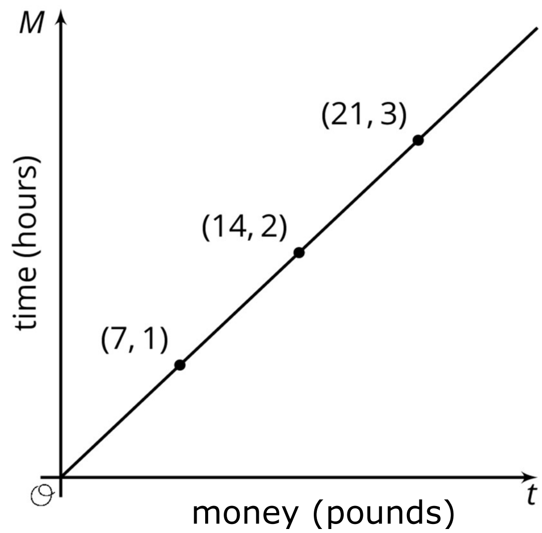
c. Draw the graph of the function. Label at least two points with input-output pairs.

Student Response

1.

a. The time, t, that Jada spent mowing lawns, is in a functional relationship with the amount of money, M, that Jada has earned. We can choose to think of t as a function of M, or vice versa.

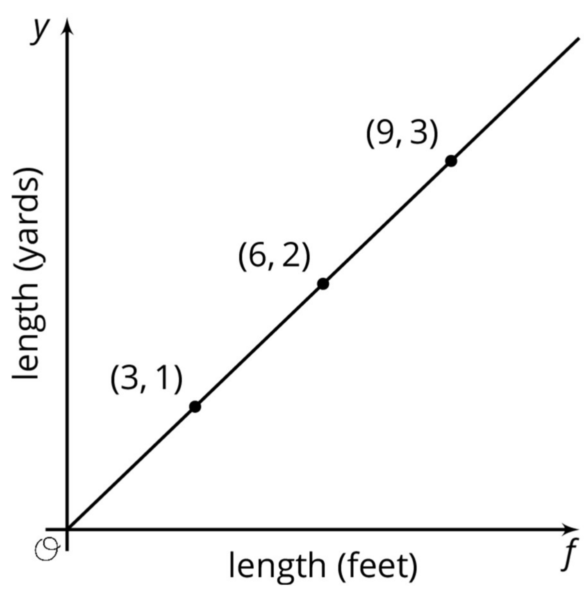
b. We write $M = 7t$ if we chose M as the dependent variable and t as the independent variable, or $t = \frac{M}{7}$ if we chose t as the dependent variable and M as the independent variable.

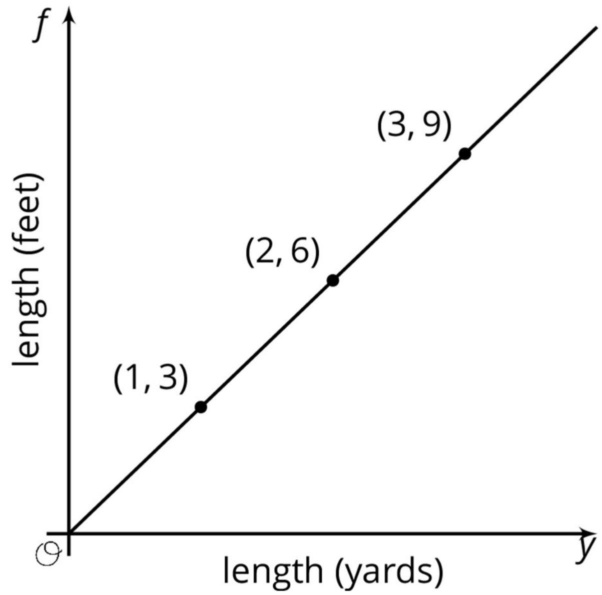
c. 

2.

a. The value of a measurement in yards, y, is in a functional relationship with the value f of that same measurement in feet. We can choose to think of y as a function of f, or vice versa.

b. We write $f = 3y$ if we choose f as the dependent variable and y as the independent variable, or $y = \frac{f}{7}$ if we choose y as the dependent variable and f as the independent variable.

c. 



Activity Synthesis

The purpose of this discussion is for students to understand that proportional relationships are functions and to connect the parts of functions to what they know about proportional relationships. Select previously identified students to share their equations and graphs. Sequence student work so that at least one example of each representation is shown for each of the problems, starting with the most common representation. Display their responses for all to see. Consider asking some of the following questions to help students make connections between the different representations:

- "For the first problem, if we wanted to know how many hours Jada needs to work to make a certain amount of money, which equation would make more sense to use? Why?" ($t = \frac{1}{7}M$, because in that equation, time worked, t, is expressed as a function of money earned, M.)
- "For the second problem, when would we want to use the equation $f = 3y$?" (When we know the number of yards and need to calculate the number of feet.)
- "How do we know that each of these situations are represented by functions?" (For each valid input, there is only one output. For example, no matter which equation I use for the relationship between feet and yards, a specific number of feet will always equal the same number of yards.)

8.3 Is it Filling Up or Draining Out?

10 minutes

The purpose of this activity is to connect features of an equation representing a function to what that means in a context.

Students start with two functions that represent a tank being filled up and another being drained out and are asked to determine which equations represent which situation. This gives students the opportunity to connect initial value and gradient, which they learned about in a previous unit, to the general form of the linear equation and to the fact that linear relationships are functions.

Instructional Routines

- Co-Craft Questions

Launch

Arrange students in groups of 2. Give students 3–5 minutes of quiet work time and then time to share their responses with their partner and reach agreement on their answers. Encourage partners to talk about specific parts of the graph and equation that indicate whether the tank is filling up or draining out. Follow with a whole-class discussion.

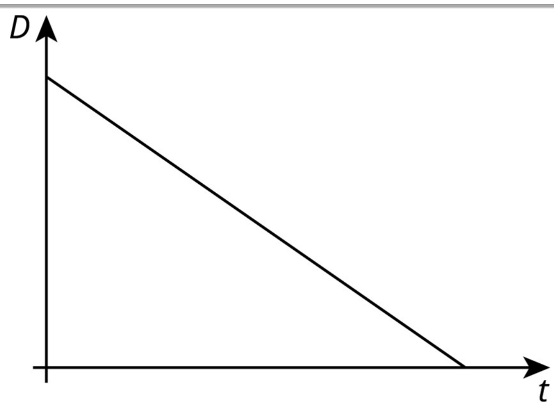
Action and Expression: Internalise Executive Functions. Chunk this task into more manageable parts to support students who benefit from support with organisation and problem solving. For example present one question at a time.
Supports accessibility for: Organisation; Attention Writing, Conversing: Co-Craft Questions. Display the task statement without the questions that follow. Ask pairs of students to write possible mathematical questions about the situation. Then, invite pairs to share their questions with the class. Look for questions that ask students to determine which tank is increasing or decreasing in volume. This helps students produce the language of mathematical questions and talk about the relationship between gradient and the situation of water flowing in or out of a tank prior to solving the questions in this activity.
Design Principle(s): Maximise meta-awareness; Support sense-making

Student Task Statement

There are four tanks of water.

- The amount of water in gallons, A, in tank A is given by the function $A = 200 + 8t$, where t is in minutes.
- The amount of water in gallons, B, in tank B starts at 400 gallons and is decreasing at 5 gallons per minute. These functions work when $t \geq 0$ and $t \leq 80$.

1. Which tank started out with more water?
2. Write an equation representing the relationship between B and t.
3. One tank is filling up. The other is draining out. Which is which? How can you tell?
4. The amount of water in gallons, C, in tank C is given by the function $C = 800 - 7t$. Is it filling up or draining out? Can you tell just by looking at the equation?
5. The graph of the function for the amount of water in gallons, D, in tank D at time t is shown. Is it filling up or draining out? How do you know?



Student Response

1. Tank B. The two equations tell us that when $t = 0$, the volume of water in tank A is 200 gallons, and the volume of water in tank B is 400 gallons.

2. $B = 400 - 5t$

3. Tank A is filling up, and tank B is draining out. As time goes on, corresponding to larger values of t, the value of A gets bigger, but the value of B gets smaller.

4. Draining out. As t increases, the value of C decreases, since we are subtracting larger values from 800. In short, it is because we are subtracting multiples of t instead of adding them that we can quickly see that tank C is draining.

5. Draining out. As time increases, the value of D goes down.

Are You Ready for More?

- Pick a tank that was draining out. How long did it take for that tank to drain? What percent full was the tank when 30% of that time had elapsed? When 70% of the time had elapsed?

- What point in the plane is 30% of the way from (0,15) to (5,0)? 70% of the way?

- What point in the plane is 30% of the way from (3,5) to (8,6)? 70% of the way?

Student Response

- Answers vary. Sample response: It takes 80 minutes for tank B to drain, because $400 \div 5 = 80$. The tank is 70% full after 30% of that time has elapsed. The tank is 30% full after 70% of that time has elapsed.

- The point (1.5,10.5) is 30% of the way from (0,15) to (5,0), because $0.3 \times 5 = 1.5$ and $0.7 \times 15 = 10.5$.
  The point (3.5,4.5) is 70% of the way from (0,15) to (5,0), because $0.7 \times 5 = 3.5$ and $0.3 \times 15 = 4.5$.

- The point (4.5,5.3) is 30% of the way from (3,5) to (8,6). The x-coordinates are 5 units apart, because $8 - 3 = 5$. 30% of 5 is 1.5 and $3 + 1.5 = 4.5$. The y-coordinates are 1 unit apart, because $6 - 5 = 1$. 30% of 1 is 0.3 and $5 + 0.3 = 5.3$.

The point (6.5,5.7) is 70% of the way from (3,5) to (8,6). 70% of 5 is 3.5 and $3 + 3.5 = 6.5$. 70% of 1 is 0.7 and $5 + 0.7 = 5.7$.

Activity Synthesis

Consider asking some of the following questions to begin the discussion:

- "For the second problem, what in the equation tells you that the gradient is decreasing? Increasing?" (Decreasing: the gradient in the equation is negative; Increasing: the gradient in the equation is positive.)
- "For the third problem, what is similar between the equation in this problem and the decreasing equation in the previous problem?"
  (Both gradients are negative.)
- "For the last problem, what in the graph tells you that tank D is draining out? What would a graph that has a tank filling up look like? What would be different?" (I know it is draining out because the graph is going down from left to right. If the tank were filling up, the graph would be going up from left to right.)

Tell students that a linear function can always be represented with an equation of the form $y = mx + c$. The gradient of the line, m, is the rate of the change of the function and the initial value of the function is c.

If time allows, give students the following scenario to come up with a possible equation for tank D:

"Tank D started out with more water than tank B but less water than tank C. The water is draining from tank D faster than from tank B but slower than tank C. What is a possible equation for the graph of the function for the amount of water D in tank D over time t?" (Students should choose an initial value between 400 and 800, and a constant rate of change between -7 and -5. One possible such equation might be $D = 600 - 6t$.)

8.4 Which is Growing Faster?

Optional: 10 minutes

The purpose of this activity is for students to connect their work with linear equations to functions. The two linear functions in this activity are represented differently and students are asked to compare various features of each representation.

Identify students who use different methods to answer the questions. For example, students may write an equation to represent Noah's account, and others may make a table to show the value in each account at different numbers of weeks by reasoning about the rate of change and the amount in each account when they were opened.

Instructional Routines

- Stronger and Clearer Each Time

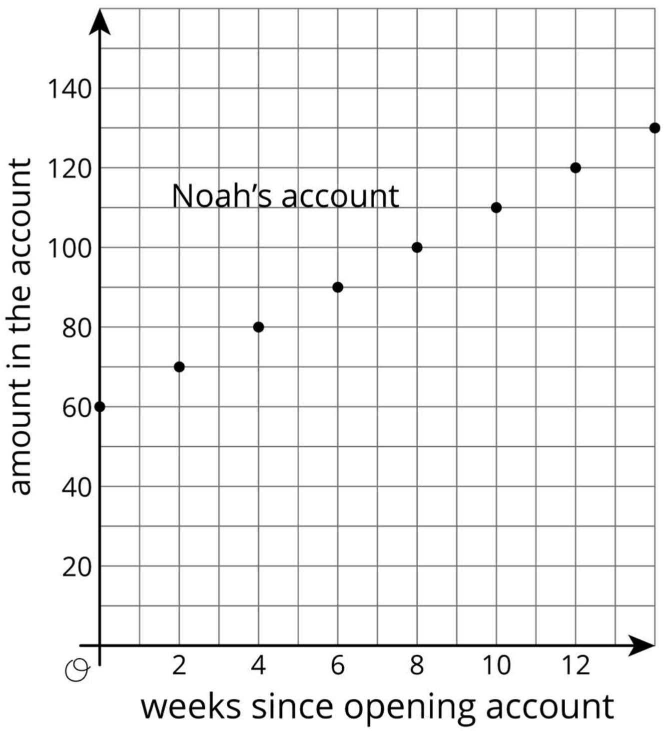
Launch

Give students 1–2 minutes of quiet work time. Follow with a whole-class discussion.

Student Task Statement

Noah is depositing money in his account every week to save money. The graph shows the amount he has saved as a function of time since he opened his account.

Elena opened an account the same day as Noah. The amount of money E in her account is given by the function $E = 8w + 60$, where w is the number of weeks since the account was opened.



1. Who started out with more money in their account? Explain how you know.
2. Who is saving money at a faster rate? Explain how you know.
3. How much will Noah save over the course of a year if he does not make any withdrawals? How long will it take Elena to save that much?

Student Response

1. They are the same. At the left edge of the graph, representing the time when they opened the accounts, Noah had £60. When t is 0, the money in Elena's account when it was opened is found by $E = 8 \times 0 + 60$, so she also had £60.
2. Elena is saving money at a faster rate. Every 2 weeks, Noah's account increases by £10 while Elena's account goes up by £8 each week, so she makes £16 in two weeks.

3. Noah will save £260 over a year in addition to the £60 he opened the account with, since he saves £10 every 2 weeks and there are 52 weeks in the year. It will take Elena just 33 weeks to save the same amount since she also started with £60 ($260 \div 8 = 32.5$, so rounding up, it will take 33 weeks).

Activity Synthesis

Display the graph of Noah's savings over time and the equation for the amount of money in Elena's account for all to see. Select students previously identified to share their responses.

Consider asking the following questions to help student make connections between the different representations:

- "How did you determine the amount Noah saved in a year?" (I used the graph to figure out that Noah saves £5 each week and multiplied that by 52 weeks.)
- "What equations could you use to solve the last question?" (I could use the equation $N = 60 + 5w$ for the amount of money in Noah's account after w weeks. When $w = 52$, Noah has £320. If I solve the equation $320 = 8w + 60$ for w, I would know how many weeks it would take Elena to have £320 in her account.)
- "How could you solve the last question without using an equation?"
(I could extend the graph out to 52 weeks and plot the value of each account over the year.)

Engagement: Develop Effort and Persistence. Break the class into small discussion groups and then invite a representative from each group to report back to the whole class.
Supports accessibility for: Attention; Social-emotional skills Writing, Speaking: Stronger and Clearer Each Time. Use this routine to give students a structured opportunity to refine their explanation of how they determined who started out with more money in their account. Ask each student to meet with 2–3 other partners in a row for feedback. Provide students with prompts for feedback that will help them strengthen their ideas and clarify their language (e.g., "What did you do first?", and "How did you determine how much money they each had at the start?", etc.). Students can borrow ideas and language from each partner to strengthen the final product. They can return to the first partner and revise their initial response.
Design Principle(s): Optimise output (for explanation)

Lesson Synthesis

To help students make connections between work they have done previously with linear equations and functions, consider asking some of the following questions:

- "How can we tell if a linear function is increasing from an equation? From a graph?" (In a linear equation $y = mx + c$, if m is positive, the linear function is increasing. In a graph, if the line is going up from left to right, then the function is increasing.)

- "How can we tell from the graph if the initial value of the function is positive? How can I tell from the equation?" (If the graph of the function crosses the vertical axis above 0, then the initial value is positive. In the equation, $y = mx + c$ of a linear function, the c is positive when the initial value is positive.)

8.5 Beginning to See Daylight

Cool Down: 5 minutes

Launch

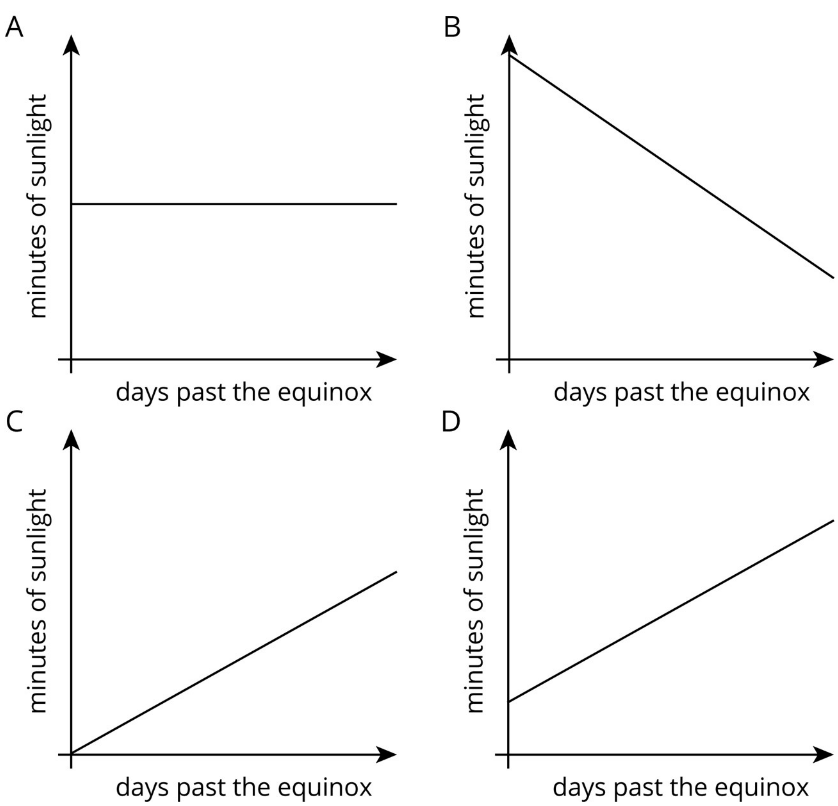
Explain that an equinox is the day when there is an approximately equal amount of daylight and darkness.

Anticipated Misconceptions

Some students may think graph C is an acceptable choice, since some places do get zero hours of sunlight sometimes, but this never happens near the equinox or for any cities in France.

Student Task Statement

In a certain city in France, they gain 2 minutes of daylight each day after the spring equinox (usually in March), but after the autumnal equinox (usually in September) they lose 2 minutes of daylight each day.



1. Which of the graphs is most likely to represent the graph of daylight for the month after the spring equinox?

2. Which of the graphs is most likely to represent the graph of daylight for the month after the autumnal equinox?

3. Why are the other graphs not likely to represent either month?

Student Response

1. D

2. B

3. Graph A does not make sense because there is a constant amount of daylight. Graph C does not make sense because it goes through the origin, meaning it started with 0 minutes of daylight.

Student Lesson Summary

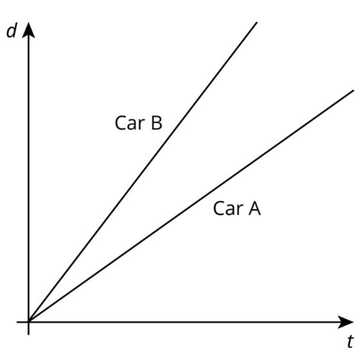
Suppose a car is travelling at 30 miles per hour. The relationship between the time in hours and the distance in miles is a proportional relationship. We can represent this relationship with an equation of the form $d = 30t$, where distance is a function of time (since each input of time has exactly one output of distance). Or we could write the equation $t = \frac{1}{30}d$ instead, where time is a function of distance (since each input of distance has exactly one output of time).

More generally, if we represent a linear function with an equation like $y = mx + c$, then c is the initial value (which is 0 for proportional relationships), and m is the rate of change of the function. If m is positive, the function is increasing. If m is negative, the function is decreasing. If we represent a linear function in a different way, say with a graph, we can use what we know about graphs of lines to find the m and c values and, if needed, write an equation.

Lesson 8 Practice Problems

1. **Problem 1 Statement**

   Two cars drive on the same road in the same direction. The graphs show the distance, d, of each one as a function of time, t. Which car drives faster? Explain how you know.



Solution

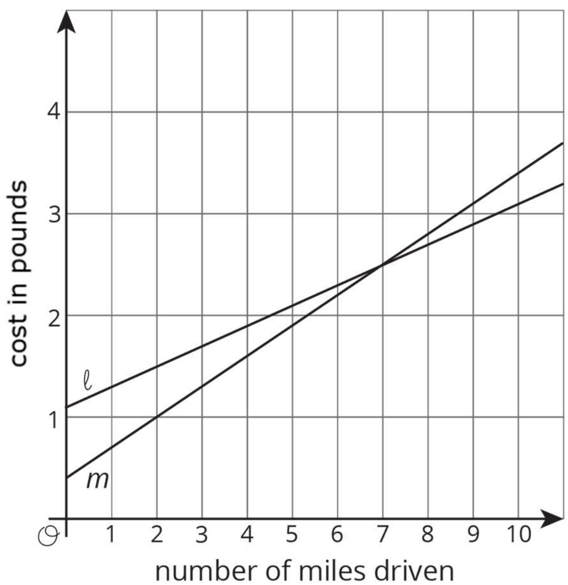
Car B drives faster. The two cars began at the same place, but after any amount of time, car B has travelled farther than car A. Graphically, the gradient of the line corresponding to car B is greater than the gradient of the line corresponding to car A, so the rate of change of distance per time (speed) is higher for car B.

2. **Problem 2 Statement**

Two car services offer to pick you up and take you to your destination. Service A charges 40p to pick you up and 30p for each mile of your trip. Service B charges £1.10 to pick you up and charges c pence for each mile of your trip.

a. Match the services to the lines ℓ and m.

b. For Service B, is the additional charge per mile greater or less than 30p per mile of the trip? Explain your reasoning.



Solution

a. Service A is represented by Line m. Service B is represented by Line ℓ.

b. Less than 30p per mile since Line ℓ is not increasing as quickly as Line m.

3. **Problem 3 Statement**

Kiran and Clare like to race each other home from school. They run at the same speed, but Kiran's house is slightly closer to school than Clare's house. On a graph, their distance from their homes in metres is a function of the time from when they begin the race in seconds.

a. As you read the graphs left to right, would the lines go up or down?

b. What is different about the lines representing Kiran's run and Clare's run?

c. What is the same about the lines representing Kiran's run and Clare's run?

Solution

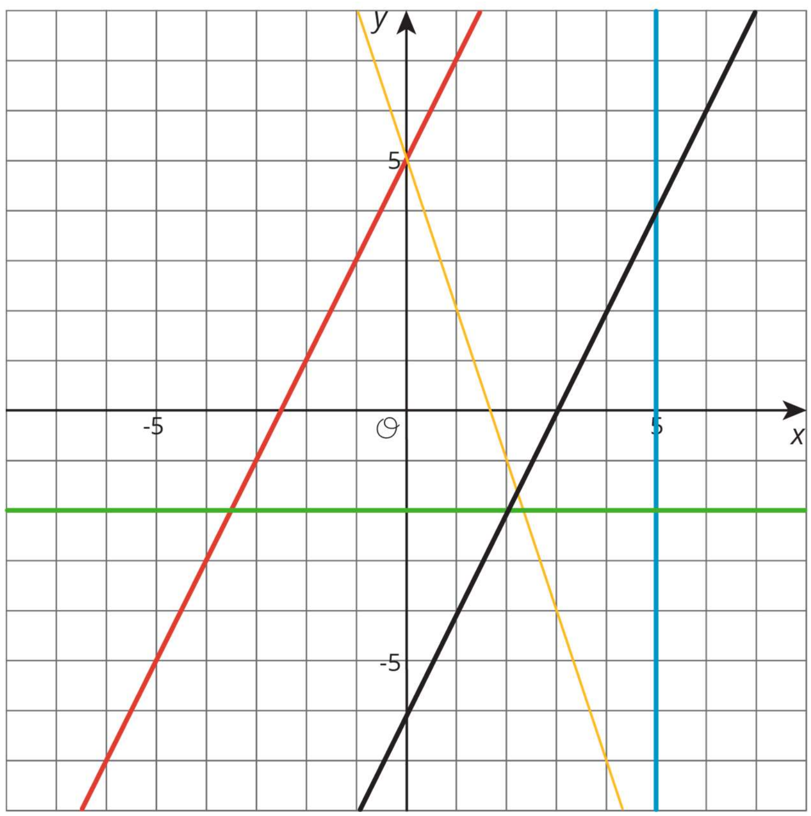
a. Down

b. Answers vary. Sample response: Clare's line would be higher up since she started farther away from her house.

c. Answers vary. Sample response: The lines would have the same gradient since they run at the same speed.

4. **Problem 4 Statement**

Write an equation for each line.



Solution

Green line: y = -2, blue line: $x = 5$, black line: $y = 2x - 6$, yellow line: $y = -3x + 5$, red line: $y = 2x + 5$

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