# The Mathematics of Mazes

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1 author:

Hollis Williams
The University of Warwick

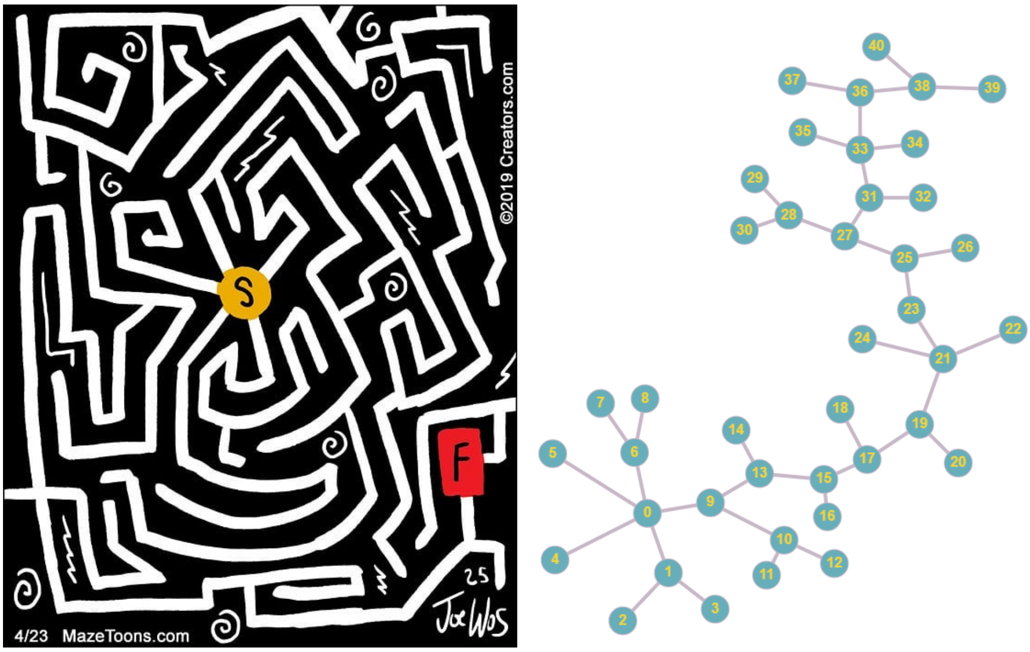


*Figure 1: Example of a perfect maze. In terms of graph theory, the maze is a connected acyclic undirected graph with* 41 *vertices where the first vertex is the starting point and the last vertex is the end point. The maze was reproduced with permission of Joe Wos.*

# *The Mathematics of Mazes*

***Hollis Williams*** [a]

[a] *School of Engineering, University of Warwick, Coventry CV4 7AL, UK*

We discuss the two main perspectives from which a mathematician might view a maze: that of graph theory and that of problem-solving.

Mathematics is often compared to a landscape, or even an entire universe. Another analogy which some people like to use is that mathematics is a maze: one can go in circles for years without realising it or take hundreds of turns and unexpectedly find a new route into familiar territory. Let's turn this around and instead think about the mathematics of mazes. The first person to seriously investigate plane mazes from a mathematical point of view was Euler. These investigations led him to make some of the first contributions to the subjects now known as graph theory and topology. Mazes typically look very messy, so the first thing a mathematician might want to do is tidy a maze up and abstract it out into a collection of nodes (or vertices) joined by paths (or edges). This object is known as a graph in pure mathematics.

If a maze can be represented by a graph which has vertices and edges such that each pair of vertices is joined by only one edge, then it is known as a perfect maze. If we further require that none of the edges have directions associated to them, then the graph is known as a tree. A common example of such a graph is a family tree. A labyrinth is trivially a perfect maze as it can be represented by one pair of vertices joined by a single edge (having one path which can be followed from beginning to end). Figure 1 shows an example of a perfect maze along with the result when we ignore all the artistic details and only draw the edges and vertices. We can see that the maze is a tree because each pair of vertices is joined by one path only and none of the edges have directions associated to them. Figure 2 shows an example of a non-perfect maze represented by a graph which contains pairs of vertices joined by more than one edge.

A type of maze which is particularly appealing to graph theorists is known as the Hamilton maze [1]. The goal of a Hamilton maze is to find the unique path which visits each vertex once apart from the start and end points ie. a path which visits each vertex once and then ends up back at the beginning. Such a path is known as a Hamiltonian cycle. A Hamiltonian path visits each vertex once without being required to end up back at

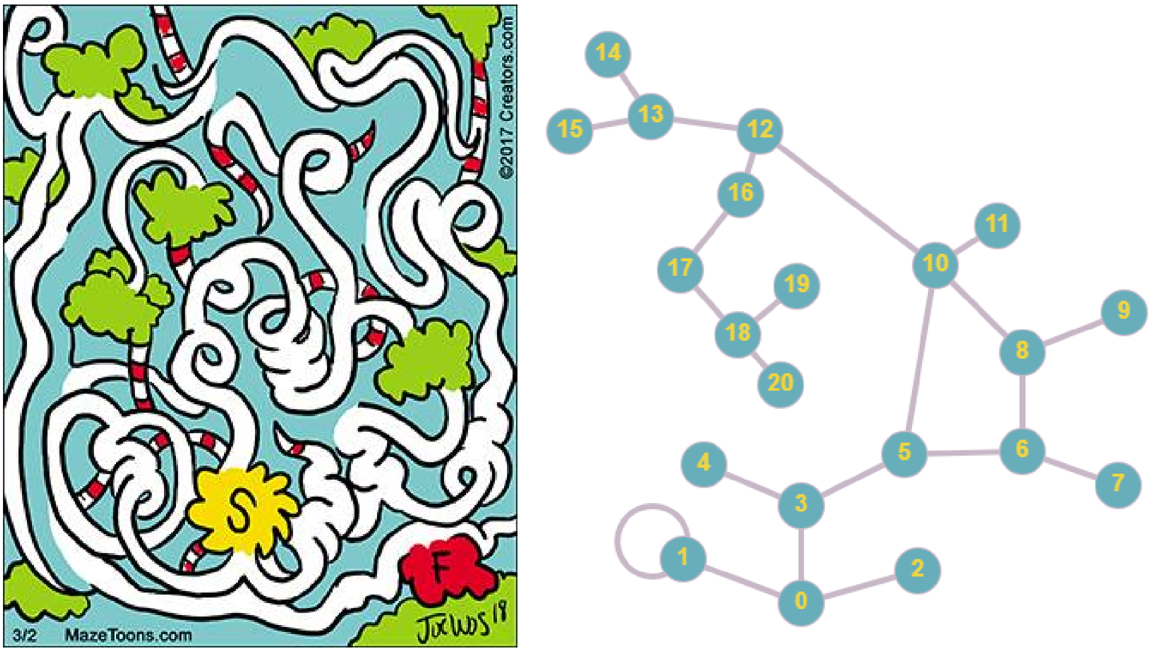


*Figure 2: Example of a non-perfect maze. In graph theory terms, this maze is not a tree because it contains cycles. The maze was reproduced with permission of Joe Wos*

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the start point. Hamilton originally considered the problem of finding a Hamiltonian cycle in the edge graph of the dodecahedron (known as Hamilton's puzzle, or the icosian game). The general Hamiltonian problem is to find out whether a graph admits a Hamiltonian path or a Hamiltonian cycle. Depending on the type of graph, it might be possible to use a computer algorithm to solve this problem in a reasonable amount of time. For example, it can be shown that 4-connected planar graphs (graphs with more than 4 vertices which stay connected when fewer than 4 vertices are removed) always admit Hamiltonian cycles, and that one can always use a computer algorithm to find the cycle in linear time using linear computer storage space [2]. In Figure 3 (top), we show an example of a Hamilton maze (a maze which admits a cycle such that every vertex is visited once). This is a Hamilton maze because if we start at vertex 2, there is a path which goes through each vertex once and ends up back at vertex 2. In Figure 3 (bottom), we show a non-Hamilton maze. One can check that for this graph it is impossible to start from and end up at the same vertex whilst visiting every other vertex only once.

Chess players might see a parallel between the Hamilton maze and the knight's tour problem, where a player makes a sequence of moves with the knight such that every square on the chessboard is visited once. A closed knight's tour is a path which ends up at the beginning square and an open knight's tour is a path which ends at a different square. These are clearly special cases of a Hamiltonian cycle and a Hamiltonian path, respectively, where the graph of all the possible moves of the knight on a chessboard is called a knight's graph. In Figure 4, we show the knight's graph associated with a $4 \times 4$ chessboard. This graph does not admit a Hamiltonian cycle, so one cannot visit every square once with the knight and end up back at the beginning. It is an interesting fact that a knight's tour does not exist on a smaller $4 \times 4$ chessboard, but it does exist for a regular $8 \times 8$ chessboard [3].

Euler was originally considering a similar scenario when he solved the problem of the Seven Bridges of Königsberg in the negative. The problem in this case was whether someone could take a walk through the city and walk across all seven bridges without ever crossing the same bridge twice. Euler was able to show that this was impossible by converting the seven bridges into a graph (the bridges are edges and the landmasses are vertices) and noting that in order to move across each edge only once, the number of edges adjacent to each vertex must be an even number. However, all four of the vertices in the problem have an odd number of adjacent edges, and since only two vertices can be the start point and the end point, it follows that there is no solution. The corresponding graph is shown in Figure 5.

In stripping the problem down to a graph (much as we have down for mazes) and discarding irrelevant geometric properties such as the lengths or curvatures of the bridges, Euler effectively made the first gestures towards the subject which we now call topology. One must bear in mind that this idea of taking a maze or a

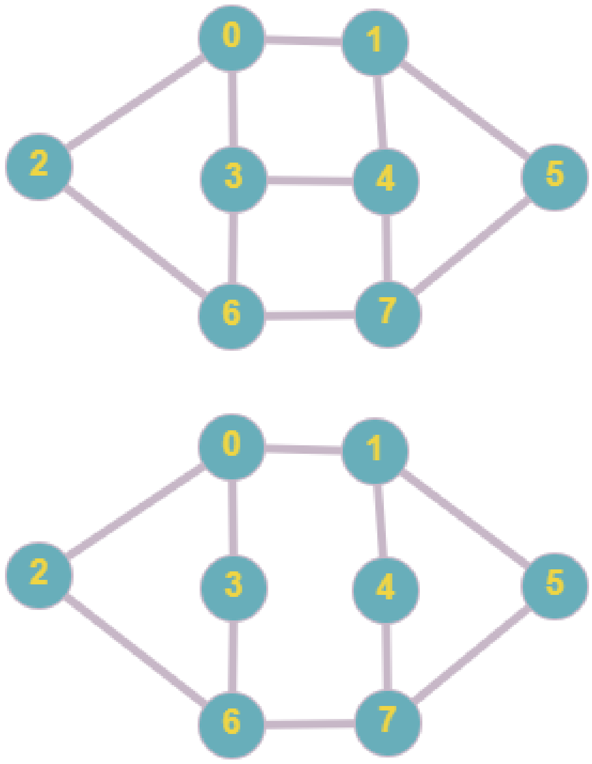


*Figure 3: Example of a Hamilton maze and a non-Hamilton maze.*

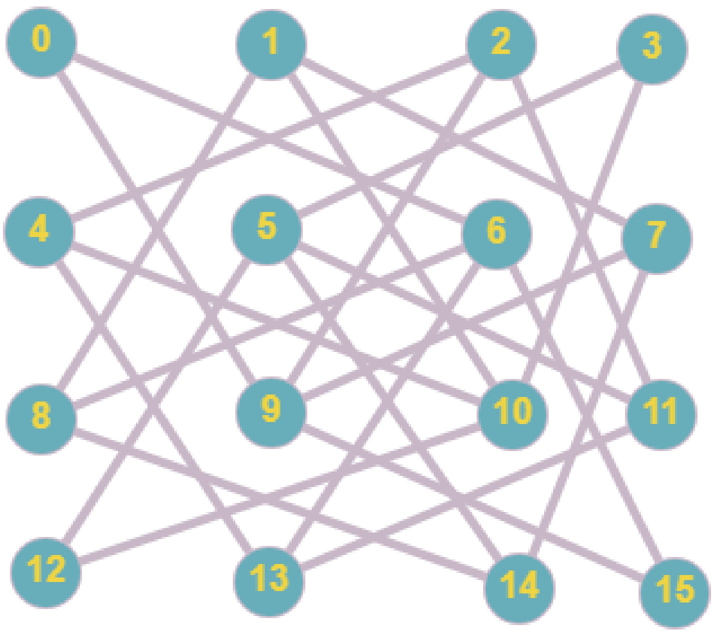


*Figure 4: Example of a knight's graph which does not admit a knight's tour.*

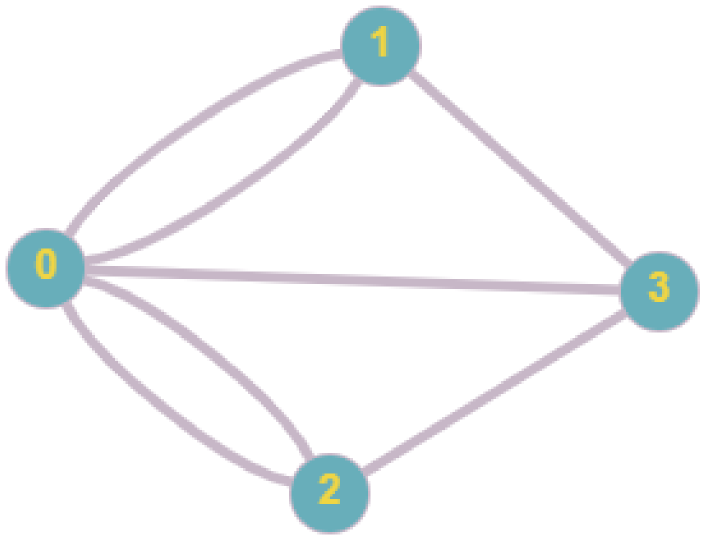


*Figure 5: Graph corresponding to the Seven Bridges of Königsberg.*

sequence of bridges and boiling it down to its essential information was revolutionary at the time, as the subject of topology as we know it did not exist.

Besides the topological/graph theory interpretation, the main other perspective on mazes sees them as puzzles or 'problems' to be solved. How challenging can a standard maze actually be as a problem? In Figure 6 (top), a maze is shown which was drawn by Lewis Carroll for the purpose of being a difficult puzzle (Carroll is of course better known as the author of *Alice's Adventures in Wonderland*). In terms of our earlier terminology, both mazes in Figure 6 are non-perfect because they both contain repeated vertices where a path loops from a vertex back to the same vertex. In particular, there are pairs of vertices in Papa's Maze which are joined by multiple paths. Although Carroll's maze looks complex, it is actually quite easy and can be solved in a matter of minutes. Probably the most complicated example of a maze ever drawn without computer assistance emerged in 2014 (known as Papa's Maze, or the Impossible Maze), shown in Figure 6 (bottom). This was solved by the author and M. Justice independently in 2014. A second, much simpler maze of the same size appeared in 2015 (known as Papa's Maze 2). This was solved by the author in 2015. Finally, a third maze equal in complexity to the first one appeared in 2020 (known as Papa's Maze 3). This was also solved by the author in 2020 [4].

Papa's Maze is quite difficult to solve both because of the high number of dead ends and the number of possible path choices (other mazes typically create difficulty using only one of these). There are countless ways of making mazes more challenging as puzzles (some of them probably inspired by video games like *Tomb Raider*): doors which need keys, paths and walls which change, one-way paths or any other method one can think of to restrict the movement of the player. We mentioned that Papa's Maze was created without computer assistance, so one might wonder whether it is possible to computer generate difficult mazes using algorithms based on graph theory. That turns out to be the case, and there are many of these with their own advantages and disadvantages.

The usual basic approach is to start with a connected graph and then make a subgraph which connects two points in the most confusing way possible, interspersing random edges to get plenty of loops. Probably the most interesting way to computer generate a maze is to use a cellular automaton where each generation of cells survives if it has a certain number of neighbouring cells (similar to Conway's Game of Life), eventually building a maze structure. Most (or all) of these algorithms so far result in mazes which are quite predictable and uniform compared to a maze which a human can create. The final question is whether computer algorithms can be written which solve mazes, and the answer is yes, with limitations specific to each algorithm. This is clearly possible on some trivial level, since one could write an algorithm which simply tells the computer to continually walk across edges crossing off dead ends until the end point is reached.

Needless to say, this is an incredibly slow process and far from optimal as the best way of finding a path from the start point to the end point. Another common way to solve mazes is to use the left-hand rule, where one always keeps the boundary of the maze on the left-hand side, and keeps travelling along that edge until the end point is reached. This works on a topological level because most mazes are simply connected and have boundaries which can be morphed continuously into a circle (any polygon is homeomorphic to a circle, to use

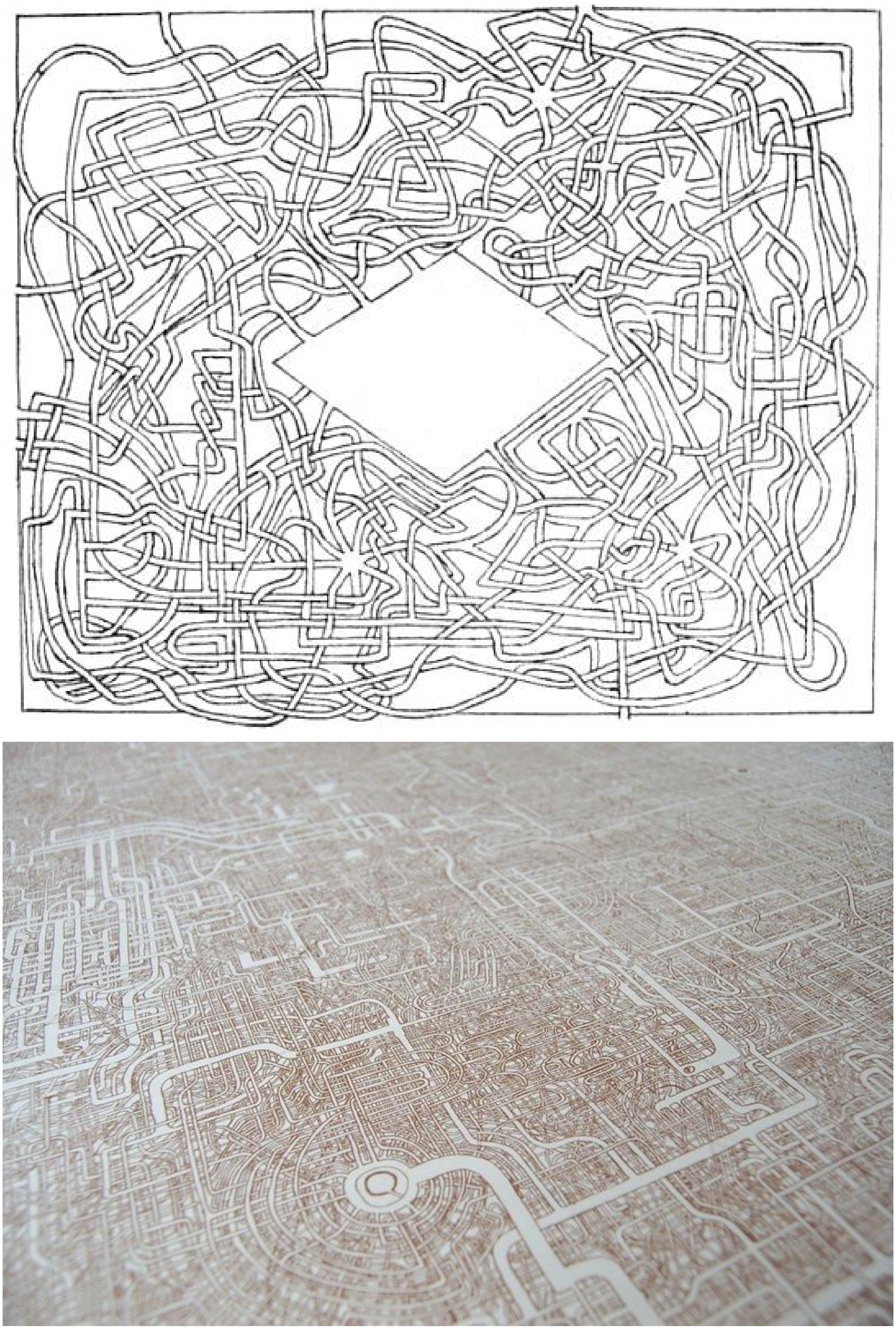

*Figure 6: Examples of complicated mazes. The top figure shows Lewis Carroll's maze, and the bottom figure shows a section of Papa's Maze (reproduced with permission of Spoon & Tamago).*

topology jargon). Following the left-hand wall is topologically the same as walking around a circle from the start point to the end point if we forget the irrelevant geometric details. It is doubtful that Papa's Maze is simply connected, so this would likely not work. In fact, one starts inside the maze in that case, and all it takes for the algorithm to fail is that the start point not be connected to a section of the maze which is connected to the outermost boundary.

A more effective algorithm due to Trémaux uses as its basic principle the idea that one should cross off dead ends, but also mark the floor in such a way that paths which close up loops on a node are treated by the computer as dead ends. If ones does this, all the loops will be cut away, and the maze will eventually become simply connected again such that it can be solved with the left-hand rule. This was similar to the technique which the author used to solve Papa's Maze in a reasonable amount of time, whereas Justice seems to have used a technique which is basically what a computer would do when carrying out the brute force algorithm which we mentioned.

## References

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