# Lesson 2: Ratios and rates with fractions

## Goals

- Compare and contrast (orally and in writing) different strategies for solving a problem involving equivalent ratios with fractional quantities.
- Explain (orally and in writing) how to find and use a unit rate to solve a problem involving fractional quantities.

## Learning Targets

- I can solve problems about ratios of fractions and decimals.

## Lesson Narrative

Before this unit, students worked with ratios of whole numbers and with whole number percentages. Now they start to work with ratios of fractions and fractional percentages. In this lesson they encounter situations where a ratio of fractions arises naturally. They calculate scale factors and unit rates associated with ratios of fractions. They consider a situation involving a ratio where the second number is 100, in order to prepare for thinking about a percentage as a particular type of rate, and they compare rates associated with different ratios. The representations they use—bar models and double number lines—are the same as they have used previously, but in the context of more complicated ratios.

The Mona Lisa task has more than one reasonable answer, and students must make sense of the situation in order to choose one.

### Building On

- Interpret and calculate quotients of fractions, and solve word problems involving division of fractions by fractions, e.g., by using visual fraction models and equations to represent the problem. For example, create a story context for $\left(\frac{2}{3}\right) \div \left(\frac{3}{4}\right)$ and use a visual fraction model to show the quotient; use the relationship between multiplication and division to explain that $\left(\frac{2}{3}\right) \div \left(\frac{3}{4}\right) = \frac{8}{9}$ because $\frac{3}{4}$ of $\frac{8}{9}$ is $\frac{2}{3}$. (In general, $\left(\frac{a}{b}\right) \div \left(\frac{c}{d}\right) = \frac{ad}{bc}$.) How much chocolate will each person get if 3 people share $\frac{1}{2}$ lb of chocolate equally? How many $\frac{3}{4}$ cup servings are in $\frac{2}{3}$ of a cup of yogurt? How wide is a rectangular strip of land with length $\frac{3}{4}$ mi and area $\frac{1}{2}$ square mi?
- Use ratio and rate reasoning to solve real-world and mathematical problems, e.g., by reasoning about tables of equivalent ratios, bar models, double number line diagrams, or equations.

- Solve problems involving scale drawings of geometric figures, including calculating actual lengths and areas from a scale drawing and reproducing a scale drawing at a different scale.

### Addressing

- Calculate unit rates associated with ratios of fractions, including ratios of lengths, areas and other quantities measured in like or different units. For example, if a person walks $\frac{1}{2}$ mile in each $\frac{1}{4}$ hour, calculate the unit rate as the complex fraction $\frac{\frac{1}{2}}{\frac{1}{4}}$ miles per hour, equivalently 2 miles per hour.

### Building Towards

- Analyse proportional relationships and use them to solve real-world and mathematical problems.
- Calculate unit rates associated with ratios of fractions, including ratios of lengths, areas and other quantities measured in like or different units. For example, if a person walks $\frac{1}{2}$ mile in each $\frac{1}{4}$ hour, calculate the unit rate as the complex fraction $\frac{\frac{1}{2}}{\frac{1}{4}}$ miles per hour, equivalently 2 miles per hour.

### Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Collect and Display
- Compare and Connect
- Discussion Supports
- Number Talk
- Think Pair Share

### Required Preparation

For the activity Scaling the Mona Lisa, consider showing a picture of the Mona Lisa painting.

### Student Learning Goals

Let's calculate some rates with fractions.

## 2.1 Number Talk: Division

**Warm Up: 5 minutes**

The purpose of this number talk is to elicit strategies and understandings students have for dividing a fraction by a fraction. Later in this lesson, students will need to be able to divide a fraction by a fraction to solve problems in contexts.

Four problems are given. It may not be possible to share every strategy. Consider gathering only two or three different strategies per problem, saving most of the time for the final question.

### Instructional Routines

- Discussion Supports
- Number Talk

### Launch

Reveal one problem at a time. Give students 30 seconds of quiet think time for each problem and ask them to give a signal when they have an answer and a strategy. Keep all previous problems displayed throughout the talk. Follow with a whole-class discussion.

*Representation: Internalise Comprehension.* To support working memory, provide students with sticky notes or mini whiteboards.
*Supports accessibility for: Memory; Organisation*

### Anticipated Misconceptions

Students may get stuck trying to remember a procedure to divide fractions or may also think the answer to each problem is just the whole number divided by 3. Help students reason about dividing a whole number by a unit fraction by asking what number if divided into thirds gives us an answer of ___ ($5, 2, \frac{1}{2}, 2\frac{1}{2}$)?

If students get stuck on the last problem, help them see that previous problems can be used to figure out an answer to this last one. Since $2\frac{1}{2}$ is half of 5 the answer is going to be half of the answer to $5 \div \frac{1}{3}$. They may also apply the distributive property to use the answer to $2 \div \frac{1}{3}$ and $\frac{1}{2} \div \frac{1}{3}$ to figure out the answer.

### Student Task Statement

Find each quotient mentally.

$5 \div \frac{1}{3}$

$2 \div \frac{1}{3}$

$\frac{1}{2} \div \frac{1}{3}$

$2\frac{1}{2} \div \frac{1}{3}$

### Student Response

- 15
- 6
- $1\frac{1}{2}$
- $7\frac{1}{2}$

(Equivalent answers are also acceptable.)

### Activity Synthesis

Ask students to share their strategies for each problem. Record and display their explanations for all to see. To involve more students in the conversation, consider asking:

- “Who can restate ___’s reasoning in a different way?”
- “Did anyone have the same strategy but would explain it differently?”
- “Did anyone solve the problem in a different way?”
- “Does anyone want to add on to _____’s strategy?”
- “Do you agree or disagree? Why?”

*Speaking: Discussion Supports.*: Display sentence frames to support students when they explain their strategy. For example, "First, I _____ because . . ." or "I noticed _____ so I . . . ." Some students may benefit from the opportunity to rehearse what they will say with a partner before they share with the whole class.
*Design Principle(s): Optimise output (for explanation)*

## 2.2 A Train is Travelling at . . .

**10 minutes**

The purpose of this activity is to review different strategies for working with ratios and to prepare students to use these strategies with ratios involving fractions. The activity also foreshadows percentages by asking about the distance travelled in 100 minutes.

Monitor for different strategies like these:

- $\frac{15}{2} \div 6$ to find the distance travelled in 1 minute, and then multiply it by 100.
- draw a double number line.

- create a table of equivalent ratios.

Depending on their prior learning, students might lean towards the first strategy.

### Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Compare and Connect

### Launch

Give students 3 minutes of quiet work time. Encourage them to find more than one strategy if they have time. Follow with whole-class discussion around the various strategies they used.

### Anticipated Misconceptions

Students might calculate the unit rate as $6 \div \frac{15}{2}$. Ask students what this number would mean in this problem? (This number means that it takes $\frac{4}{5}$ of a minute to travel 1 kilometre.) In this case, students should be encouraged to create a table or a double number line, since it will help them make sense of the meaning of the numbers.

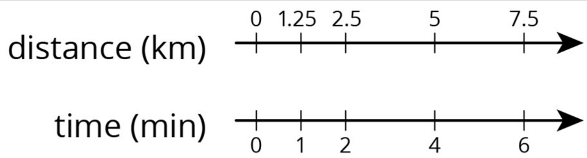
### Student Task Statement

A train is travelling at a constant speed and goes 7.5 kilometres in 6 minutes. At that rate:

1. How far does the train go in 1 minute?
2. How far does the train go in 100 minutes?


### Student Response

1. $\frac{5}{4}$ kilometres or equivalent. Possible strategies:
   - Double Number Line:




– Table:

| distance (km) | time (min) |
|---|---|
| 7.5 | 6 |
| $\frac{5}{4}$ | 1 |

– Unit Rate: $\frac{15}{2} \div \frac{6}{1} = \frac{5}{4}$ or $\frac{15}{2} \div \frac{6}{1} = 1\frac{1}{4}$

2. The train goes 125 kilometres (or equivalent) in 100 minutes. Possible strategies:

– Table:

| distance (km) | time (min) |
|---|---|
| 7.5 | 6 |
| $\frac{5}{4}$ | 1 |
| $\frac{500}{4}$ | 100 |

– Unit Rate: $\frac{5}{4} \times 100$ or $1\frac{1}{4} \times 100$

## Activity Synthesis

Select students to share the strategies they used. To the extent possible, there should be one student per strategy listed. If no students come up with one or more representations, create them so that students can compare and contrast.

- Divide ($\frac{15}{2} \div 6$) to find the number of kilometres travelled in 1 minute, and multiply by 100
- Double Number Line
- Table

Display strategies for all to see throughout the discussion.

Help students connect the strategies by asking:

- Was there a place in your solution where you calculated $\frac{15}{2} \div 6$?
- How can we see this value being used in the double number line? Table?

*Engagement: Develop Effort and Persistence.* Encourage and support opportunities for peer interactions. Prior to the whole-class discussion, invite students to share their work with a partner. Display sentence frames to support student conversation such as: "First, I _____ because...", "I noticed _____ so I...", "Why did you...?", "I agree/disagree because…."
*Supports accessibility for: Language; Social-emotional skills Speaking: Compare and Connect.* Use this routine when students present the strategies they used to determine the distance travelled. Ask students to consider what is the same and what is different about each approach. Draw students' attention to the different ways the unit rate and total distance can be seen in each representation (i.e., bar model, double number line and table). Listen for and amplify students' correct use of the term "unit rate." These exchanges can strengthen students' mathematical language use as they reason to make sense of strategies used to calculate unit rates and distance travelled.
*Design Principle(s): Maximise meta-awareness*

## 2.3 Comparing Running Speeds

**10 minutes**

The purpose of this activity is to provide another context that leads students to calculate a unit rate from a ratio of fractions. This work is based on students' work in Year 7 on dividing fractions.

Students notice and wonder about two statements and use what they wonder to create questions that are collected for all to see. Each student picks a question secretly and calculates the answer, then shares the answer with their partner. The partner tries to guess the question. Most of the time in this activity should be spent on students engaging in partner discussion.

### Instructional Routines

- Discussion Supports

### Launch

Arrange students into groups of 2. Display the two statements for all to see. Ask students to write down what they notice and wonder, and then use what they wonder to come up with questions that can be answered using the given information. Create a list of questions and display for all to see. Here are suggested questions to listen for:

- Who ran faster, Noah or Lin?
- How far would Lin run in 1 hour?
- How far did Noah run in 1 hour?
- How long would it take Lin to run 1 mile at that rate?
- How long would it take Noah to run 1 mile at that rate?

*Action and Expression: Internalise Executive Functions.* Chunk this task into more manageable parts to support students who benefit from support with organisation and problem solving. For example, present one question at a time and monitor students to ensure they are making progress throughout the activity.
*Supports accessibility for: Organisation; Attention Speaking: Discussion Supports.* Provide sentence frames to students to support their explanations of their process for finding the answer to their selected question. For example, "First, I _____. Then, I ____."
*Design Principle(s): Support sense-making, Optimise output (for explanation)*

### Anticipated Misconceptions

The warm-up was intended to remind students of some strategies for dividing fractions by fractions, but students may need additional support working with the numbers in this task.

Students might have a hard time guessing their partner's question given only the answer. Ask their partners to share the process they used to calculate the solution, they might leave out numbers and describe in general the steps they took to find the answer first. If their partner is still unable to guess the question, have them share the specific number they used. If they need additional support to guess the question, have their partner show them their work on paper (without sharing the question they answered) and see if this helps them figure out the question.

### Student Task Statement

Lin ran $2\frac{3}{4}$ miles in $\frac{2}{5}$ of an hour. Noah ran $8\frac{2}{3}$ miles in $\frac{4}{3}$ of an hour.

1. Pick one of the questions that was displayed, but don't tell anyone which question you picked. Find the answer to the question.
2. When you and your partner are both done, share the answer you got (do not share the question) and ask your partner to guess which question you answered. If your partner can't guess, explain the process you used to answer the question.
3. Switch with your partner and take a turn guessing the question that your partner answered.

### Student Response

Answers vary. Sample responses: Lin ran faster than Noah, because $2\frac{3}{4} \div \frac{2}{5} = 6\frac{7}{8}$ and $8\frac{2}{3} \div \frac{4}{3} = 6\frac{1}{2}$.

Lin ran $6\frac{7}{8}$ miles in one hour since $2\frac{3}{4} \div \frac{2}{5} = 6\frac{7}{8}$. Noah ran $6\frac{1}{2}$ miles in one hour since $8\frac{2}{3} \div \frac{4}{3} = 6\frac{1}{2}$.

It took Lin $\frac{8}{55}$ of an hour to run a mile since $\frac{2}{5} \div 2\frac{3}{4} = \frac{8}{55}$. That is $8\frac{8}{11}$ minutes since $\frac{8}{55} \times 60 = 8\frac{8}{11}$. It took Noah $\frac{2}{13}$ of an hour to run a mile since $\frac{4}{3} \div 8\frac{2}{3} = \frac{2}{13}$. That is $9\frac{3}{13}$ minutes since $\frac{2}{13} \times 60 = 9\frac{3}{13}$.

### Are You Ready for More?

Nothing can go faster than the speed of light, which is 299 792 458 metres per second. Which of these are possible?

1. Travelling a billion metres in 5 seconds.
2. Travelling a metre in 2.5 nanoseconds. (A nanosecond is a billionth of a second.)
3. Travelling a parsec in a year. (A parsec is about 3.26 light years and a light year is the distance light can travel in a year.)

### Student Response

1. Yes (200 000 000 metres per second)
2. No (400 000 000 metres per second)
3. No, because travelling 1 parsec in 1 year means travelling 3.26 times faster than the speed of light

### Activity Synthesis

After both partners have a chance to guess each other's question, ask a few different students to share their strategies for guessing which question their partner answered.

## 2.4 Scaling the Mona Lisa

**Optional: 10 minutes (there is a digital version of this activity)**

The purpose of this activity is to provide a context where a ratio of fractions arises naturally, and students need to find an equivalent ratio to solve the problem. The ratio $2\frac{1}{2} : 1\frac{3}{4}$ is equivalent to $10 : 7$, so a scaled copy of the Mona Lisa that is 10 inches by 7 inches would fit on the cover of the notebook. Other answers are possible. Some students might try to find the biggest possible copy that will fit on the cover, which would result in a different scale factor.

The digital version of the student materials includes an applet so that students can experiment with the context, because there are many related measurements within the context that can be hard to visualise. For example, the applet makes it clear that you can't simply scale down the Mona Lisa and make it perfectly fit on the notebook, since the notebook and the Mona Lisa are not scaled copies of each other. It also serves to remind students that the length and width of the Mona Lisa have to be scaled by the same factor, or the image becomes distorted.

Students discuss how they found their solution with a partner and determine if the scale factors they came up with are reasonable. Students must think about whether it makes sense to try to scale the picture so that it fills as much of the page as possible, or whether it makes more sense to leave room for a title. As students discuss with their partner, identify pairs of students who have a good argument that a certain scale factor makes more sense to use than another. Ask these students to share during the whole class discussion.

### Instructional Routines

- Collect and Display
- Think Pair Share

### Launch

If desired, show students an image of the Mona Lisa.

If using the print version, and appropriate technology is available, consider displaying the applet for everyone to see to help students better understand the situation https://ggbm.at/j8B9vZKV.

The purpose of the applet is for experimenting and understanding the situation. If using it, demonstrate how it works, and ask students to think about:

- How to use the applet to create scale copies of the Mona Lisa. (Both dimensions have to be adjusted by the same factor.)
- Is it possible to scale down the Mona Lisa so that it perfectly covers the notebook? (No, choices have to be made about what the final product will look like.)

Arrange students in groups of 2. Give 3–5 minutes of quiet work time to do the problem. Then, ask them to take turns sharing with their partner the method used to calculate scale factor and reasonableness of their answers.

If using the digital activity, still have students work in groups of 2 and have them work individually on the problem with the applet, before sharing their method(s) to calculate scale factor with their partner.

*Conversing, Representing, Writing: Collect and Display.* As students discuss the scale factor they used, listen for and collect mathematical language students use to describe the strategies they used to find scale factors. Throughout the remainder of the lesson, continue to update collected student language and remind students to borrow language from the display as needed. This will help students use mathematical language during paired and group discussions.
*Design Principle(s): Optimise output (for explanation); Maximise meta-awareness*

### Anticipated Misconceptions

Students might get stuck thinking the scaled copy needs to measure 11 inches by 9 inches. Ask students:

- Does the copy of the painting have to cover the entire notebook?
- What are some other options if the image doesn't cover the entire notebook?
- What if the image is bigger than the notebook cover? What if it is smaller?

### Student Task Statement

In real life, the Mona Lisa measures $2\frac{1}{2}$ feet by $1\frac{3}{4}$ feet. A company that makes office supplies wants to print a scaled copy of the Mona Lisa on the cover of a notebook that measures 11 inches by 9 inches.

1. What size should they use for the scaled copy of the Mona Lisa on the notebook cover?
2. What is the scale factor from the real painting to its copy on the notebook cover?
3. Discuss your thinking with your partner. Did you use the same scale factor? If not, is one more reasonable than the other?

### Student Response

Answers vary. Sample responses:

- Converting to inches by multiplying the number of feet by 12, the dimensions $1\frac{3}{4}$ feet by $2\frac{1}{2}$ feet are equivalent to 21 inches by 30 inches. Multiplying both of these by $\frac{1}{3}$, the dimensions of the scaled copy would be 7 inches by 10 inches. This size would fit on the notebook with some space around the edges. (The scale factor is $\frac{1}{3}$.)
- The ratio of $1\frac{3}{4}$ and $2\frac{1}{2}$ is equivalent to 0.7 to 1, which is found by dividing $1\frac{3}{4}$ by $2\frac{1}{2}$. If we multiply both of these by 11, the dimensions could be 7.7 inches by 11 inches. (The scale factor needs to turn 30 inches into 11 inches, so the scale factor is $\frac{11}{30}$.)

### Activity Synthesis

Select previously identified students to share the arguments they had with their partners. Some guiding questions:

- What scale factor did you and your partner agree upon? How did you both agree upon this?
- Were there any similarities between the methods you and your partner used? Were there any differences?

## Lesson Synthesis

In this lesson, we worked with ratios of fractions.

- "What are strategies we can use to find solutions to ratio problems that involve fractions?" (double number line, tables, calculating unit rate)
- "How are those strategies different from and similar to ways we previously solved ratio problems that didn't involve fractions?" (They are structurally the same, but the arithmetic might take more time.)

## 2.5 Comparing Orange Juice Recipes

**Cool Down: 5 minutes**

### Student Task Statement

- Clare mixes $2\frac{1}{2}$ cups of water with $\frac{1}{3}$ cup of orange juice concentrate.
- Han mixes $1\frac{2}{3}$ cups of water with $\frac{1}{4}$ cup of orange juice concentrate.

Whose orange juice mixture tastes stronger? Explain or show your reasoning.

### Student Response

Han's mixture tastes stronger. Clare uses $7\frac{1}{2}$ cups of water per cup of orange juice concentrate, because $2\frac{1}{2} \div \frac{1}{3} = 7\frac{1}{2}$. Han uses $6\frac{2}{3}$ cups of water per cup of orange juice concentrate, because $1\frac{2}{3} \div \frac{1}{4} = 6\frac{2}{3}$. Han's mixture has less water for the same amount of orange juice concentrate.

## Student Lesson Summary

There are 12 inches in a foot, so we can say that for every 1 foot, there are 12 inches, or the ratio of feet to inches is $1 : 12$. We can find the **unit rates** by dividing the numbers in the ratio:

$1 \div 12 = \dfrac{1}{12}$
so there is $\frac{1}{12}$ foot per inch.

$12 \div 1 = 12$
so there are 12 inches per foot.

The numbers in a ratio can be fractions, and we calculate the unit rates the same way: by dividing the numbers in the ratio. For example, if someone runs $\frac{3}{4}$ mile in $\frac{11}{2}$ minutes, the ratio of minutes to miles is $\frac{11}{2} : \frac{3}{4}$.

$\frac{11}{2} \div \frac{3}{4} = \frac{22}{3}$, so the person's pace is $\frac{22}{3}$ minutes per mile.

$\frac{3}{4} \div \frac{11}{2} = \frac{3}{22}$, so the person's speed is $\frac{3}{22}$ miles per minute.

## Glossary

- unit rate

## Lesson 2 Practice Problems

**Problem 1 Statement**

A cyclist rode 3.75 miles in 0.3 hours.

a. How fast was she going in miles per hour?

b. At that rate, how long will it take her to go 4.5 miles?

**Solution**

a. 12.5 miles per hour

b. 0.36 hours or 21.6 minutes

**Problem 2 Statement**

A recipe for sparkling grape juice calls for $1\frac{1}{2}$ quarts of sparkling water and $\frac{3}{4}$ quart of grape juice.

a. How much sparkling water would you need to mix with 9 quarts of grape juice?

b. How much grape juice would you need to mix with $\frac{15}{4}$ quarts of sparkling water?

c. How much of each ingredient would you need to make 100 quarts of punch?

**Solution**

Notice that the ratio $1\frac{1}{2}$ quarts of sparkling water to $\frac{3}{4}$ quarts of grape juice is equivalent to the ratio 2 quarts of sparkling water to 1 quart of grape juice. While not needed, this ratio with whole numbers can help answer all three questions.

a. 18 quarts

b. $\frac{15}{8}$ quarts or equivalent

c. $\frac{200}{3}$ quarts of sparking water and $\frac{100}{3}$ quarts of grape juice (or equivalent).

**Problem 3 Statement**

At a deli counter,

- Someone bought $1\frac{3}{4}$ pounds of ham for £14.50.
- Someone bought $2\frac{1}{2}$ pounds of turkey for £26.25.
- Someone bought $\frac{3}{8}$ pounds of roast beef for £5.50.

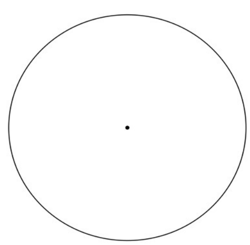
Which meat is the least expensive per pound? Which meat is the most expensive per pound? Explain how you know.

**Solution**

Ham is the least expensive. It costs about £8.29 per pound, because $14.50 \div 1\frac{3}{4} = 8\frac{2}{7} \approx 8.29$. Roast beef is the most expensive. It costs about £14.67 per pound, because $5.50 \div \frac{3}{8} = 14\frac{2}{3} \approx 14.67$. Turkey costs about £10.50 per pound, because $26.25 \div 2\frac{1}{2} = 10.50$. While these prices per pound are not exact, they are far enough apart to put the costs in order with certainty.

**Problem 4 Statement**

a. Draw a scaled copy of the circle using a scale factor of 2.

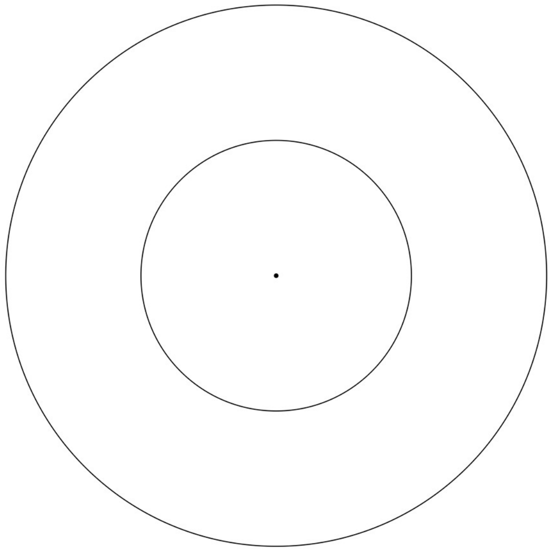


b. How does the circumference of the scaled copy compare to the circumference of the original circle?

c. How does the area of the scaled copy compare to the area of the original circle?

**Solution**

a. The outer circle is a scaled copy of the inner circle using scale factor 2.



b. The circumference of the scaled copy is twice the circumference of the original.

c. The area of the scaled copy is four times the area of the original.

**Problem 5 Statement**

Jada has a scale map of Kansas that fits on a page in her book. The page is 5 inches by 8 inches. Kansas is about 210 miles by 410 miles. Select **all** scales that could be a scale of the map. (There are 2.54 centimetres in an inch.)

a. 1 in to 1 mi

b. 1 cm to 1 km

c. 1 in to 10 mi

d. 1 ft to 100 mi

e. 1 cm to 200 km

f. 1 in to 100 mi

g. 1 cm to 1000 km

**Solution** ["E", "F"]