Lesson 11: Finding distances on the coordinate grid

Goals

- Calculate the distance between two points on the coordinate grid by using Pythagoras' theorem and explain (orally) the solution method.
- Generalise (orally) a method for calculating the length of a line segment on the coordinate grid using Pythagoras' theorem.

Learning Targets

- I can find the distance between two points on the coordinate grid.
- I can find the length of a diagonal line segment on the coordinate grid.

Lesson Narrative

In this lesson, students continue to apply Pythagoras' theorem to find distances between points on the coordinate grid.

Students who successfully answer the problems in the second activity use the structure of the coordinate grid to draw a right-angled triangle, an example of looking for and making use of structure on the coordinate grid.

Addressing

- Apply Pythagoras' theorem to find the distance between two points in a coordinate system.

Building Towards

- Apply Pythagoras' theorem to find the distance between two points in a coordinate system.

Instructional Routines

- Stronger and Clearer Each Time
- Clarify, Critique, Correct
- Compare and Connect
- Poll the Class
- Think Pair Share

Student Learning Goals

Let's find distances on the coordinate grid.

11.1 Closest Distance

Warm Up: 5 minutes

The purpose of this warm-up is for students to find the distance between two points on the same horizontal or vertical line on the coordinate grid. Students are given only the coordinates and no graph to encourage them to notice that to find the distance between two points on the same horizontal or vertical line, they subtract the coordinate that is not the same in both points. (This is an idea students should have encountered earlier in KS3.) This understanding will be important for students in upcoming lessons as they begin using the distance formula between two points in the plane as they apply Pythagoras' theorem.

Launch

Arrange students in groups of 2. Give students 2 minutes of quiet work time followed by 1 minute to compare their responses with a partner. Follow with a whole-class discussion.

Student Task Statement

1. Order the following pairs of coordinates from closest to farthest apart. Be prepared to explain your reasoning.
   a. (2,4) and (2,10)
   b. (-3,6) and (5,6)
   c. (-12,-12) and (-12,-1)
   d. (7,0) and (7,-9)
   e. (1,-10) and (-4,-10)
2. Name another pair of coordinates that would be closer together than the first pair on your list.
3. Name another pair of coordinates that would be farther apart than the last pair on your list.

Student Response

1. E (5), A (6), B (8), D (9), C (11)
2. Answers vary. Sample response: (2,4) and (2,8)
3. Answers vary. Sample response: (12,-10) and (-4,-10)

Activity Synthesis

Invite students to share their order of the pairs of coordinates from closest to furthest apart. Record and display the list for all to see. After the class agrees on the correct order, ask students to share the distance between a few of the pairs of coordinates and their strategy for finding that distance. Ask 2–3 students to share pairs of coordinates they found that would have a closer or further distance than the ones in the list.

If the following ideas do not arise during the discussion, consider asking the following questions:

- "Why does each pair have one coordinate that is the same?"
- "How did you decide on which coordinate to subtract?"
- "Why didn't you need to subtract the other?"
- "Could we represent this distance with a line segment? How do you know?"
- "Would your strategy work for any pair of coordinates?"
- "Which pairs would it work for? Which pairs are we not sure if it would work for?"

11.2 How Far Apart?

10 minutes

In this activity, students find distances between points on the coordinate grid. The three points are the vertices of a right-angled triangle, helping students to see that they can find the distance between two points that are not on the same vertical or horizontal line by creating a right-angled triangle.

Instructional Routines

- Clarify, Critique, Correct
- Think Pair Share

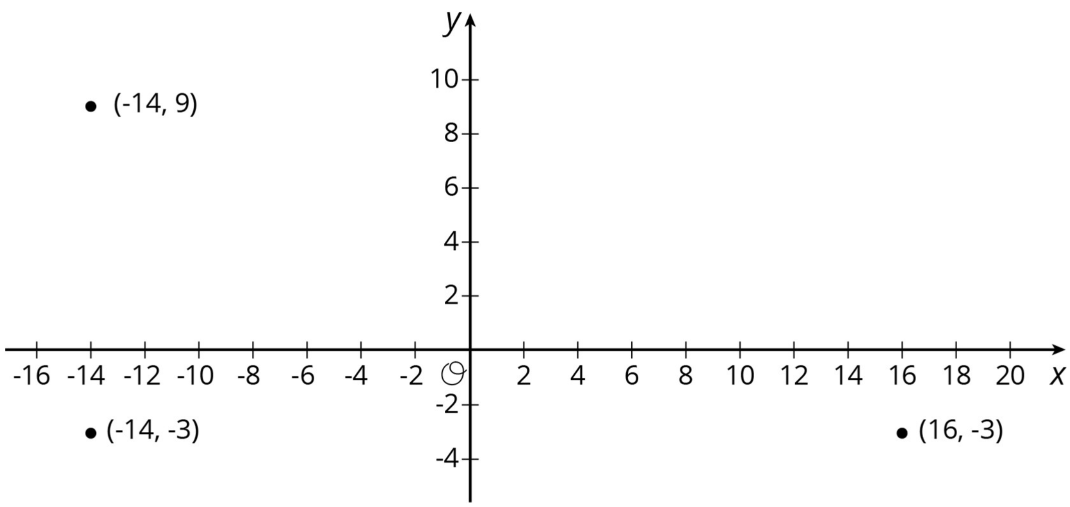
Launch

Arrange students in groups of 2. Give students 2 minutes of quiet work time followed by partner and whole-class discussions.

Engagement: Internalise Self-Regulation. Chunk this task into more manageable parts to differentiate the degree of difficulty or complexity. For example, consider beginning with a graph showing the grid lines and only 2 points. After students have successfully found the distance between the 2 points, add the other point.
Supports accessibility for: Organisation; Attention

Student Task Statement

Find the distances between the three points shown.



Student Response

12, 30, $\sqrt{1044}$

Activity Synthesis

Invite groups to share their solutions. Then draw two points on the coordinate grid, for example (-3,-4) and (2,7). Ask students how we can use the problem we just solved to find the distance between these two points. If no students suggest drawing the point (2,-4), draw it and ask how we can use it to find the distance between the first two points.

Reading, Writing, Speaking: Clarify, Critique, Correct. Before students share their solutions, display an incorrect solution based on a common error you observe for finding the distance between points with negative coordinates. For example, "The distance between (-14,9) and (-14,-3) is 6, because $9 - 3 = 6$. The distance between (-14,-3) and (16,-3) is 2, because $16 - 14 = 2$." Ask students to identify the error, critique the reasoning, and revise the original statement. As students discuss in partners, listen for students who reference the points on the coordinate grid to show that the distances are incorrect. Amplify the language students use to clarify that the horizontal distance between two points is the difference between the x-values and the vertical distance between two points is the difference between the y-values. This routine will engage students in meta-awareness as they critique and correct a common error when finding the distance between points.
Design Principles(s): Cultivate conversation; Maximise meta-awareness

11.3 Perimeters with Pythagoras

Optional: 15 minutes

In this optional activity, students calculate the perimeters of two triangles on the coordinate grid using Pythagoras' theorem. Use this activity if time allows and your students need practice calculating the distance between two points in a coordinate grid.

Instructional Routines

- Stronger and Clearer Each Time
- Poll the Class
- Think Pair Share

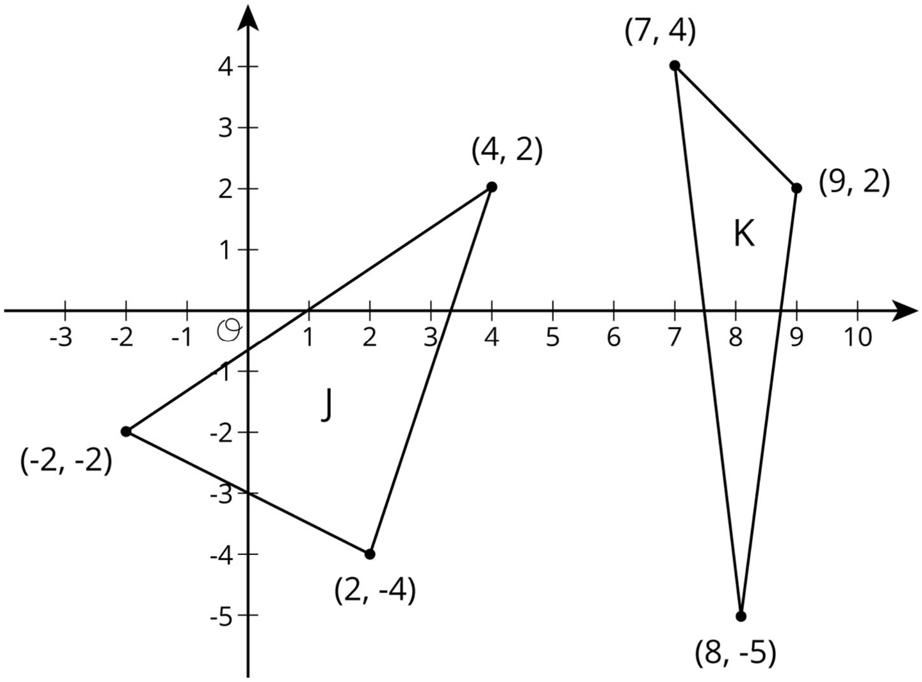
Launch

Arrange students in groups of 2. Display the image of the two shapes for all to see and then poll the class and ask which shape they think has the longer perimeter. Display the results of the poll for all to see during the activity. Tell the class that each partner will now calculate the perimeter of one of the shapes.

Representation: Internalise Comprehension. Activate or supply background knowledge. Allow students to use calculators to ensure inclusive participation in the activity. *Supports accessibility for: Memory; Conceptual processing Writing, Speaking, Listening: Stronger and Clearer Each Time.* Use this routine to help students consider audience when preparing to share their work. Ask students to prepare a visual display that shows how they calculated the perimeter of their triangle. Students should consider how to display their calculations so that another student can interpret them. Some students may wish to add notes or details to their drawings to help communicate their thinking. Ask each student to meet with 2–3 other partners for feedback. Display prompts for feedback that will help students strengthen their ideas and clarify their language. For example, "How did you find the distance between these points?", "What is the perimeter of a triangle?", and "How did you find the perimeter of the triangle?" Students can borrow ideas and language from each partner to refine and clarify their original explanation. This will help students refine both their ideas, and their verbal and written output.
Design Principles(s): Optimise output (for explanation); Maximise meta-awareness

Student Task Statement

1. Which shape do you think has the longer perimeter?
2. Select one shape and calculate its perimeter. Your partner will calculate the perimeter of the other. Were you correct about which shape had the longer perimeter?



Student Response

1. Answers vary.

2. Shape J has a perimeter of approximately 18 units. The perimeter is the sum $\sqrt{2^2+6^2}+\sqrt{2^2+4^2}+\sqrt{4^2+6^2}=\sqrt{40}+\sqrt{20}+\sqrt{52}\approx 18$. Shape K has a perimeter of approximately 19 units. The perimeter is the sum $\sqrt{9^2+1^2}+\sqrt{7^2+1^2}+\sqrt{2^2+2^2}=\sqrt{82}+\sqrt{50}+\sqrt{8}\approx 19$.

Are You Ready for More?

Quadrilateral $ABCD$ has vertices at $A=(-5,1)$, $B=(-4,4)$, $C=(2,2)$, and $D=(1,-1)$.

1. Use Pythagoras' theorem to find the lengths of sides AB, BC, CD, and AD.

2. Use Pythagoras' theorem to find the lengths of the two diagonals, AC and BD.

3. Explain why quadrilateral $ABCD$ is a rectangle.

Student Response

1. Use Pythagoras' theorem to find the length of each segment. Segment AB has length $\sqrt{10}$ because $AB^2=1^2+3^2$. Segment CD also has length $\sqrt{10}$ because the right-angled triangle used to find AB is congruent to the right-angled triangle used to find CD. The length of AD is $\sqrt{40}$ because $AD^2=6^2+2^2$. The triangle used to calculate BC is congruent to the one used to calculate AD, so the length of BC is also $\sqrt{40}$.

2. The length of AC is $\sqrt{50}$ because $AC^2=7^2+1^2$. The length of BD is also $\sqrt{50}$ because $BD^2=5^2+5^2$.

3. The shape $ABCD$ is a rectangle because it has four right angles. For example, the angle at A is a right angle by the converse of Pythagoras' theorem, since we have that $AD^2 + AB^2 = BD^2$.

Activity Synthesis

Repeat the poll from the start of the activity (perimeter of the triangles) to see how results have changed. Select 1–2 students to share their calculations.

11.4 Finding the Right Distance

15 minutes

The purpose of this task is for students to think about a general method for finding the distance between two points on the coordinate grid. Students do not need to formalise this into a more traditional representation of the distance formula. In groups of 4, each student will find the distance between two coordinate pairs and then share how they completed their calculations. The coordinates are carefully chosen so that the distances are all equal since each pair represents a possible diameter for a circle centred at (2,-2) with radius 5, though students do not need to know this in order to complete their calculations.

Identify students who clearly explain their thinking as they work with their group. Notice any groups that discover the points are all on the perimeter of a circle.

Instructional Routines

- Compare and Connect

Launch

Arrange students in groups of 4. Tell students that once everyone in their group has calculated the distance between their points they will share how they did their calculations and then answer the problems. Encourage students to listen carefully to the ideas of other members of their group in order to write a clear explanation for the second question.

Engagement: Develop Effort and Persistence. Encourage and support opportunities for peer interactions. Invite students to talk about their ideas with a partner before writing them down. Display sentence frames to support students when they explain their strategy. For example, "I noticed _____ so I . . .", "By comparing triangles I . . .", or "Another strategy could be _____ because . . . ."
Supports accessibility for: Language; Social-emotional skills

Student Task Statement

Have each person in your group select one of the sets of coordinate pairs shown here. Then calculate the length of the line segment between those two coordinates. Once the values are calculated, have each person in the group briefly share how they did their calculations.

- $(6, -5)$ and $(-2, 1)$

- (-1,-6) and (5,2)
- (-1,2) and (5,-6)
- (-2,-5) and (6,1)

1. How does the value you found compare to the rest of your group?
2. In your own words, write an explanation to another student for how to find the distance between any two coordinate pairs.

Student Response

1. All lengths are 10 units.
2. Answers vary. Sample response: For the coordinate pairs, (-2,1) and (6,-5), a right-angled triangle can be drawn with the coordinate pairs as vertices. The shorter sides of this triangle are 6 and 8. This means the distance between the coordinate pairs is given by c in the equation $6^2 + 8^2 = c^2$. Then $c = 10$ since $c^2 = 36 + 64 = 100$.

Activity Synthesis

The purpose of this discussion is for students to compare their methods for finding the distance between two points. Select 2–3 previously identified students to share how they found the distance between their points.

If any groups figured out that the points lie on a circle, ask them to share how they did so. Then, ask students to find the distance between one (or both) of their points and the point (2,-2) using the method they described in the second problem. If students calculations are correct, they should get a distance of 5 units, which is the radius of the circle.

Speaking, Listening: Compare and Connect. Ask students to prepare a visual display of their work along with a written explanation for how to find the distance between any two coordinate pairs. As students investigate each other's work, ask them to share what worked well in a particular approach. Listen for and amplify the language students use to explain how they used Pythagoras' theorem to calculate distances. Then encourage students to explain how the terms a and b in Pythagoras' theorem are related to the coordinate pairs. For example, a could represent the vertical distance between the coordinate pairs and b could represent the horizontal distance between the coordinate pairs. This will foster students' meta-awareness and support constructive conversations as they compare strategies for finding the distance between coordinate pairs and make connections between the terms a and b in Pythagoras' theorem and the coordinate pairs.
Design Principles(s): Cultivate conversation; Maximise meta-awareness

Lesson Synthesis

The purpose of the discussion is to check student understanding of how to use Pythagoras' theorem to calculate distances between points on the coordinate grid. Ask:

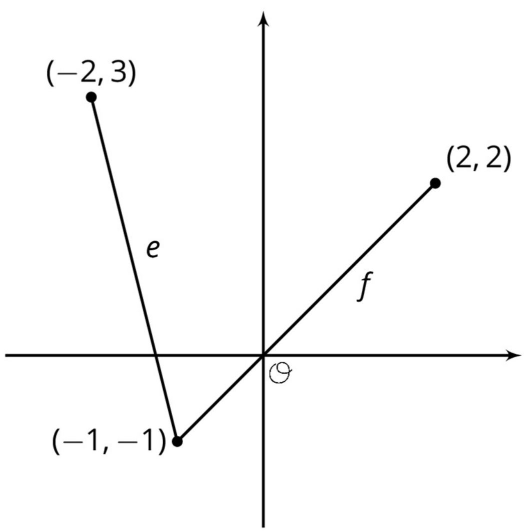
- "How can you find the distance between points on the coordinate grid?" (If they are on the same horizontal or vertical line, we just subtract the coordinates that are different. If they aren't, we can construct a right-angled triangle and use Pythagoras' theorem.)
- "What advice would you give someone finding the distance between two points on the coordinate grid using Pythagoras' theorem?" (Make a sketch!)

11.5 Lengths of Line Segments

Cool Down: 5 minutes

Student Task Statement

Here are two line segments with lengths e and f. Calculate the exact values of e and f. Which is larger?

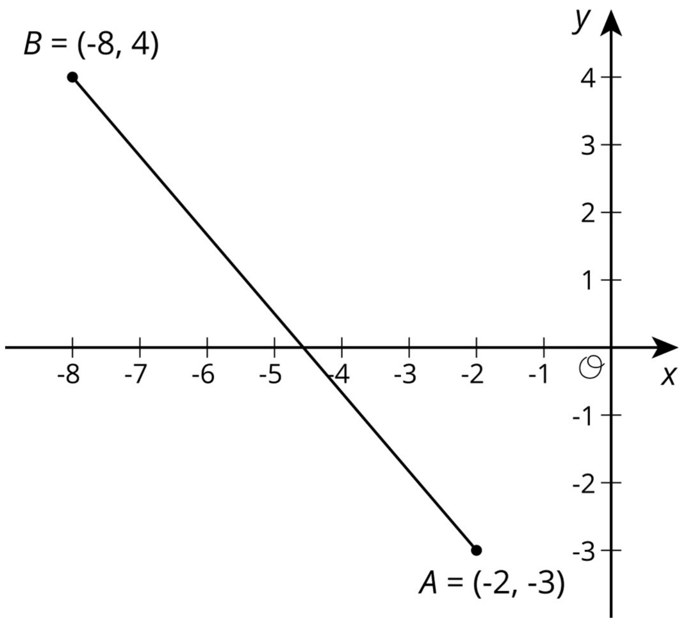


Student Response

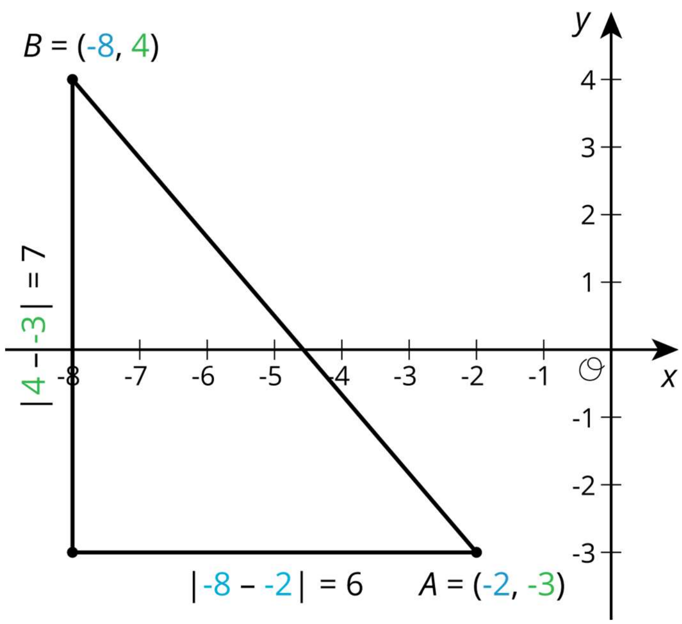
The length of e is $\sqrt{17}$ units, and the length of f is $\sqrt{18}$ units. $e = \sqrt{1^2 + 4^2} = \sqrt{1 + 16} = \sqrt{17}$. $f = \sqrt{3^2 + 3^2} = \sqrt{9 + 9} = \sqrt{18}$. Line segment f is longer.

Student Lesson Summary

We can use Pythagoras' theorem to find the distance between any two points on the coordinate grid. For example, if the coordinates of point A are (-2,-3), and the coordinates of point B are (-8,4), let's find the distance between them. This distance is also the length of line segment AB. It is a good idea to plot the points first.



Think of the distance between A and B, or the length of segment AB, as the hypotenuse of a right-angled triangle. The lengths of the shorter sides can be deduced from the coordinates of the points.



The length of the horizontal shorter side is 6, which can be seen in the diagram, but it is also the distance between the x-coordinates of A and B since $|\text{-}8 - \text{-}2| = 6$. The length of the vertical shorter side is 7, which can be seen in the diagram, but this is also the distance between the y-coordinates of A and B since $|4 - \text{-}3| = 7$.

Once the lengths of the shorter sides are known, we use Pythagoras' theorem to find the length of the hypotenuse, AB, which we can represent with c. Since c is a positive number, there is only one value it can take:

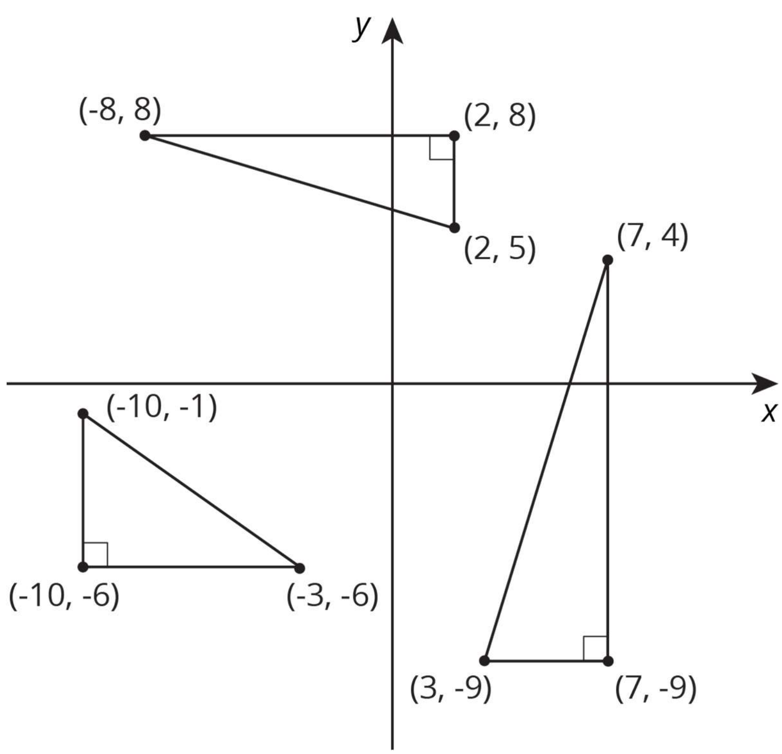
$$\begin{aligned} 6^2 + 7^2 &= c^2 \\ 36 + 49 &= c^2 \\ 85 &= c^2 \\ \sqrt{85} &= c \end{aligned}$$

This length is a little longer than 9, since 85 is a little longer than 81. Using a calculator gives a more precise answer, $\sqrt{85} \approx 9.22$.

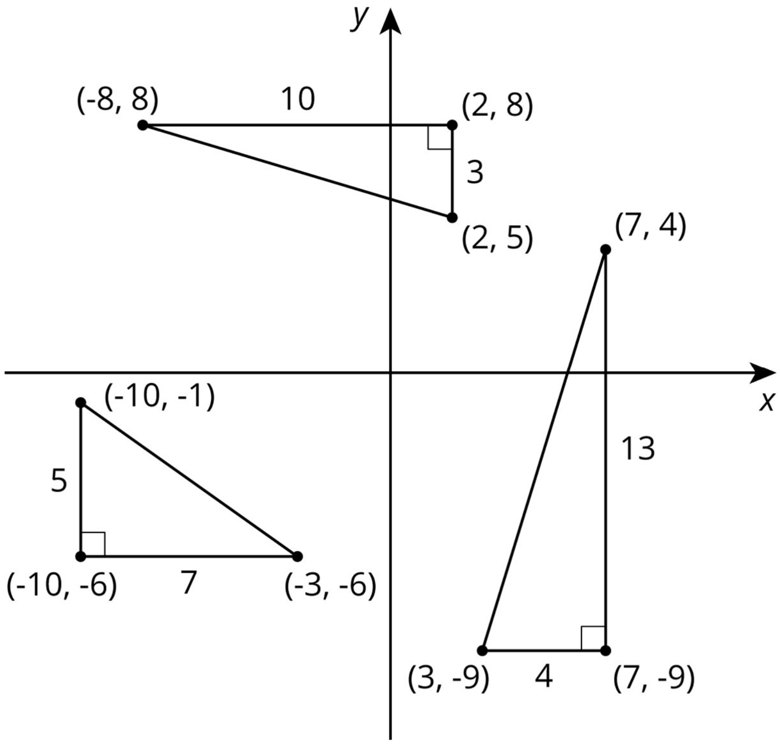
Lesson 11 Practice Problems

1. **Problem 1 Statement**

The right-angled triangles are drawn on the coordinate grid, and the coordinates of their vertices are labelled. For each right-angled triangle, label each shorter side with its length.



Solution



2. **Problem 2 Statement**

   Find the distance between each pair of points. If you get stuck, try plotting the points on graph paper.

   a. $M = (0,\text{-}11)$ and $P = (0,2)$

   b. $A = (0,0)$ and $B = (\text{-}3,\text{-}4)$

   c. $C = (8,0)$ and $D = (0,\text{-}6)$

   Solution

   a. 13

   b. 5

   c. 10

3. **Problem 3 Statement**

   a. Find an object that contains a right angle. This can be something in nature or something that was made by humans or machines.

   b. Measure the two sides that make the right angle. Then measure the distance from the end of one side to the end of the other.

   c. Draw a diagram of the object, including the measurements.

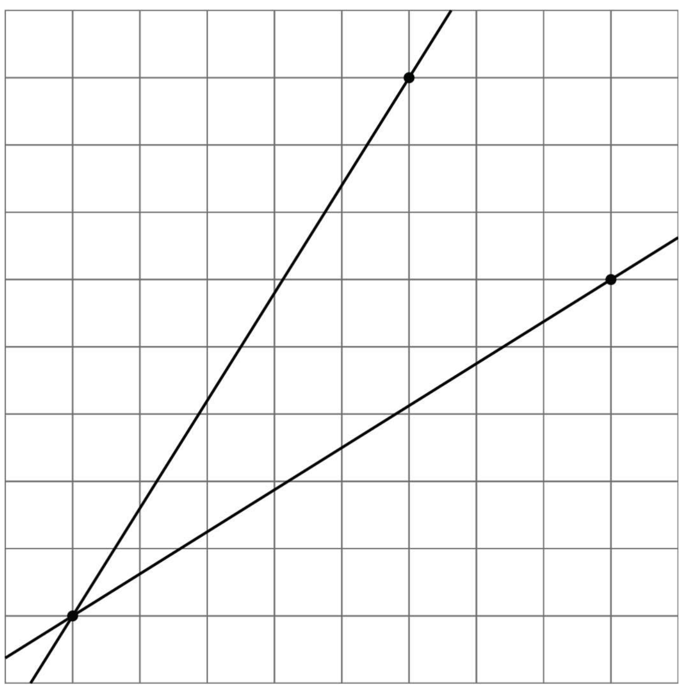
d. Use Pythagoras' theorem to show that your object really does have a right angle.

Solution

Answers vary. A correct response will include a labelled diagram and the three measurements inserted into $a^2 + b^2 = c^2$ with enough work to show that the three measurements make this equation true. (Or close-enough, accounting for measurement error.)

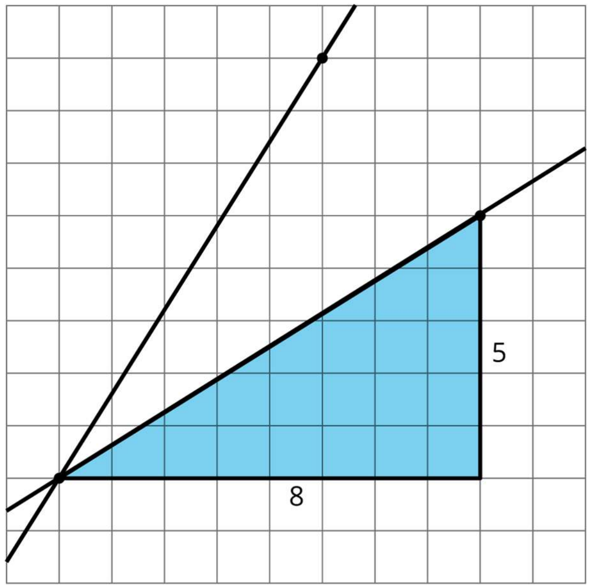
4. **Problem 4 Statement**

Which line has a gradient of 0.625, and which line has a gradient of 1.6? Explain why the gradients of these lines are 0.625 and 1.6.

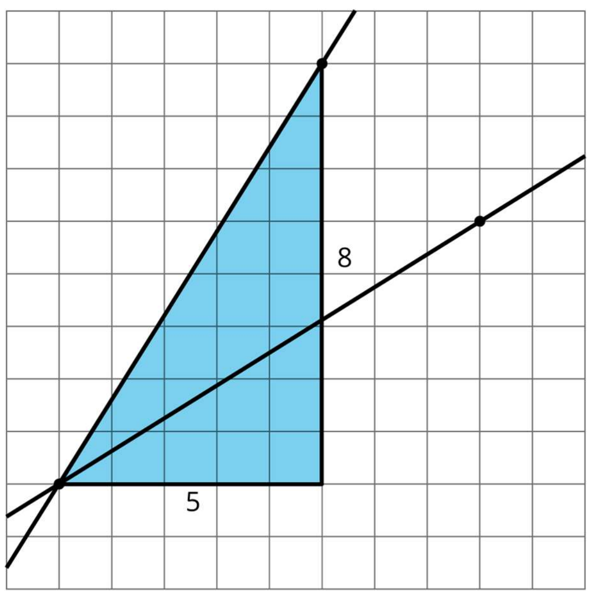


Solution

Gradient of 0.625:



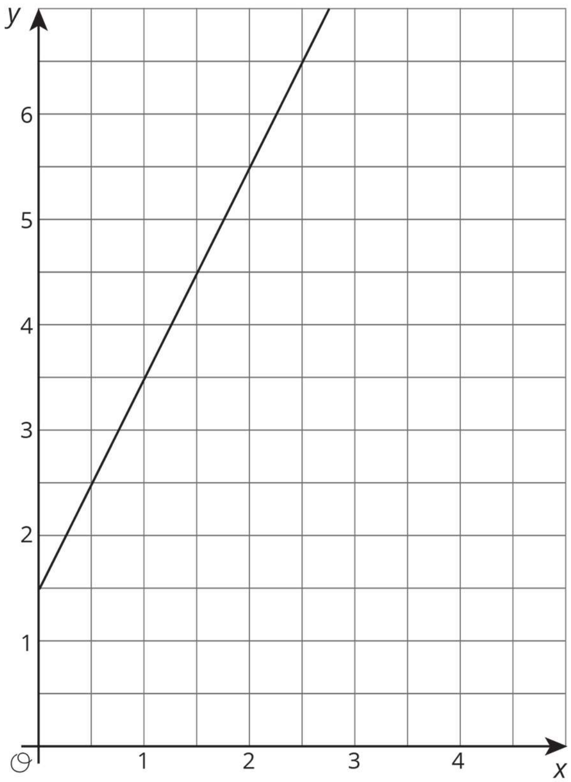
Gradient of 1.6:



Construct triangles perpendicular to the axes whose hypotenuses lie on their line to find the gradients. The gradients of the lines are then the quotient of the length of the vertical edge by the length of the horizontal edge.

5. **Problem 5 Statement**

Write an equation for the graph.



Solution

$y = 2x + 1.5$