Lesson 16: Finding cone dimensions

Goals

- Calculate the value of one dimension of a cylinder, and explain (orally and in writing) the reasoning.
- Compare volumes of a cone and cylinder in context, and justify (orally) which volume is a better value for a given price.
- Create a table of dimensions of cylinders, and describe (orally) patterns that arise.

Learning Targets

- I can find missing information of about a cone if I know its volume and some other information.

Lesson Narrative

As they did with cylinders in a previous lesson, students in this lesson use the formula $V = \frac{1}{3}\pi r^2 h$ to find the radius or height of a cone given its volume and the other dimension. Then they apply their understanding about the volumes of cylinders and cones to decide which popcorn container and price offers the best deal. Depending on the amount of guidance students are given, this last activity can be an opportunity to explain their reasoning and critique the reasoning of others.

Building On

- Understand solving an equation or inequality as a process of answering a question: which values from a specified set, if any, make the equation or inequality true? Use substitution to determine whether a given number in a specified set makes an equation or inequality true.

Addressing

- Know the formulae for the volumes of cones, cylinders, and spheres and use them to solve real-world and mathematical problems.

Instructional Routines

- Co-Craft Questions
- Discussion Supports
- Number Talk
- Think Pair Share

Student Learning Goals

Let’s figure out the dimensions of cones.

16.1 Number Talk: Thirds

Warm Up: 5 minutes

The purpose of this number talk is to elicit understandings and review strategies students have for finding the unknown value in an equation that involves the fraction $\frac{1}{3}$. These understandings will be helpful later in this lesson when students are solving for the unknown length of the radius or height of a cone given its volume.

While four problems are given, it may not be possible to share every strategy. Consider gathering only two or three different strategies per problem, saving most of the time for the final question.

Instructional Routines

- Discussion Supports
- Number Talk

Launch

Reveal one problem at a time. Give students brief quiet think time for each problem, and ask them to give a signal when they have an answer and a strategy. Keep all previous problems displayed throughout the talk. Follow with a whole-class discussion.

Representation: Internalise Comprehension. To support working memory, provide students with sticky notes or mini whiteboards.
Supports accessibility for: Memory; Organisation

Anticipated Misconceptions

Students might want to divide 27 by 3 or multiply 27 by $\frac{1}{3}$ to solve for h. Ask students what number you can multiply by $\frac{1}{3}$ (or divide by 3) to result in the number 27.

Student Task Statement

For each equation, decide what value, if any, would make it true.

$$27 = \frac{1}{3}h$$

$$27 = \frac{1}{3}r^2$$

$$12\pi = \frac{1}{3}\pi a$$

$$12\pi = \frac{1}{3}\pi b^2$$

Student Response

- 81
- 9
- 36
- 6

Activity Synthesis

Ask students to share their strategies for each problem, in particular highlighting the ways students worked with the fraction $\frac{1}{3}$. Record and display their explanations for all to see. To involve more students in the conversation, consider asking:

- “Who can restate ___’s reasoning in a different way?”
- “Did anyone have the same strategy but would explain it differently?”
- “Did anyone solve the problem in a different way?”
- “Does anyone want to add on to _____’s strategy?”
- “Do you agree or disagree? Why?”

Speaking: Discussion Supports.: Display sentence frames to support students when they explain their strategy. For example, "First, I _____ because . . ." or "I noticed _____ so I . . . ." Some students may benefit from the opportunity to rehearse what they will say with a partner before they share with the whole class.
Design Principle(s): Optimise output (for explanation)

16.2 An Unknown Radius

Optional: 5 minutes

The purpose of this activity is for students to calculate the radius of the cone given the volume and height. This activity is similar to an activity in a previous lesson where students calculated the radius of a cylinder given the volume and height. A difference here is that solving for the unknown takes an additional step to deal with the $\frac{1}{3}$.

Encourage students to connect the strategies that they used in the warm-up to this problem. Identify students who make the connections and ask them to share during the discussion.

Instructional Routines

- Discussion Supports

Launch

Arrange students in groups of 2. Give students 2–3 minutes of quiet work time followed by time to share their strategies with their partner. Follow with a whole-class discussion.

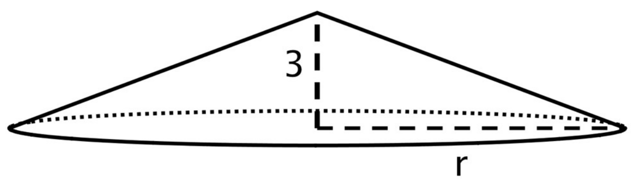
Speaking: Discussion Supports. Display sentence frames for students to use when they share their strategies with their partners. For example, "The radius is ______ because _____.", or "To find the radius, first I _____. Then, I_____." or "I used the $\frac{1}{3}$ by _____." Ask students to further clarify their strategy with operational words, such as "multiply by the reciprocal," or "divide by." This will help students practise and develop language for describing their strategies for working with fractions. *Design Principle(s): Support sense-making*

Anticipated Misconceptions

Students might struggle with the $\frac{1}{3}$ while solving for the unknown radius length. Encourage students to think about the strategies they used in the warm-up to work with the $\frac{1}{3}$.

Student Task Statement

The volume V of a cone with radius r is given by the formula $V = \frac{1}{3}\pi r^2 h$.



The volume of this cone with height 3 units and radius r is $V = 64\pi$ cubic units. This statement is true:

$64\pi = \frac{1}{3}\pi r^2 \times 3$ What does the radius of this cone have to be? Explain how you know.

Student Response

The radius must be 8 units because the equation simplifies to $64 = r^2$, so r must be 8.

Activity Synthesis

Select previously identified students to share the strategies they used to calculate the radius. Ask students, "Which do you think is more challenging: calculating the radius of a cone or the radius of the cylinder when the height and volume of the shapes are known? Why?" (I think they are the same, you just have to work with $\frac{1}{3}$ when calculating the radius of the cone, which adds an extra step.)

16.3 Cones with Unknown Dimensions

15 minutes

The purpose of this activity is for students to use the structure of the volume formula for cones to calculate missing dimensions of a cone given other dimensions. Students are given the image of a generic cone with marked dimensions for the radius, diameter, and height to help their reasoning about the different rows in the table.

While completing the table, students work with approximations and exact values of π as well as statements that require reasoning about squared values. The final row of the table asks students to find missing dimensions given an expression representing volume that uses letters to represent the height and the radius. This requires students to manipulate expressions consisting only of variables representing dimensions.

Encourage students to make use of work done in some rows to help find missing information in other rows. Identify students who use this strategy, and ask them to share during the discussion.

Instructional Routines

- Discussion Supports

Launch

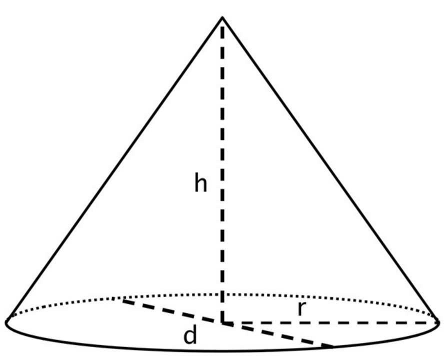
Give students 6–8 minutes of quiet work time, followed by a whole-class discussion.

Action and Expression: Internalise Executive Functions. Chunk this task into more manageable parts to support students who benefit from support with organisation and problem solving. For example ask students to complete two lines of the table at a time and then assess for accuracy and comprehension.
Supports accessibility for: Organisation; Attention

Anticipated Misconceptions

Students might try to quickly fill in the missing dimensions without the proper calculations. Encourage students to use the volume of a cone equation and the given dimensions to figure out the unknown dimensions.

Student Task Statement



Each row of the table has some information about a particular cone. Complete the table with the missing dimensions.

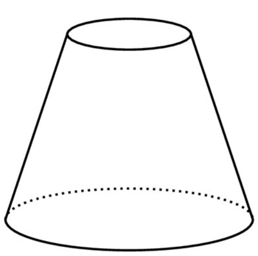
diameter (units)	radius (units)	area of the base (square units)	height (units)	volume of cone (cubic units)
	4		3	
	$\frac{1}{3}$		6	
		144π	$\frac{1}{4}$	
20				200π
			12	64π
			3	3.14

Student Response

diameter (units)	radius (units)	area of the base (square units)	height (units)	volume of cone (cubic units)
8	4	16π	3	16π
$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{9}\pi$	6	$\frac{2}{9}\pi$
24	12	144π	$\frac{1}{4}$	12π
20	10	100π	6	200π
8	4	16π	12	64π
2	1	π	3	3.14

Are You Ready for More?

A *frustum* is the result of taking a cone and slicing off a smaller cone using a cut parallel to the base.



Find a formula for the volume of a frustum, including deciding which quantities you are going to include in your formula.

Student Response

Answers vary. Sample response:

Imagine the original cone before the top piece is cut off. Then we will let R be the larger radius of the frustum, r be the smaller radius of the frustum, H be the height of the original cone, and h be the height of the conical piece cut off. Then the formula for the frustum is the volume of a cone with radius R and height H minus the volume of the removed top, which has radius r and height h: $V = \frac{1}{3}(\pi R^2 H - \pi r^2 h)$

Alternatively, let x be the height of the piece cut off, and h be the height of the frustum. Then we can calculate volume using $V = \frac{1}{3}\pi(R^2(h+x) - r^2 x)$

Activity Synthesis

Select previously identified students to share the strategies they used to fill in the missing information. Ask students:

- "Which information, in your opinion, was the hardest to calculate?"
- "If you had to pick two pieces of information given in the table which information would you want? Why?"

Select a few rows of the table, and ask students how they might find the volume of a cylinder with the same radius and height as the cone. (Multiply by 3)

Make sure students understand that when working with the volume formula for either a cylinder or cone, if they know two out of three for radius, height and volume, they can always calculate the third.

Speaking: Discussion Supports. Use this routine to support whole-class discussion. For each strategy that is shared, ask students to restate and/or revoice what they heard using mathematical language. Consider providing students time to restate what they hear to a partner, before selecting one or two students to share with the class. Ask the original speaker if their peer was accurately able to restate their thinking. Call students' attention to any words or phrases that helped to clarify the original statement. This will help students to produce and make sense of the language needed to communicate their own ideas when calculating missing dimensions of a cone.
Design Principle(s): Support sense-making; Optimise output (for explanation)

16.4 Popcorn Deals

10 minutes

The purpose of this activity is for students to reason about the volume of popcorn that a cone- and cylinder-shaped popcorn cup holds and the price they pay for it. Students start by picking the popcorn container they would buy (without doing any calculations) and then work with a partner to answer the question of which container is a better value. This activity is designed for the discussion to happen around a few different concepts:

- The volume is a lower estimate because there is still some popcorn coming out the top of the containers.
- The fact that one container (the cone) looks like it has more coming out of the top might sway students to think that the cone is a better deal. However, the difference in the volume amounts should support the fact that even with a taller and wider diameter the cone still holds less volume than the cylinder.
- Why would the movie theatre purposefully sell a container that has less volume for more price?

Identify groups who see these connections while they are working on the task, and ask them to share their arguments during the discussion.

Instructional Routines

- Co-Craft Questions
- Think Pair Share

Launch

Arrange students in groups of 2. Give students 1 minute of quiet think time to decide which popcorn container they would purchase. Students are not asked to use any written calculations to determine this answer. Poll the class and display the results for all to see which containers students chose. Keep this information displayed to refer back to during the discussion.

Give students 2–3 minutes of time to work with their partner to determine which container is a better value. Follow with a whole-class discussion.

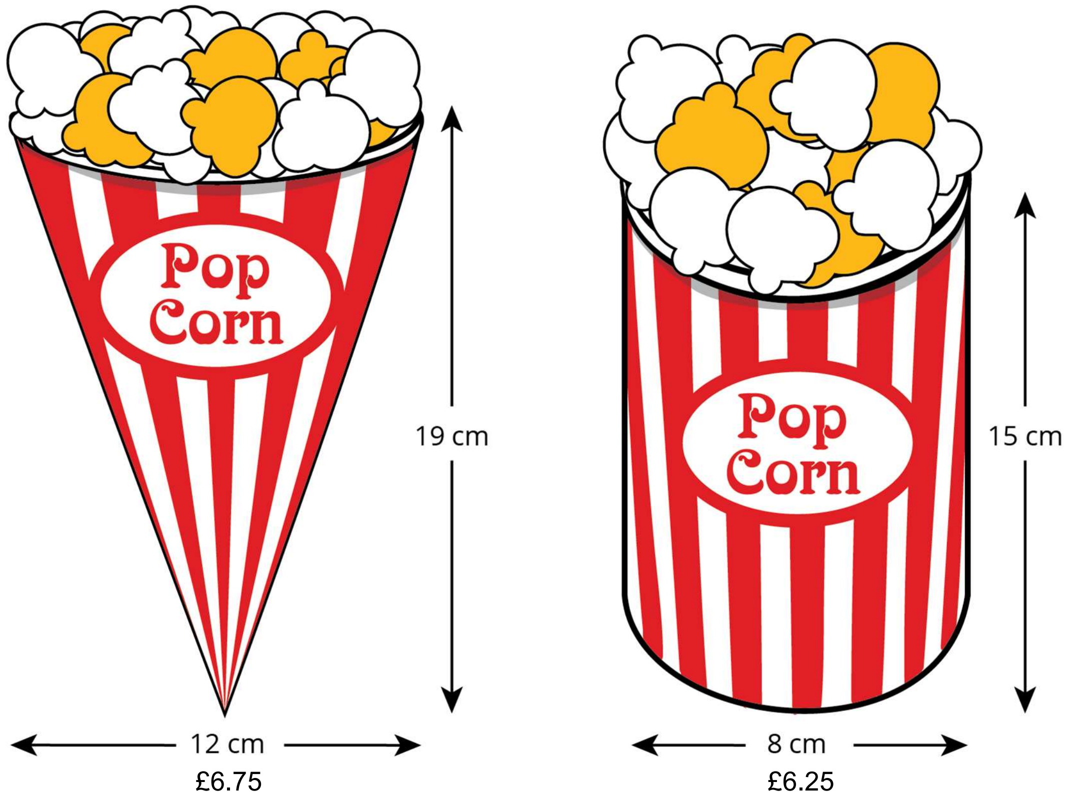
Action and Expression: Develop Expression and Communication. Invite students to talk about their ideas with a partner before writing them down. Display sentence frames to support students when they explain their ideas. For example, “First, I _____ because. . .,” “How did you get. . .?” and “Can you say more about . . .?”
Supports accessibility for: Language; Organisation Writing, Conversing: Co-Craft Questions. Display only the opening statement (i.e., “A movie theatre offers two containers.”) and the two images of the popcorn containers. Ask pairs of students to write possible questions that could be answered with mathematics. Invite 2–3 groups to share their questions with the class. Look for questions that ask students to compare the volumes of the two different containers. Next, reveal the question of the activity. This routine helps students consider the context of this problem and increases awareness about language used to talk about volume of cones and cylinders.
Design Principle(s): Maximise meta-awareness; Support sense-making

Anticipated Misconceptions

Students may try to use just the image to reason about which is a better deal. Make sure they understand that the image of the two containers is not to scale.

Student Task Statement

A movie theatre offers two containers:



Which container is the better value? Use 3.14 as an approximation for π.

Student Response

The cylinder cup is a better value. The cone's volume is about 715.92 cubic centimetres and using the cost of the conical cup is about 106.07 cubic centimetres per pound ($715.92 \div 6.75 \approx 106.07$). The cylinder's volume is about 753.6 cubic centimetres and using the cost of the cylindrical cup is about 120.58 cubic centimetres per pound ($753.6 \div 6.25 \approx 120.58$). Since the cylinder gives you more volume per pound, it is a better value.

Activity Synthesis

Poll the class again for which container they would purchase. Display the new results next to the original results for all to see during the discussion.

Select previously identified groups to share their arguments for which container has a better value. Encourage groups to share details of their calculations and record these for all to see in order to mark any similarities and differences between the groups' arguments. Consider asking some of the following questions to help students think deeper about the situation:

- "Do you think your volume calculations overestimate or underestimate the amount of popcorn each container can hold?" (I think the calculations underestimate because the popcorn piles higher than the lip of the container.)
- "Why do you think movie theatres charged more for the cone?" (It may look like it has more volume to some people since it has a larger diameter and height. It may be easier to place into a cup holder.)
- "Do you think a lot of people would buy the cone over the cylinder?" (Yes. The 50p difference is not a lot, and since it is taller and wider, people might think the cone is bigger. Although the cylinder is a better value, there may be other considerations like how easy the cup is to hold or put in the cup holder in the seat, or whether you have time in line to calculate the value.)

Lesson Synthesis

Display the image of the popcorn cone from the activity Popcorn Deals, including the dimensions of the cone. Ask students, "What size of cylinder cup would you need to have the same volume as the cone?"

Working in groups of 2, tell partners to determine the height and radius of a possible cylinder of equivalent volume, including making a sketch with labels on the dimensions. Display sketches and invite students to share the strategies they used to find their cylinders.

16.5 A Square Radius

Cool Down: 5 minutes

Launch

Provide students with access to calculators.

Student Task Statement

Noah and Lin are making paper cones to hold popcorn to hand out at parent maths night. They want the cones to hold 9π cubic inches of popcorn. What are two different possible values for height h and radius r for the cones?

Student Response

Answers vary. Sample responses:

- Height and radius both 3 inches, since $\frac{1}{3}\pi \times 3^2 \times 3 = 9\pi$.

- Radius 2 inches and height 6.75 inches, since $\frac{1}{3}\pi \times 2^2 \times 6.75 = 9\pi$.
- Radius 1 inch and height 27 inches, since $\frac{1}{3}\pi \times 1^2 \times 27 = 9\pi$.
- Radius 9 inches and height $\frac{1}{3}$ inches, since $\frac{1}{3}\pi \times 9^2 \times \frac{1}{3} = 9\pi$. (This cone may look more like a plate, but it solves the problem.)

Student Lesson Summary

As we saw with cylinders, the volume V of a cone depends on the radius r of the base and the height h:

$$V = \frac{1}{3}\pi r^2 h$$

If we know the radius and height, we can find the volume. If we know the volume and one of the dimensions (either radius or height), we can find the other dimension.

For example, imagine a cone with a volume of 64π cm^3, a height of 3 cm, and an unknown radius r. From the volume formula, we know that

$$64\pi = \frac{1}{3}\pi r^2 \times 3$$

Looking at the structure of the equation, we can see that $r^2 = 64$, so the radius must be 8 cm.

Now imagine a different cone with a volume of 18π cm^3, a radius of 3 cm, and an unknown height h. Using the formula for the volume of the cone, we know that

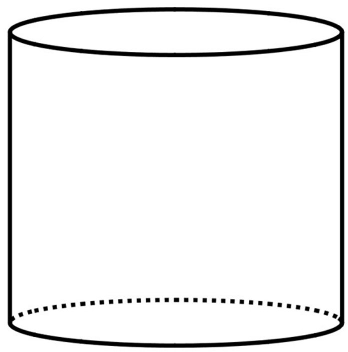
$$18\pi = \frac{1}{3}\pi 3^2 h$$

so the height must be 6 cm. Can you see why?

Lesson 16 Practice Problems

1. **Problem 1 Statement**

The volume of this cylinder is 175π cubic units.



What is the volume of a cone that has the same base area and the same height?

Solution

$\frac{175}{3}\pi$, about 183 cubic units (The volume of the cone is exactly one-third the volume of the corresponding cylinder.)

2. **Problem 2 Statement**

A cone has volume 12π cubic inches. Its height is 4 inches. What is its radius?

Solution

3 inches. (The volume of a cone is given by $V = \frac{1}{3}\pi r^2 h$. With the information given: $12\pi = \frac{1}{3}\pi r^2 \times 4$; $12 = \frac{4}{3}r^2$; $9 = r^2$ Then $r = 3$.)

3. **Problem 3 Statement**

A cone has volume 3π.

a. If the cone's radius is 1, what is its height?

b. If the cone's radius is 2, what is its height?

c. If the cone's radius is 5, what is its height?

d. If the cone's radius is $\frac{1}{2}$, what is its height?

e. If the cone's radius in r, then what is the height?

Solution

a. 9 units

b. $\frac{9}{4}$ units

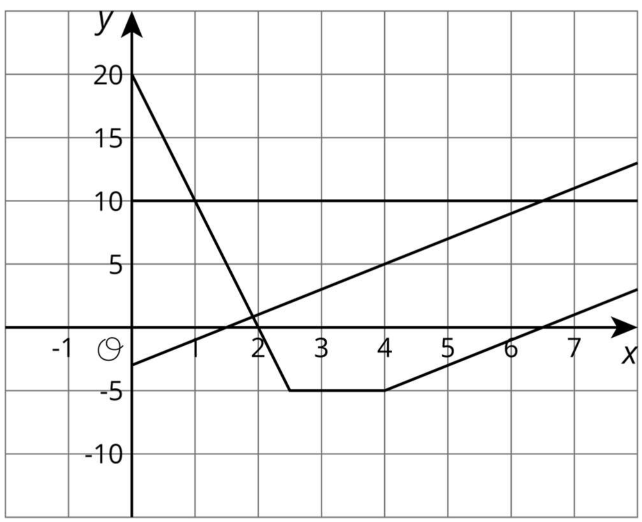
c. $\frac{9}{25}$ units

d. 36 units

e. $\frac{9}{r^2}$ units

4. **Problem 4 Statement**

Three people are playing near the water. Person A stands on the dock. Person B starts at the top of a pole and ziplines into the water, then climbs out of the water. Person C climbs out of the water and up the zipline pole. Match the people to the graphs where the horizontal axis represents time in seconds and the vertical axis represents height above the water level in feet.



Solution

A is the constant graph at 10. B is the graph that includes the point $(0,20)$. C is the graph that starts negative and increases. (Students may label the graphs A, B, and C or describe which person's story matches each graph.)

5. **Problem 5 Statement**

A room is 15 feet tall. An architect wants to include a window that is 6 feet tall. The distance between the floor and the bottom of the window is b feet. The distance between the ceiling and the top of the window is a feet. This relationship can be described by the equation $a = 15 - (b + 6)$

a. Which variable is independent based on the equation given?

b. If the architect wants b to be 3, what does this mean? What value of a would work with the given value for b?

c. The customer wants the window to have 5 feet of space above it. Is the customer describing a or b? What is the value of the other variable?

Solution

a. b

b. It means the architect wants the bottom of the window to be 3 feet above the floor. a is 6.

c. a. b is 4.