Lesson 16: Parallel lines and the angles in a triangle

Goals

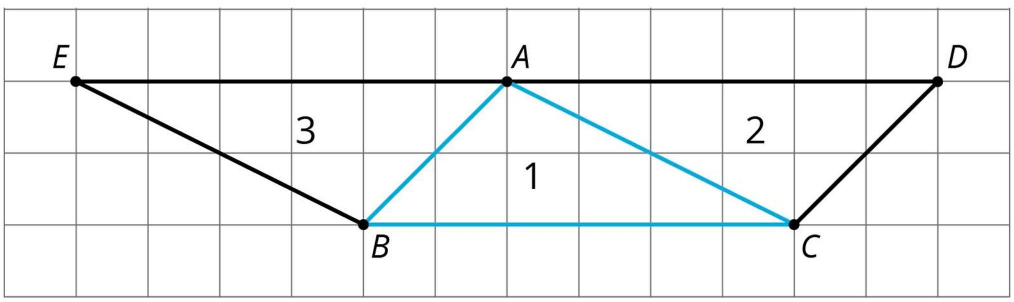
- Create diagrams using 180-degree rotations of triangles to justify (orally and in writing) that angles in a triangle sum up to 180 degrees.
- Generalise the Triangle Sum Theorem using translations, rotations and reflections or the congruence of alternate angles of parallel lines cut by a transversal.

Learning Targets

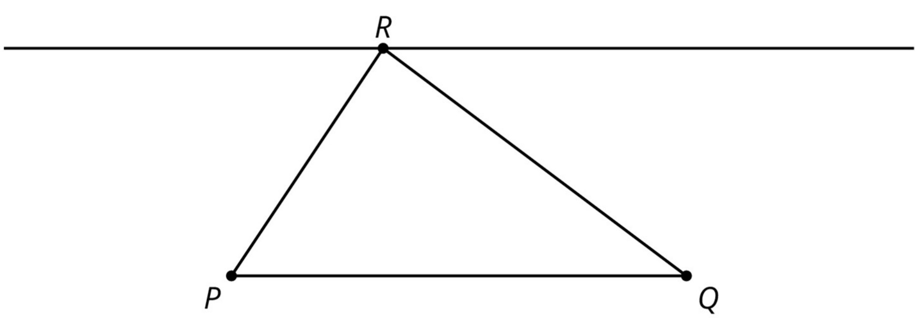
- I can explain using pictures why the sum of the angles in any triangle is 180 degrees.

Lesson Narrative

Earlier in this unit, students learned that when a 180° rotation is applied to a line ℓ, the resulting line is parallel to ℓ. Here is a picture students worked with earlier in the unit:



The picture was created by applying 180° rotations to $\triangle ABC$ with centres at the midpoints of line segments AC and AB. Notice that E, A, and D all lie on the same grid line that is parallel to line BC. In this case, we have the structure of the grid to help see why this is true.



In this lesson, students begin by examining the argument using grid lines described above. Then they examine a triangle off of a grid, PQR. Here an auxiliary line plays the role of the grid lines: the line parallel to line PQ through the opposite vertex R. The three angles sharing vertex R make a line and so they add to 180 degrees. Using what they have learned earlier in this unit (either congruent alternate angles for parallel lines cut by a transverse or applying translations, rotations and reflections explicitly), students argue that the sum of the angles in triangle PQR is the same as the sum of the angles meeting at vertex R. This shows that the sum of the angles in *any* triangle is 180 degrees.

Building On

- Apply and extend previous understandings of operations with fractions to add, subtract, multiply, and divide rational numbers.
- Angles are taken to angles of the same size.

Addressing

- Use informal arguments to establish facts about the angle sum and exterior angles of triangles, about the angles created when parallel lines are cut by a transversal, and the angle-angle criterion for similarity of triangles. For example, arrange three copies of the same triangle so that the sum of the three angles appears to form a line, and give an argument in terms of transversals why this is so.

Building Towards

- Explain a proof of Pythagoras' Theorem and its converse.

Instructional Routines

- Co-Craft Questions
- Discussion Supports
- True or False

Required Materials

Geometry toolkits

tracing paper, graph paper, coloured pencils, scissors, and an index card to use as a straightedge or to mark right angles, plus a ruler and protractor. Clear protractors with no holes and with radial lines printed on them are recommended.

Student Learning Goals

Let's see why the angles in a triangle add to 180 degrees.

16.1 True or False: Computational Relationships

Warm Up: 5 minutes

This warm-up encourages students to reason algebraically about various computational relationships and patterns. While students may evaluate each side of the equation to determine if it is true or false, encourage students to think about the properties of arithmetic operations in their reasoning.

Instructional Routines

- True or False

Launch

Display one problem at a time. Tell students to give a signal when they have an answer and a strategy. After each problem, give students 1 minute of quiet think time and follow with a whole-class discussion.

Anticipated Misconceptions

In the first question, students may think you can round or adjust numbers in a subtraction problem in the same way as in addition problems. For example, when adding $62 + 28$, taking 2 from the 62 and adding it to the 28 does not change the sum. However, using that same strategy when subtracting, the distance between the numbers on the number line changes and the difference does not remain the same.

Student Task Statement

Is each equation true or false?

$62 - 28 = 60 - 30$

$3 \times -8 = (2 \times -8) - 8$

$\frac{16}{-2} + \frac{24}{-2} = \frac{40}{-2}$

Student Response

- False. Explanations vary. Possible response: Think about a number line. The difference between numbers is how far apart they are. 62 and 28 are further apart than 60 and 30.
- True. Explanations vary. Possible response: Rewrite (3×-8) as $(2 \times -8) + (1 \times -8)$.
- True. Explanations vary. Possible response: Since $16 + 24 = 40$ both sides of the equation are equal to $\frac{40}{-2}$.

Activity Synthesis

Ask students to share their strategies for each problem. Record and display their explanations for all to see. Ask students how they decided upon a strategy. To involve more students in the conversation, consider asking:

- Do you agree or disagree? Why?
- Who can restate ___’s reasoning in a different way?
- Does anyone want to add on to _____’s reasoning?

After each true equation, ask students if they could rely on the reasoning used on the given problem to think about or solve other problems that are similar in type. After each false equation, ask students how we could make the equation true.

16.2 Angle Plus Two

15 minutes (there is a digital version of this activity)

In the previous lesson, students conjectured that the interior angles of a triangle add up to 180 degrees. The purpose of this activity is to explain this structure in some cases. Students apply 180° rotations to a triangle in order to calculate the sum of its three angles.
They have applied these transformations earlier in the context of building shapes using translations, rotations and reflections. Here they exploit the structure of the coordinate grid to see in a particular case that the sum of the three angles in a triangle is 180 degrees. The next activity removes the grid lines and gives an argument that applies to all triangles.

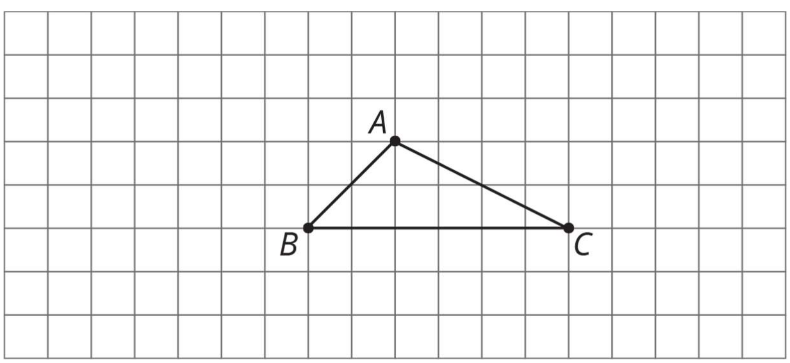
Instructional Routines

- Discussion Supports

Launch

Arrange students in groups of 2–3. Provide access to geometry toolkits. Give 5 minutes work time building the diagram and measuring angles. Then allow for a short whole-class check-in about angle measurement error, and then provide time for students to complete the task.

Important note for classes using the digital activity: The applet measures angles in the standard direction, anti-clockwise. Students need to select points in order. For example, to measure angle ACB in this triangle, students would select the angle measure tool and then click on A, then C, and then B. If they click on B, then C, and then A, they get the reflex angle.



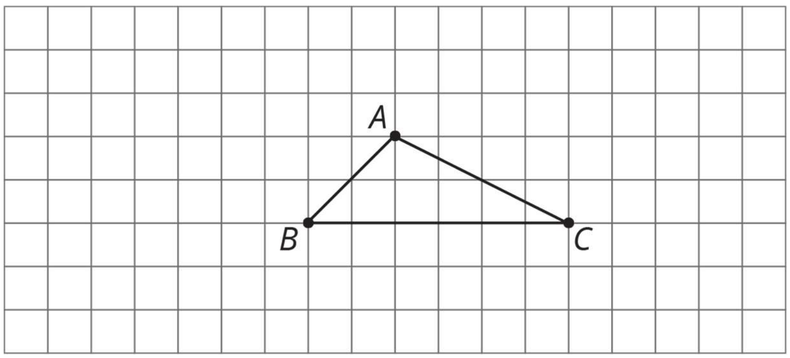
Action and Expression: Internalise Executive Functions. Chunk this task into more manageable parts to support students who benefit from support with organisational skills in problem solving. For example, present 2–3 questions at a time and monitor students to ensure they are making progress throughout the activity.
Supports accessibility for: Organisation; Attention

Anticipated Misconceptions

Some students may have trouble with the rotations. If they struggle, remind them of similar work they did in a previous lesson. Help them with the first rotation, and allow them to do the second rotation on their own.

Student Task Statement

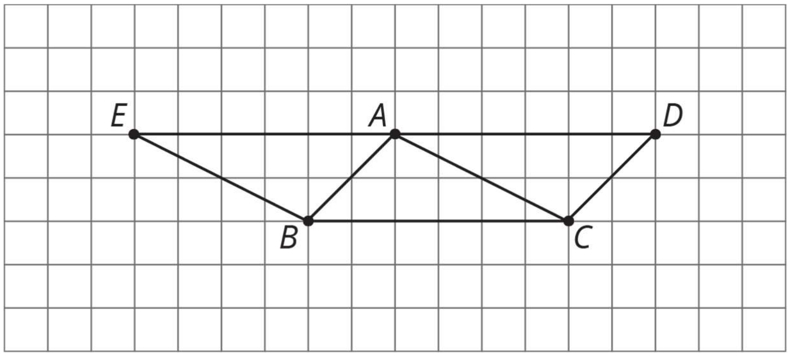
Here is triangle ABC.



1. Rotate triangle ABC 180° around the midpoint of side AC. Label the new vertex D.
2. Rotate triangle ABC 180° around the midpoint of side AB. Label the new vertex E.
3. Look at angles EAB, BAC, and CAD. Without measuring, write what you think is the sum of these angles. Explain or show your reasoning.
4. Is angle EAB equal to any angle in triangle ABC? If so, which one? If not, how do you know?
5. Is angle CAD equal to any angle in triangle ABC? If so, which one? If not, how do you know?
6. What is the sum of angles ABC, BAC, and ACB?

Student Response

1. Rotating by 180° around the midpoint of line segment AC, C and A swap places and B goes to the new point labelled D in the picture. Rotating 180° around the midpoint of line segment AB, points A and B trade places and C goes to the new point labelled E in the picture.



2. They look like they will add to 180°, because they appear to form a straight angle and there are 180° in a straight angle.

3. Yes, angle ABC. When triangle ABC is rotated 180 degrees with centre the midpoint of line segment AB, $\angle ABC$ goes to $\angle EAB$.

4. Yes, angle ACB. When triangle ABC is rotated 180 degrees with centre the midpoint of line segment AC, $\angle ACB$ goes to $\angle CAD$.

5. The sum of these angles should be 180°, because it is the same as the sum of angles EAB, BAC, and CAD and these angles add up to 180°.

Activity Synthesis

Ask students how the grid lines helped to show that the sum of the angles in this triangle is 180 degrees. Important ideas to bring out include:

- A 180 rotation of a line BC (with centre the midpoint of AB or the midpoint of AC) is parallel to line BC and lays upon the horizontal grid line that point A is on.
- The grid lines are parallel so the rotated angles lie on the (same) grid line.

Consider asking students, "Is it always true that the sum of the angles in a triangle is 180°?" (Make sure students understand that the argument here does not apply to most triangles, since it relies heavily on the fact that BC lies on a grid line which means we know the images of BC, AE and DA, also lie on grid line that point A is on.)

It turns out that the key to showing the more general result lies in studying the rotations that were used to generate the three triangle picture. This investigation is the subject of the next activity.

Speaking: Discussion Supports. Use this routine to support whole-class discussion. For each observation that is shared, ask students to restate and/or revoice what they heard using mathematical language. Consider providing students time to restate what they hear to a partner, before selecting one or two students to share with the class. Ask the original speaker if their peer was accurately able to restate their thinking. Call students' attention to any words or phrases that helped to clarify the original statement. This will provide more

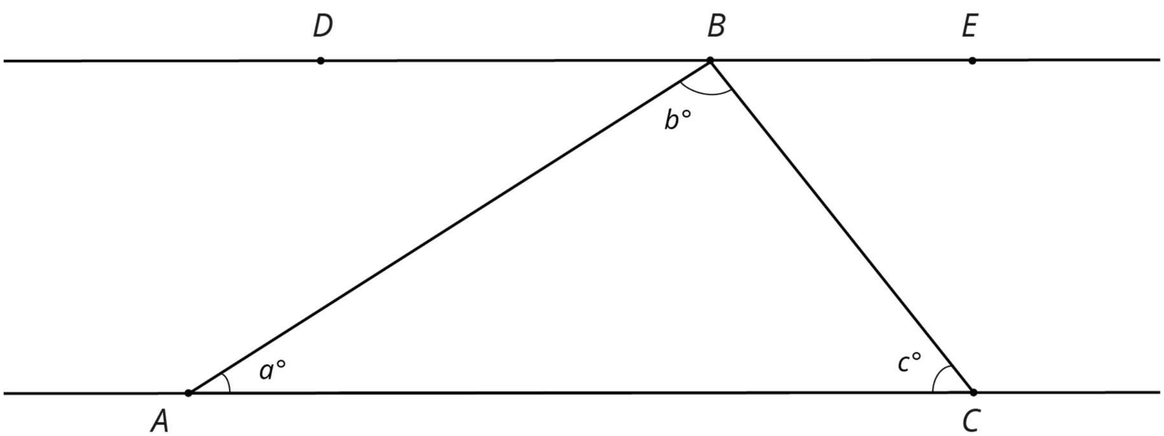
students with an opportunity to produce language about transformations.
Design Principle(s): Support sense-making

16.3 Every Triangle in the World

15 minutes

The previous activity recalls the 180° rotations used to create an important diagram. This diagram shows that the sum of the angles in a triangle is 180° if the triangle happens to lie on a grid with a horizontal side. The purpose of this activity is to provide a complete argument, not depending on the grid, of why the sum of the three angles in a triangle is 180°.

In this activity, rather than building a complex shape from a triangle and its rotations, students begin with a triangle and a line parallel to the base through the opposite vertex:



In this image, lines AC and DE are parallel. The advantage to this situation is that we *know* that points D, B, and E all lie on a line. In order to calculate angles DBA and EBC, students can use either the rotation idea of the previous activity or congruence of alternate angles in parallel lines cut by a transversal. In either case, they need to analyse the given constraints and decide on a path to pursue to show the congruence of angles. Students then conclude from the fact that D, B, and E lie on the same line that $a + b + c = 180$.

There is a subtle distinction in the logic between this lesson and the previous. The previous lesson suggests that the sum of the angles in a triangle is 180° using direct measurements of a triangle on a grid. This activity *shows* that this is the case by using a generic triangle and reasoning about parallel lines cut by a transversal. But, in order to do so, we needed to know to draw the line parallel to line AC through B and that this idea came through experimenting with rotating triangles.

Instructional Routines

- Co-Craft Questions

Launch

Keep students in the same groups. Tell students they'll be working on this activity *without* the geometry toolkit.

Begin with 5 minutes of quiet work time. Give groups time to compare their arguments, then have a whole-class discussion.

Action and Expression: Develop Expression and Communication. Invite students to talk about their ideas with a partner before writing them down. Display sentence frames to support students when they explain their ideas. For example, "The angles in a triangle always add up to 180 degrees because…", "To show this, first I _____ because...", "That could/couldn't be true because…", "This method works/doesn't work because…."
Supports accessibility for: Language; Organisation Conversing, Representing, Writing: Co-Craft Questions. Display only the image and the first line of this task without revealing the questions that follow. Ask students to write possible mathematical questions about the representation. Then, invite groups to share their responses with the class. Listen for and amplify any questions involving the relationships between the sum of angles in a triangle and in a straight angle. This will help students produce the language of mathematical questions prior to being asked to analyse another's reasoning for the task.
Design Principle(s): Cultivate conversation; Support sense-making

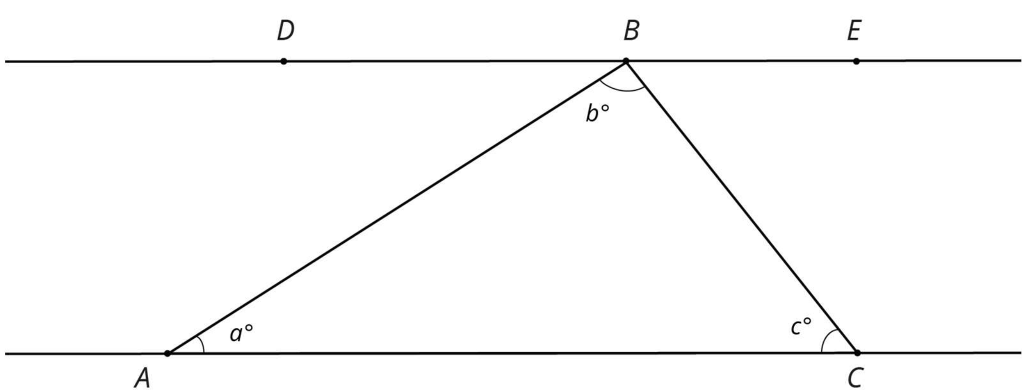
Anticipated Misconceptions

Some students may say that a, b, and c are the three angles in a triangle, so they add up to 180. Make sure that these students understand that the goal of this activity is to explain why this must be true. Encourage them to use their answer to the first question and think about what they know about different angles in the diagram.

For the last question students may not understand why their work in the previous question only shows $a + b + c = 180$ for one particular triangle. Consider drawing a different triangle (without the parallel line to one of the bases), labelling the three angles a, b, c, and asking the student why $a + b + c = 180$ for *this* triangle.

Student Task Statement

Here is $\triangle ABC$. Line DE is parallel to line AC.



1. What is $\angle DBA + b + \angle CBE$? Explain how you know.

2. Use your answer to explain why $a + b + c = 180$.

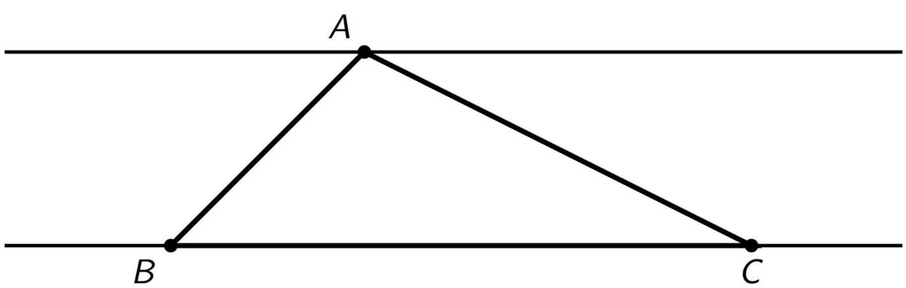
3. Explain why your argument will work for *any* triangle: that is, explain why the sum of the angles in *any* triangle is 180°.

Student Response

1. Angles DBA, ABC, and CBE make a line. So, the sum of their angles is 180°.

2. Angles DBA and CAB are congruent because these are alternate angles in the parallel lines AC and DE with transversal AB. Angles EBC and BCA are congruent because these are alternate angles in the parallel lines AC and DE with transversal BC. Angles DBA, ABC, and CBE make a line, and so their angles add up to 180°. Then $a + b + c = 180$.

3. For any triangle, draw a line parallel to one side, containing the opposite vertex:



With this picture, use the same argument to show that the sum of the three angles of the triangle is 180°. This works for *every* triangle.

Are You Ready for More?

1. Using a ruler, create a few quadrilaterals. Use a protractor to measure the four angles inside the quadrilateral. What is the sum of these four angles?

2. Come up with an explanation for why anything you notice must be true (hint: draw one diagonal in each quadrilateral).

Student Response

1. The sum of the interior angles in any quadrilateral is 360°. Note that since protractors are imprecise, physical measurements may range by a few degrees away from this.

2. In any quadrilateral, you can draw a diagonal that partitions the quadrilateral into two triangles. The sum of the angles in each triangle is 180°, and the six angles in the two triangles comprise all of the angles in the quadrilateral.

Activity Synthesis

Ask students how this activity differs from the previous one, where $\triangle ABC$ had a horizontal side lying on a grid line. Emphasise that:

- This argument applies to *any* triangle ABC.
- The prior argument relies on having grid lines and having the base of the triangle lie on a grid line.
- This argument relies heavily on having the parallel line to AC through B drawn, something we can always add to a triangle.

The key inspiration in this activity is putting in the line DE through B parallel to AC. Once this line is drawn, previous results about parallel lines cut by a transversal allow us to see why the sum of the angles in a triangle is 180°. Tell students that the line DE is often called an 'auxiliary construction' because we are trying to show something about $\triangle ABC$ and this line happens to be very helpful in achieving that goal. It often takes experience and creativity to hit upon the right auxiliary construction when trying to prove things in mathematics.

16.4 Four Triangles Revisited

Optional: 5 minutes

This activity revisits a picture that students have seen earlier and that they will see again later when they investigate Pythagoras' Theorem. The four right triangles around the boundary make a quadrilateral inside which happens to be a square. Since the four triangles surrounding the inner quadrilateral are congruent by construction, this makes the inner shape a rhombus. The angle calculations that students make here justify why it is actually a square. The fact that quadrilateral $ACEG$ is a square is not asked for in the activity, but if any students notice this, encourage them to share their observations and reasoning.

Students are making use of structure throughout this activity, most notably by:

- Using the fact that congruent triangles have congruent angles.
- Using the fact that angles that make a line add up to 180 degrees.

Instructional Routines

- Discussion Supports

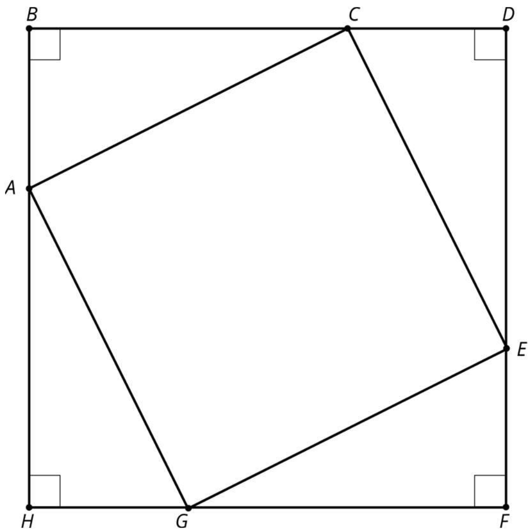
Launch

Provide 3 minutes of work time followed by a whole-class discussion.

Action and Expression: Develop Expression and Communication. Maintain a display of important terms and information. During the launch, take time to review the following facts from previous lessons that students will need to access for this activity: congruent triangles have congruent angles, a straight angle is equal to 180 degrees, and the sum of the angles in a triangle is 180 degrees.
Supports accessibility for: Memory; Language

Student Task Statement

This diagram shows a square $BDFH$ that has been made by images of triangle ABC under translations, rotations and reflections.



Given that angle BAC measures 53 degrees, find as many other angles as you can.

Student Response

All other angles can be determined from the one given angle.

Angles ECD, GEF, and AGH all measure 53° because $\angle BAC$ is 53°. These three angles correspond to $\angle BAC$ under a rigid motion of $\triangle ABC$. To find angles ACB, CED, EGF, and GAH, notice that these are all congruent because they are all correspond to $\angle ACB$ of $\triangle ABC$ under a rotation. Angle ACB measures 37° because the angles in a triangle add to 180°. One angle in $\triangle ABC$ measures 53°, and another measures 90°, so the third angle measures $180° - 53° - 90° = 37°$.

Angles ACE, CEG, EGA, and GAC all measure 90°. Here is an argument for $\angle ACE$: We know that angle ACB measures 37° and angle ECD measures 53°. So angle ACE must measure 90° because it makes a line together with $\angle ACB$ and $\angle ECD$.

Activity Synthesis

Display the image of the 4 triangles for all to see. Invite students to share how they calculated one of the other unknown angles in the image, adding to the image until all the unknown angles are filled in.

In wrapping up, note that angles ACE, CEG, EGA, and GAC are all right angles. It's not necessary to show or tell students that $ACEG$ is a square yet. This diagram will be used in establishing Pythagoras' Theorem later in the course, so it's a good place to end.

Speaking: Discussion Supports. To help students produce statements about finding the unknown angles in the diagram, provide sentence frames such as, "Knowing ____ , helps me

find____ because . . .” or “Angle _____ corresponds to angle _____ because . . . .”
Design Principle(s): Optimise output (for explanation); Support sense-making

Lesson Synthesis

Revisit the basic steps in the proof that shows that the sum of the angles in a triangle is 180°. Consider asking a student to make a triangle that you can display for all to see and add onto it showing each step.

- We had a triangle, and a line through one vertex parallel to the opposite side.
- We knew that the three angles with their vertices on the line summed to 180°.
- We knew that two of these angles were congruent to corresponding angles in the triangle, and the third one was inside the triangle.
- Therefore, the three angles in the triangle must also sum to 180°.

Tell students that this is one of the most useful results in geometry and they will get to use it again and again in the future.

16.5 Angle Sizes

Cool Down: 5 minutes

Students sketch different triangles and list angle possibilities. Knowing that the sum of the angles in a triangle is 180 degrees establishes that each angle in an equilateral triangle measures 60 degrees.

Student Task Statement

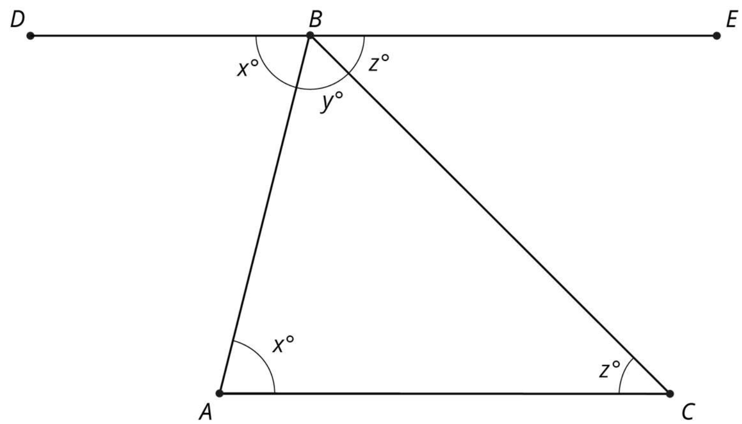
1. In an equilateral triangle, all side lengths are equal and all angles are equal. Sketch an equilateral triangle. What are its angles?
2. In an isosceles triangle, which is not equilateral, two side lengths are equal and two angles are equal. Sketch three different isosceles triangles.
3. List two different possibilities for the angles of an isosceles triangle.

Student Response

1. The three angles must add to 180° but must all be equal. Therefore each is 60°, since $60 = \frac{1}{3} \times 180$.
2. Answers vary. Triangles should each have two sides that are the same length but not three.
3. Answers vary. Sample responses: 70°, 70°, 40° and 30°, 30°, 120°.

Student Lesson Summary

Using parallel lines and rotations, we can understand why the angles in a triangle always add to 180°. Here is triangle ABC. Line DE is parallel to AC and contains B.

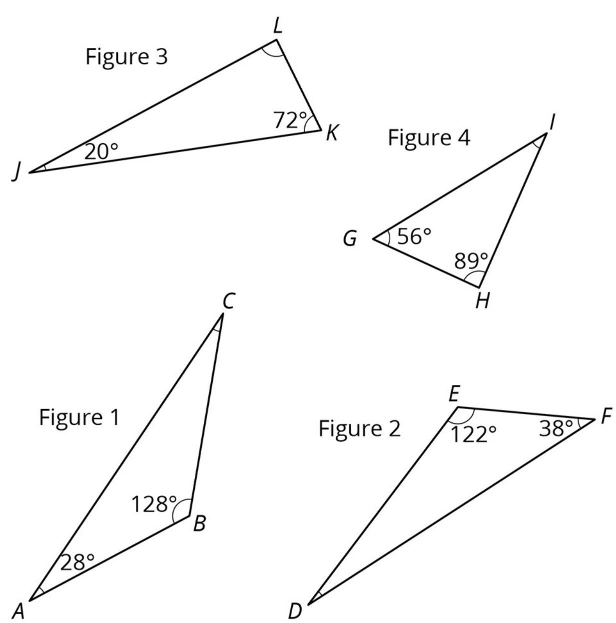


A 180 degree rotation of triangle ABC around the midpoint of AB interchanges angles A and DBA so they have the same size: in the picture these angles are marked as $x°$. A 180 degree rotation of triangle ABC around the midpoint of BC interchanges angles C and CBE so they have the same size: in the picture, these angles are marked as $z°$. Also, DBE is a straight line because 180 degree rotations take lines to parallel lines. So the three angles with vertex B make a line and they add up to 180° ($x + y + z = 180$). But x, y, z are the three angles in $\triangle ABC$ so the sum of the angles in a triangle is always 180°!

Lesson 16 Practice Problems

1. **Problem 1 Statement**

   For each triangle, find the missing angle.



Solution

a. 24 degrees $(24 + 28 + 128 = 180)$

b. 20 degrees $(20 + 38 + 122 = 180)$

c. 88 degrees $(88 + 20 + 72 = 180)$

d. 35 degrees $(35 + 56 + 89 = 180)$

2. **Problem 2 Statement**

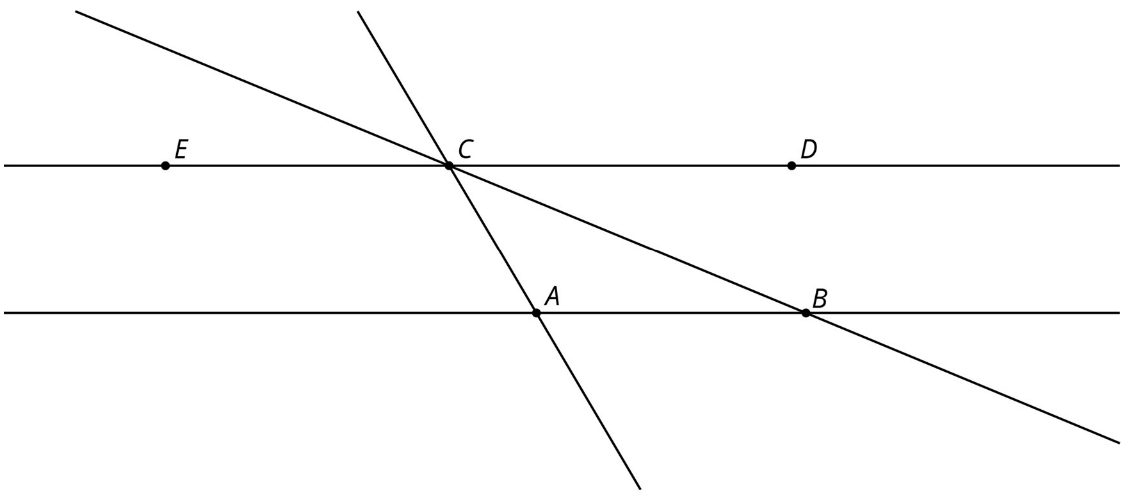
Is there a triangle with *two* right angles? Explain your reasoning.

Solution

No, the three angles in a triangle add up to 180 degrees. Two right angles would already make 180 degrees, and so the third angle of the triangle would have to be 0 degrees—this is not possible.

3. **Problem 3 Statement**

In this diagram, lines AB and CD are parallel.



Angle ABC measures 35° and angle BAC measures 115°.

a. What is $\angle ACE$?

b. What is $\angle DCB$?

c. What is $\angle ACB$?

Solution

a. 115°

b. 35°

c. 30°

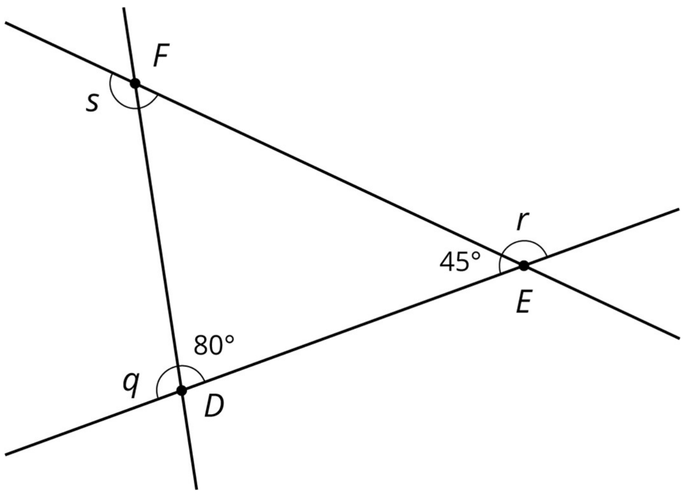
4. **Problem 4 Statement**

Here is a diagram of triangle DEF.

a. Find angles q, r, and s.

b. Find the sum of angles q, r, and s.

c. What do you notice about these three angles?



Solution

a. $q = 100, r = 135, s = 125$

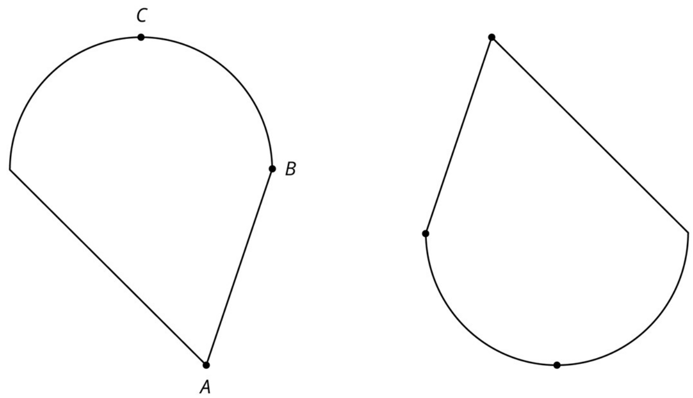
b. $q + r + s = 360$

c. Answers vary. Sample response: Those three angles together make one full revolution of a circle, or 360 degrees. If a person starts walking in the middle of line segment DE toward E, then turns and walks to F, turns again and walks to D, and turns one more time to return to the starting place, they end up facing the same direction they started.

5. **Problem 5 Statement**

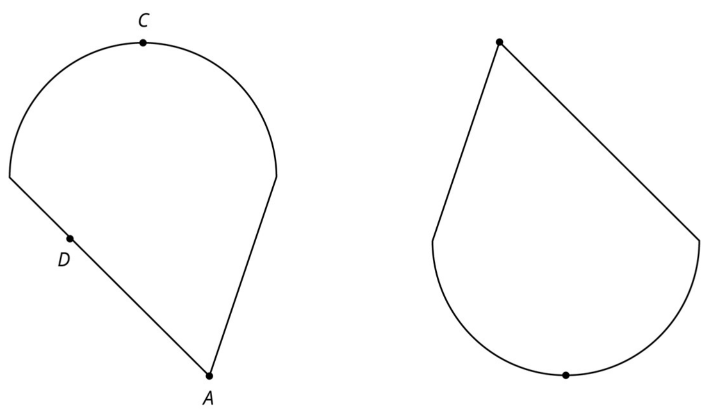
The two shapes are congruent.

a. Label the points A', B' and C' that correspond to A, B, and C in the shape on the right.



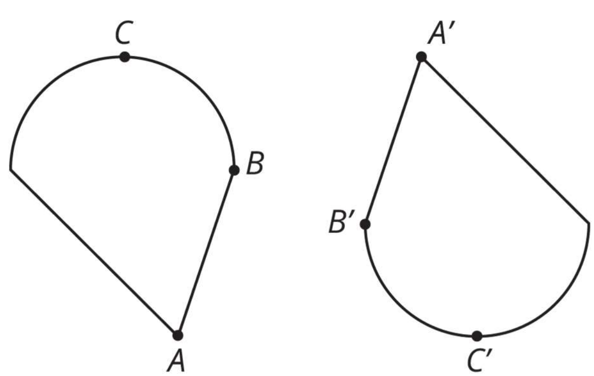
b. If line segment AB measures 2 cm, how long is line segment $A'B'$? Explain.

c. The point D is shown in addition to A and C. How can you find the point D' that corresponds to D? Explain your reasoning.



Solution

a.



b. 2 cm. The shapes are congruent, and the corresponding line segments of congruent shapes are congruent.

c. Because the shapes are congruent, the point D' will be on the corresponding side and will be the same distance from C' that D is from C. D can be found by looking for the point on the line segment going down and to the right from A' that is the appropriate distance from C'.