Lesson 16: Distinguishing between surface area and volume

Goals

- Comprehend that surface area and volume are two different attributes of three-dimensional objects and are measured in different units.
- Describe (orally and in writing) shapes built out of cubes, including observations about their surface area and volume.
- Determine the surface area and volume of shapes made out of cubes.

Learning Targets

- I can explain how it is possible for two polyhedra to have the same surface area but different volumes, or to have different surface areas but the same volume.
- I know how one-, two-, and three-dimensional measurements and units are different.

Lesson Narrative

In this optional lesson, students distinguish among measures of one-, two-, and three-dimensional attributes and take a closer look at the distinction between surface area and volume (building on students' work in earlier grades). Use this lesson to reinforce the idea that length is a one-dimensional attribute of geometric shapes, surface area is a two-dimensional attribute, and volume is a three-dimensional attribute.

By building polyhedra, drawing representations of them, and calculating both surface area and volume, students see that different three-dimensional shapes can have the same volume but different surface areas, and vice versa. This is analogous to the fact that two-dimensional shapes can have the same area but different perimeters, and vice versa. Students must attend to units of measure throughout the lesson.

Note: Students will need to bring in a personal collection of 10–50 small objects ahead of time for the first lesson of the next unit. Examples include rocks, seashells, trading cards, or coins.

Building On

- Recognise area as an attribute of plane shapes and understand concepts of area measurement.
- Know relative sizes of measurement units within one system of units including km, m, cm; kg, g; lb, oz.; l, ml; hr, min, sec. Within a single system of measurement, express measurements in a larger unit in terms of a smaller unit. Record measurement equivalents in a two-column table. For example, know that 1 ft is 12 times as long as 1 in. Express the length of a 4 ft snake as 48 in. Generate a conversion table for feet and inches listing the number pairs (1,12), (2,24), (3,36), …

- Geometric measurement: understand concepts of volume and relate volume to multiplication and to addition.
- A solid shape which can be packed without gaps or overlaps using n unit cubes is said to have a volume of n cubic units.
- Measure volumes by counting unit cubes, using cubic cm, cubic in, cubic ft, and improvised units.
- Find the volume of a cuboid with whole-number side lengths by packing it with unit cubes, and show that the volume is the same as would be found by multiplying the edge lengths, equivalently by multiplying the height by the area of the base. Represent threefold whole-number products as volumes, e.g., to represent the associative property of multiplication.

Addressing

- Represent three-dimensional shapes using nets made up of rectangles and triangles, and use the nets to find the surface area of these shapes. Apply these techniques in the context of solving real-world and mathematical problems.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Discussion Supports
- Notice and Wonder
- Think Pair Share

Required Materials

Geometry toolkits
tracing paper, graph paper, coloured pencils, scissors, and an index card to use as a straightedge or to mark right angles.

Multi-link cubes
Sticky notes

Required Preparation

- Prepare solutions to the first question of 1-2-3 Dimensional Attributes activity on a large visual display.
- Prepare sets of 16 multi-link cubes and two sticky notes for each student.

Student Learning Goals

Let’s contrast surface area and volume.

16.1 Attributes and Their Measures

Warm Up: 10 minutes

This activity strengthens students' awareness of one-, two-, and three-dimensional attributes and the units commonly used to measure them. Students decide on the units based on the attributes being measured and the size of the units and how appropriate they would be for describing given quantities.

As students work, select a few students to share their responses to the last two questions of the activity (on the quantities that could be measured in miles and in cubic metres).

Instructional Routines

- Think Pair Share

Launch

Consider a quick review of metric and imperial units of measurement before students begin work. Include some concrete examples that could help illustrate the size of each unit.

Then, pick an object in the classroom for which surface area and volume could be measured (e.g. a desk). Ask students, "What units might we use to measure the surface area of the desktop? What units might we use to measure the volume of a drawer?"

Clarify the relative sizes of the different units that come up in the conversation. For instance, discuss how a metre is a little over three feet, a yard is three feet, a kilometre is about two-thirds of a mile, a millimetre is one tenth of a centimetre, etc.

Give students 4–5 minutes of quiet think time and then a couple of minutes to share their responses with a partner. Prepare to display the answers to the first six questions for all to see.

Anticipated Misconceptions

Depending on the students' familiarity with metric and imperial units, there may be some confusion about the size of each unit. Consider displaying measuring tools or a reference sheet that shows concrete examples of items measured in different-sized units.

Student Task Statement

For each quantity, choose one or more appropriate units of measurement.

For the last two, think of a quantity that could be appropriately measured with the given units.

<u>Quantities</u>

1. Perimeter of a car park:

2. Volume of a van:

3. Surface area of a refrigerator:
4. Length of an eyelash:
5. Area of a county:
6. Volume of an ocean:
7. ______________________: miles
8. ______________________: cubic metres

Units

- millimetres (mm)
- feet (ft)
- metres (m)
- square inches (sq in)
- square feet (sq ft)
- square miles (sq mi)
- cubic kilometres (cu km)
- cubic yards (cu yd)

Student Response

1. Metres, feet
2. Cubic yards
3. Square inches, square feet
4. Millimetres
5. Square miles
6. Cubic kilometres, cubic yards
7. Answers vary. Sample responses: distance between home and school, length of a river
8. Answers vary. Sample responses: volume of a room, volume of a swimming pool

Activity Synthesis

Display the solutions to the first six questions for all to see and to use for checking. Then, select a few students to share their responses to the last two questions.

Ask students what they notice about the units for area and the units for volume. If not already mentioned by students, highlight that area is always measured in square units and volume in cubic units.

16.2 Building with 8 Cubes

Optional: 25 minutes (there is a digital version of this activity)

This activity clarifies the distinction between volume and surface area and illustrates that two polyhedra can have the same volume but different surface areas.

Students build shapes using two sets of eight cubes and determine their volumes and surface areas. Since all of the designs are made of the same number of cubes, they all have the same volume. Students then examine all of the designs and discuss what distinguishes shapes with smaller surface areas from those with greater ones.

As students work, monitor the range of surface areas for the shapes that students built. Select several students whose designs collectively represent that range.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Discussion Supports
- Notice and Wonder

Launch

Give each student (or group of 2 students) 16 multi-link cubes, two sticky notes, and 8–10 minutes of work time.

Explain that their job is to design and build two shapes—using 8 cubes for each—and find the volume and surface area of each shape. Ask them to give each shape a name or a label and then record the name, surface area, and volume on a sticky note.

Anticipated Misconceptions

Even though students are dealing with only 8 cubes at a time, they may make counting errors by inadvertently omitting or double-counting squares or faces. This is especially likely for designs that are non-cuboids. Encourage students to think of a systematic way to track the number of square units they are counting.

Some students may associate volume only with cuboids and claim that the volume of non-cuboid designs cannot be determined. Remind them of the definition of volume.

Student Task Statement

Your teacher will give you 16 cubes. Build two different shapes using 8 cubes for each. For each shape:

1. Give it a name or a label (e.g., Mai's First Shape or Diego's Steps).

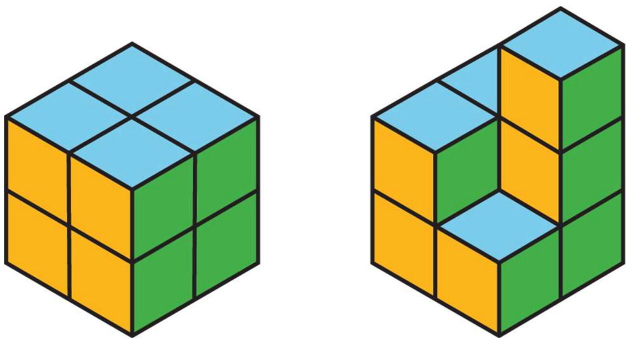
2. Determine the **volume**.

3. Determine the surface area.

4. Record the name, volume, and surface area on a sticky note.

Student Response

Designs vary. Here are two possible shapes:



They both have a volume of 8 cubic units. The first has a surface area of 24 square units. The second has a surface area of 28 square units. The smallest possible surface area for an 8-cube construction is 24 square units, and the largest is 34 square units.

Activity Synthesis

Ask all students to display their designs and their sticky notes and give them a couple of minutes to circulate and view one another's work.

Then, ask previously identified students to arrange their designs in the order of their surface area, from least to greatest, and display their designs for all to see. Record the information about the designs in a table, in the same sequence. Display the table for all to see. Here is an example.

shapes	volume	surface area
Andre's cube	8	24
Lin's steps	8	28
Jada's first shape	8	28
. . .	. . .	. . .
Noah's tower	8	34

Give students a minute to notice and wonder about the designs and the information in the table. Ask students to give a signal when they have noticed and wondered about at least one thing. Invite a few students to share their observations and questions. Then, discuss the following questions (if not already mentioned by students):

- "What do all of the shapes have in common?" (Their volume)
- "Why are all the volumes the same?" (Volume measures the number of unit cubes that can be packed into a shape. All the designs are built using the same number of cubes.)
- "Why do some shapes have larger surface areas than others? What do shapes with larger surface areas look like?" (The cubes are more spread out and have more of their faces exposed.)
- "What about those with smaller surface areas?" (They are more compact and have more of their faces hidden or shared with another cube.)
- "Is it possible to build a shape with a different volume? How?" (Yes, but it would involve using fewer or more cubes.)

If students have trouble visualising how surface area changes when the design changes, demonstrate the following:

- Make a cube made of 8 smaller cubes. Point to one cube and ask how many of its faces are exposed (3).
- Pop that cube off and move it to another place.
- Point out that, in the "hole" left by the cube that was moved, 3 previously interior faces now contribute to the surface area. At the same time, the relocated cube now has 5 faces exposed.

Speaking, Listening: Discussion Supports. To help students describe and explain their comparisons, provide language students can use (e.g., spread out vs. compact, exposed/visible vs. hidden/covered). Demonstrate the comparisons using the visuals of the cube designs. Offer sentence frames for students to help with their explanations (e.g., "The surface area of shape _____ is larger (or smaller) than shape _____ because. . ."
Design Principle(s): Optimise output (for explanation)

16.3 Comparing Cuboids Without Building Them

Optional: 20 minutes

Previously, students studied shapes with the same volume but different surface areas. Here they see that it is also possible for shapes to have the same surface area but different volumes. Students think about how the appearance of these shapes might compare visually.

Students are given the side lengths of three cuboids and asked to find the surface area and the volume of each. Some students can visualise these, but others may need to draw nets, sketch the shapes on isometric grid paper, or build physical cuboids. Prepare cubes for students to use. Each of the three cuboids can be built with 15 or fewer cubes, but 40 cubes are needed to build all three simultaneously. (If the cubes are not centimetre cubes, ask students to treat them as if the edge length of each cube was 1 cm.)

As students work, look out for errors in students' calculations in the first question, which will affect the observations they make in the second question. Select a few students who notice that the volumes of the cuboids are all different but the surface areas are the same.

Instructional Routines

- Discussion Supports

Launch

Arrange students in groups of 2. Provide access to multi-link cubes and geometry toolkits. Give students 6–7 minutes of quiet think time and then 2–3 minutes to discuss their responses with their partner. Ask partners to agree upon one key observation to share with the whole class.

Representation: Develop Language and Symbols. Use virtual or concrete manipulatives to connect symbols to concrete objects or values. Provide multi-link cubes or blocks to help students analyse the volumes and surface areas of the cuboids.
Supports accessibility for: Conceptual processing Conversing: Discussion Supports. Use this routine to help students describe what they noticed about the volume and surface areas of their cuboids. Students should take turns stating an observation and their reasoning with their partner. Invite Partner A to begin with this sentence frame: "I noticed ____, so . . ." Invite the listener, Partner B, to press for additional details referring to specific features of the cuboids. Students should switch roles until they have listed all observations.
Design Principle(s): Support sense-making; Cultivate conversation

Anticipated Misconceptions

Students may miss or double-count one or more faces of the cuboids and miscalculate surface areas. Encourage students to be systematic in their calculations and to use organisational strategies they learned when finding surface area from nets.

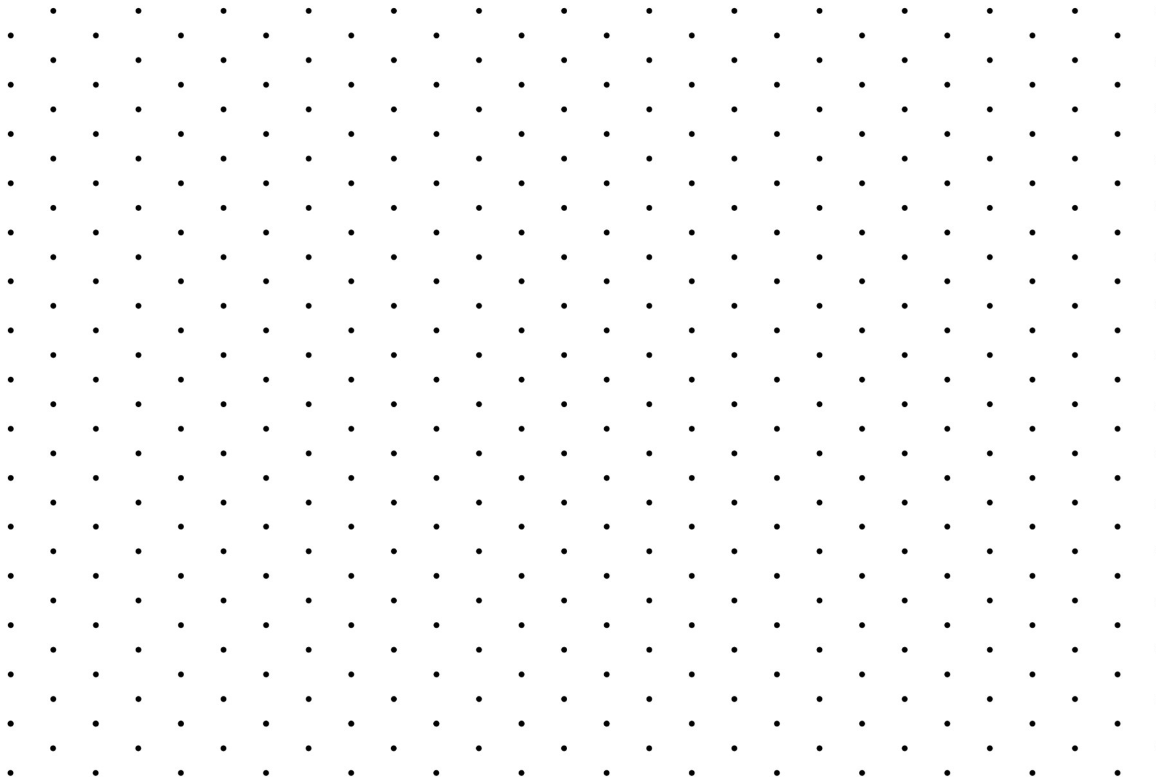
Students may need reminders to use square units for area and cubic units for volume.

Student Task Statement

Three cuboids each have a height of 1 cm.

- Cuboid A has a base that is 1 cm by 11 cm.
- Cuboid B has a base that is 2 cm by 7 cm.
- Cuboid C has a base that is 3 cm by 5 cm.

1. Find the surface area and volume of each cuboid. Use the dot paper to draw the cuboids, if needed.



2. Analyse the volumes and surface areas of the cuboids. What do you notice? Write 1 or 2 observations about them.

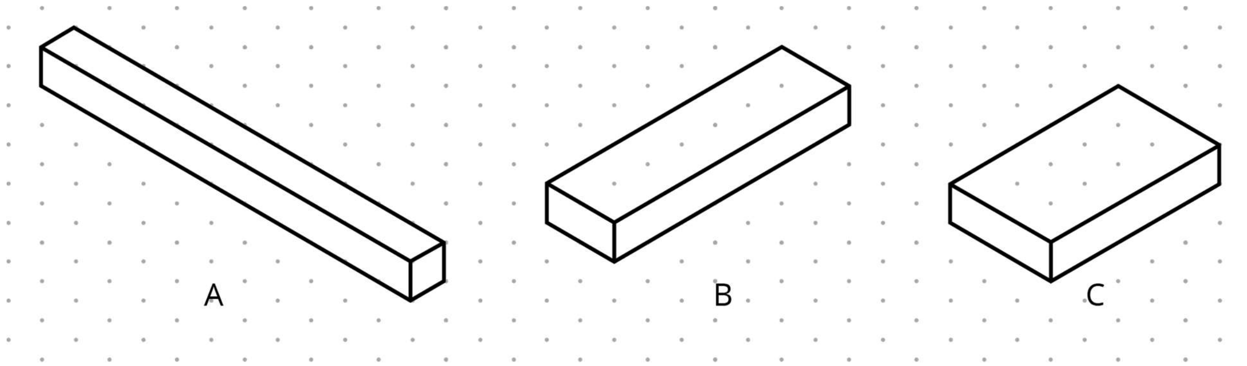
Student Response

1. Surface areas:

   - Cuboid A: $4(11 \times 1) + 2(1 \times 1) = 46$ square centimetres
   - Cuboid B: $2(7 \times 2) + 2(7 \times 1) + 2(2 \times 1) = 46$ square centimetres
   - Cuboid C: $2(5 \times 3) + 2(5 \times 1) + 2(3 \times 1) = 46$ square centimetres

   Volumes:

   - Cuboid A: 11 cubic centimetres ($11 \times 1 \times 1 = 11$)
   - Cuboid B: 14 cubic centimetres ($7 \times 2 \times 1 = 14$)
   - Cuboid C: 15 cubic centimetres ($5 \times 3 \times 1 = 15$)



2. Answers vary. Sample responses:

   - The surface areas of the cuboids are all the same, but the volumes are all different.
   - The polygons that make up the faces of each cuboid are different-sized rectangles, but their areas all add up to the same total of square centimetres.
   - Cuboid C can fit the most centimetre cubes, but because the cubes would fit together in a compact way, some of the cubes would only have two square centimetres of exposed faces.
   - Cuboid A can fit the fewest centimetre cubes, but because the cubes would be more spread out, more of their faces would be exposed.

Are You Ready for More?

Can you find more examples of cuboids that have the same surface areas but different volumes? How many can you find?

Student Response

Answers vary. Sample response: A cuboid that is 4 units by 5 units by 1 unit and one that is 2 units by 9 units by 1 unit have the same surface area but different volumes.

Generate examples by finding different pairs of factors of the same number and subtracting 1 from each factor. However, there are other ways. For example, $60 = 6 \times 10$ and $60 = 5 \times 12$. The 5-by-9-by-1 and 4-by-11-by-1 cuboids have the same surface areas but different volumes.

Activity Synthesis

Ask students to share their observations in response to the second question. Record them for all to see. For each unique observation, poll the class to see if others noticed the same thing. Highlight the following observations, or point them out if not already mentioned by students:

- The volumes of the cuboids are all different, but the surface areas are the same.

- Volume is described in terms of unit cubes and surface area in terms of the exposed faces of those unit cubes.

Explain that, in an earlier activity, we saw how different shapes could have the same volume (i.e. being made up of the same number of unit cubes) but different surface areas. Now we see that it is also possible for shapes with different volumes (i.e. consisting of different numbers of unit cubes) to have the same surface area.

If students have trouble conceptualising the idea of shapes with different volume having the same surface area, refer to the filing cabinet activity in the first lesson on surface area. The number of square sticky notes needed to cover all of the faces of the filing cabinet was its surface area. If we use all of those square notes (no more, no less) to completely cover (without overlapping sticky notes) a cabinet that has a different volume, we can say that the two pieces of furniture have the same surface area and different volumes.

Lesson Synthesis

In this lesson, we refreshed our memory of measures of one-, two-, and three-dimensional attributes. Reiterate that length is a one-dimensional attribute of geometric shapes, area is a two-dimensional attribute, and volume is a three-dimensional attribute. Revisit a few examples of units for length, area, and volume.

We also explored the surface areas and volumes of polyhedra and noticed that two shapes can have the same volumes but different surface areas, and vice versa.

- "How could two shapes with a volume of 4 cubic units have a surface area of 16 square units and 18 square units?" (Surface area and volume measure different attributes of a three-dimensional shape.)
- "What kind of measure is surface area? What kind of measure is volume?" (Surface area is a two-dimensional attribute; we measure it in square units. Volume is a three-dimensional attribute; we measure it in cubic units.)
- "Are the two measures related? Does a greater volume necessarily mean a greater surface area, and vice versa?" (No, one measure does not affect the other. A shape that has a greater volume than another may not necessarily have a greater surface area.)

16.4 Same Surface Area, Different Volumes

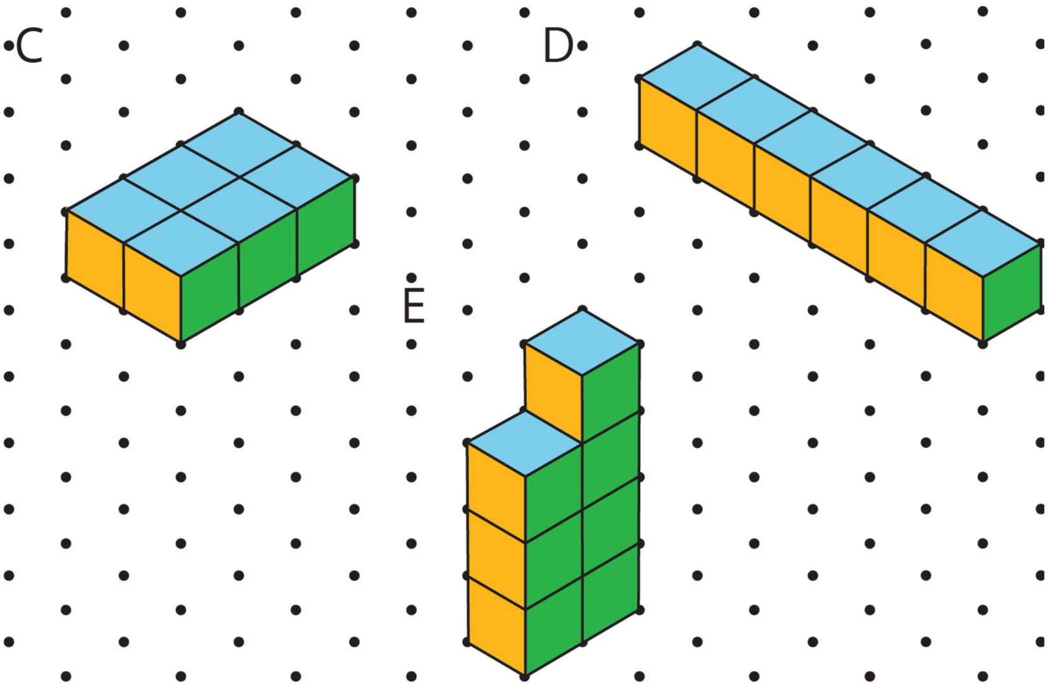
Cool Down: 5 minutes

Launch

Encourage students to refer to the class list of observations from the previous activity

Student Task Statement

Choose two shapes that have the same surface area but different volumes. Show your reasoning.

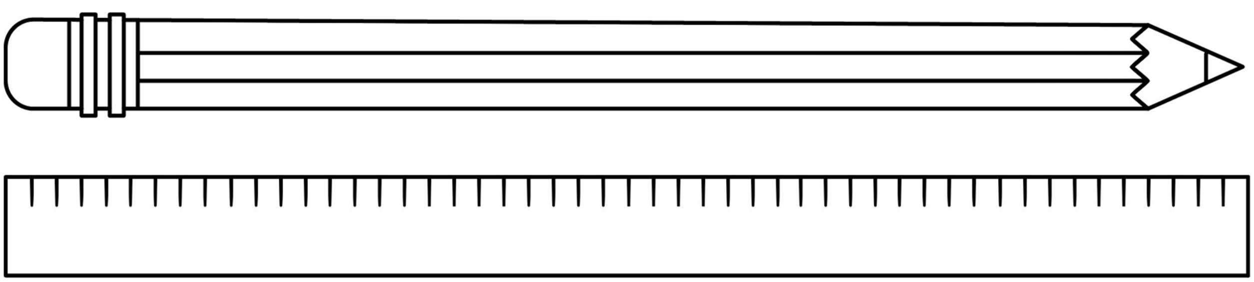


Student Response

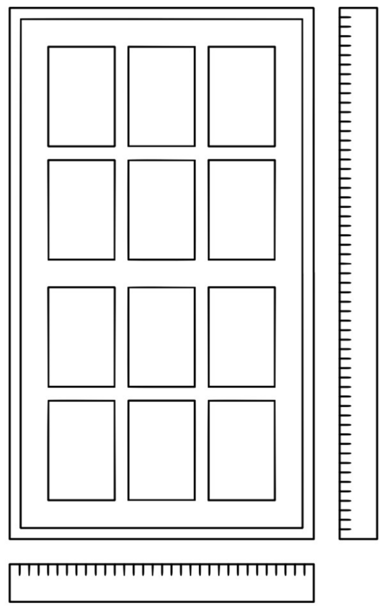
Shapes D and E both have a surface area of 26 square units, but D has a volume of 6 cubic units, and E has a volume of 7 cubic units.

Student Lesson Summary

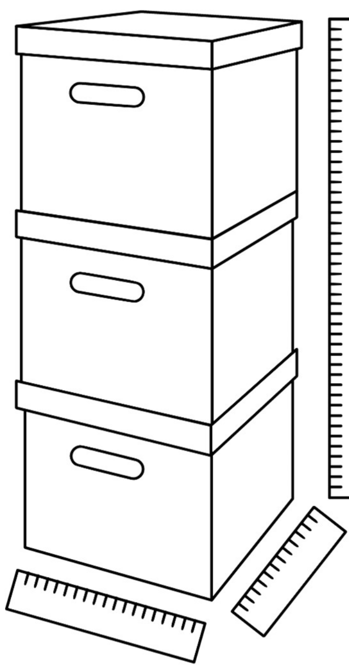
Length is a one-dimensional attribute of a geometric shape. We measure lengths using units like millimetres, centimetres, metres, kilometres, inches, feet, yards, and miles.



Area is a two-dimensional attribute. We measure area in square units. For example, a square that is 1 centimetre on each side has an area of 1 square centimetre.



Volume is a three-dimensional attribute. We measure volume in cubic units. For example, a cube that is 1 kilometre on each side has a volume of 1 cubic kilometre.

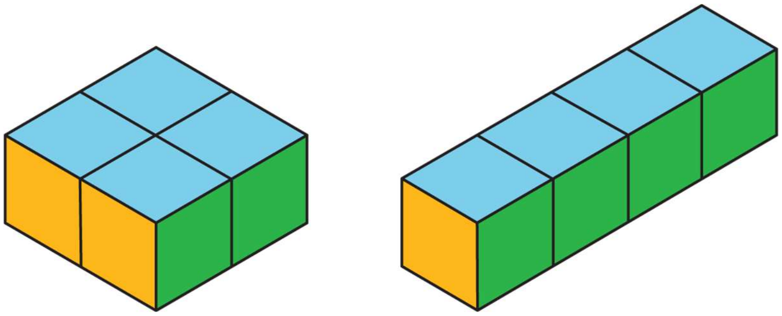


Surface area and volume are different attributes of three-dimensional shapes. Surface area is a two-dimensional measure, while volume is a three-dimensional measure.

Two shapes can have the same volume but different surface areas. For example:

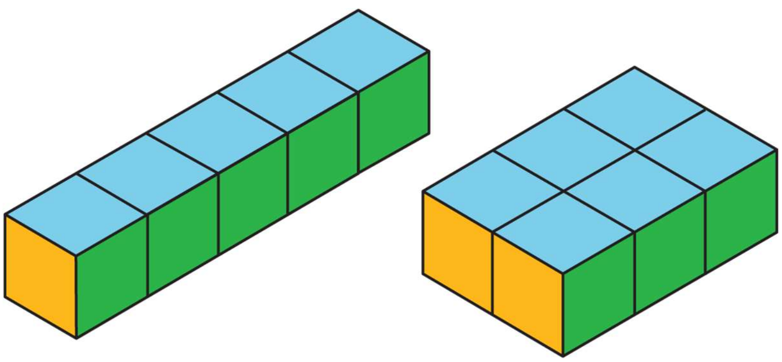
- A cuboid with side lengths of 1 cm, 2 cm, and 2 cm has a volume of 4 cm^3 and a surface area of 16 cm^2.

- A cuboid with side lengths of 1 cm, 1 cm, and 4 cm has the same volume but a surface area of 18 cm^2.



Similarly, two shapes can have the same surface area but different volumes.

- A cuboid with side lengths of 1 cm, 1 cm, and 5 cm has a surface area of 22 cm^2 and a volume of 5 cm^3.
- A cuboid with side lengths of 1 cm, 2 cm, and 3 cm has the same surface area but a volume of 6 cm^3.



Glossary

- volume

Lesson 16 Practice Problems

1. **Problem 1 Statement**

   Match each quantity with an appropriate unit of measurement.

   A. The surface area of a tissue box

   B. The amount of soil in a planter box

   C. The area of a car park

   D. The length of a football pitch

   E. The volume of a fish tank

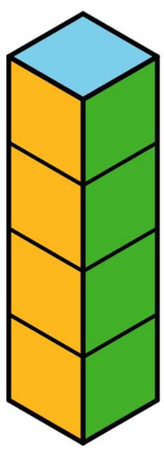
1. Square metres
2. Yards
3. Cubic inches
4. Cubic feet
5. Square centimetres

Solution

- A: 5
- B: 3
- C: 1
- D: 2
- E: 4

2. **Problem 2 Statement**

Here is a shape built from multi-link cubes.



a. Find the volume of the shape in cubic units.

b. Find the surface area of the shape in square units.

c. True or false: If we double the number of cubes being stacked, both the volume and surface area will double. Explain or show how you know.

Solution

a. 4 cubic units. $(1 \times 1 \times 4) = 4$.

b. 18 square units. $(4 \times 4) + (2 \times 1) = 18$.

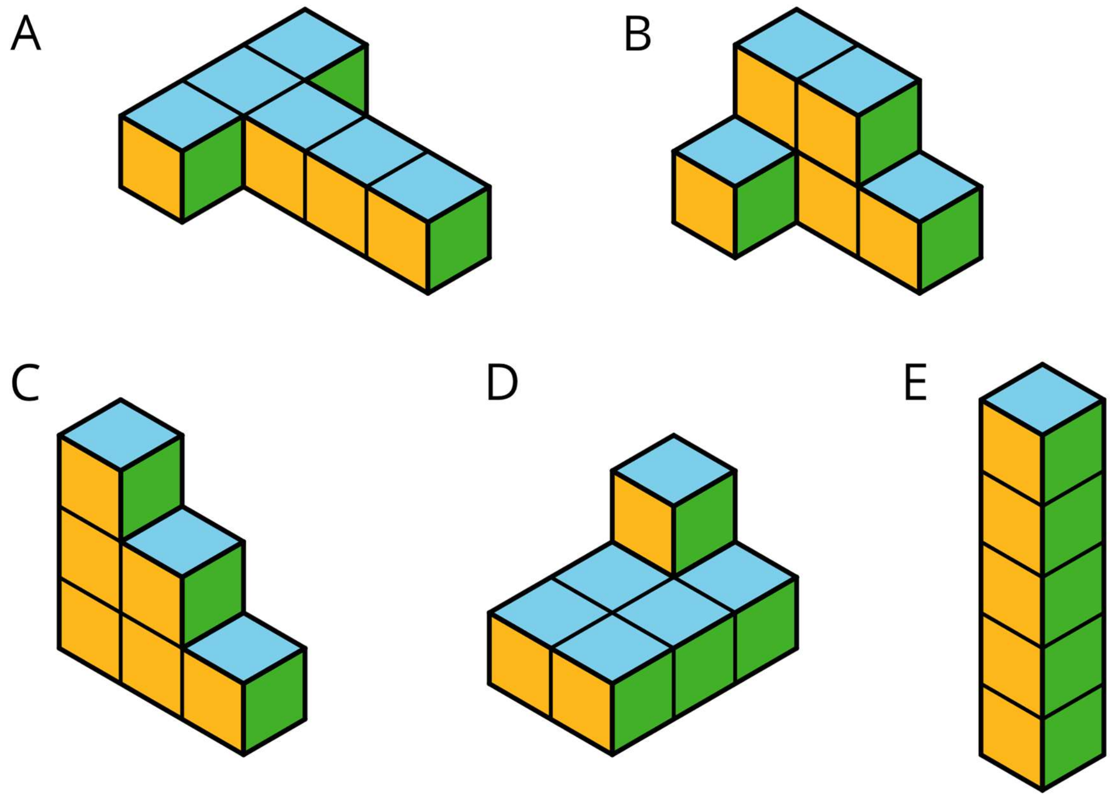
c. False. Sample reasoning: The volume will double to 8 cubic units, but the surface area will not. Only the side faces will double in area, to (4×8) or 32 square units, but the top and bottom faces will not double, so the surface area will be 34, not 36, square units.

3. **Problem 3 Statement**

Lin said, "Two shapes with the same volume also have the same surface area."

a. Which two shapes suggest that her statement is true?

b. Which two shapes could show that her statement is *not* true?



Solution

a. B and C

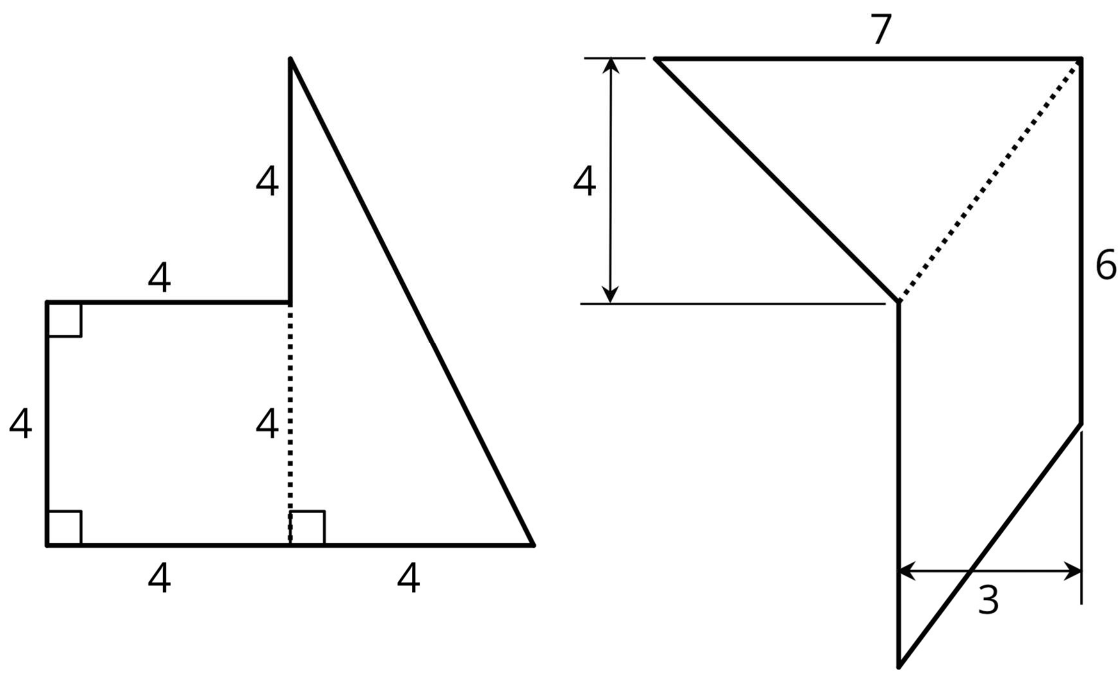
b. A and B, or A and C

4. **Problem 4 Statement**

Draw a pentagon (five-sided polygon) that has an area of 32 square units. Label all relevant sides or segments with their measurements, and show that the area is 32 square units.

Solution

Answers vary. Sample responses:

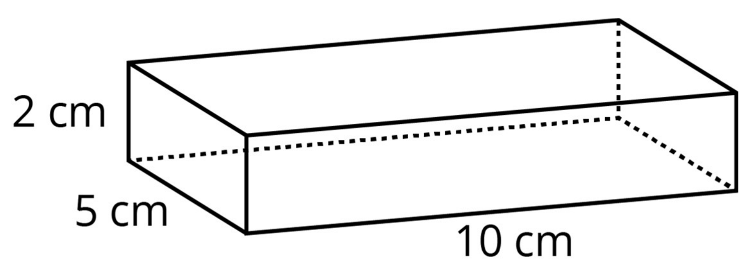


– The first pentagon is composed of a square and a right-angled triangle. The square has an area of 16 square units. The triangle has a base of 4 and a height of 8, so its area is 16 square units. The combined area is $16 + 16$ or 32 square units.

– The second pentagon is composed of a parallelogram with a base of 6 and a height of 3, and a triangle with a base of 7 and a height of 4. The area of the parallelogram is 6×3 or 18 square units. The area of the triangle is $\frac{1}{2} \times 7 \times 4$ or 14 square units. The combined area is $18 + 14$ or 32 square units.

5. **Problem 5 Statement**

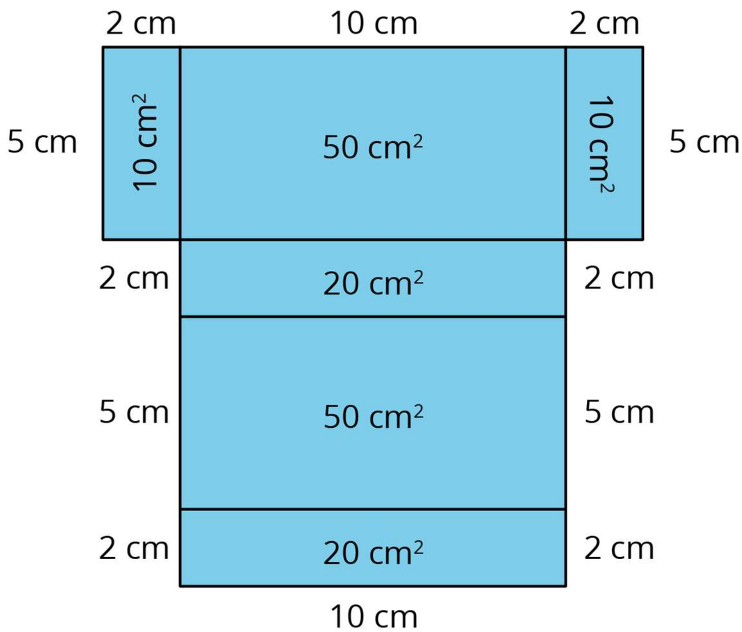
a. Draw a net for this cuboid.



b. Find the surface area of the cuboid.

Solution

a. Diagrams vary. Here is a sample net.



b. 160 square units. (There are two faces with an area of 50 square cm, two faces with an area of 20 square cm, and two faces with an area of 10 square cm.)