Lesson 14: Solving problems with rational numbers

Goals

- Apply operations with rational numbers to solve problems involving repeated gains or losses, and present (orally, in writing, and using other representations) the solution method.

Learning Targets

- I can represent situations with expressions that include rational numbers.
- I can solve problems using the four operations with rational numbers.

Lesson Narrative

In this lesson students put together what they have learned about rational number arithmetic and the interpretation of negative quantities, such as negative time or negative rates of change. They solve problems with rational numbers in the context of a negative flow rate from a tank and negative charges on an electricity bill or a bank account. The problems in this section are designed so that it is natural to solve them by filling in tables or making numerical calculations. In the next lesson, students will move towards solving algebraic equations.

Building On

- Solve real-world and mathematical problems by writing and solving equations of the form $x + p = q$ and $px = q$ for cases in which p, q and x are all nonnegative rational numbers.

Addressing

- Solve real-world and mathematical problems involving the four operations with rational numbers. Calculations with rational numbers extend the rules for manipulating fractions to complex fractions.
- Recognise and represent proportional relationships between quantities.

Building Towards

- Solve word problems leading to equations of the form $px + q = r$ and $p(x + q) = r$, where p, q, and r are specific rational numbers. Solve equations of these forms fluently. Compare an algebraic solution to an arithmetic solution, identifying the sequence of the operations used in each approach. For example, the perimeter of a rectangle is 54 cm. Its length is 6 cm. What is its width?

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Three Reads
- Compare and Connect

- Think Pair Share
- Which One Doesn't Belong?

Student Learning Goals

Let's use all four operations with directed numbers to solve problems.

14.1 Which One Doesn't Belong: Equations

Warm Up: 5 minutes

This warm-up prompts students to compare four equations. It encourages students to explain their reasoning, hold mathematical conversations, and gives you the opportunity to hear how they use terminology and talk about the equations in comparison to one another. To allow all students to access the activity, each equation has one obvious reason it does not belong.

Instructional Routines

- Which One Doesn't Belong?

Launch

Arrange students in groups of 2–4. Display the equations for all to see. Ask students to indicate when they have noticed one equation that does not belong and can explain why. Give students 1 minute of quiet think time and then time to share their thinking with their small group. In their small groups, tell each student to share their reasoning why a particular equation does not belong and together find at least one reason each equation doesn't belong.

Student Task Statement

Which equation doesn't belong?

$\frac{1}{2}x = -50$

$-60t = 30$

$x + 90 = -100$

$-0.01 = -0.001x$

Student Response

Answers vary. Sample responses:

$\frac{1}{2}x = -50$ is the only one with a fraction.

$-60t = 30$ is the only one with a different variable; is the only one with a solution whose absolute value is less than 1.

$x + 90 = -100$ is the only one with addition instead of multiplication.

$-0.01 = -0.001x$ is the only one with a positive solution; is the only one with decimals; is the only one with the variable on the right-hand side of the equation.

Activity Synthesis

Ask each group to share one reason why a particular equation does not belong. Record and display the responses for all to see. After each response, ask the class if they agree or disagree. Since there is no single correct answer to the question of which one does not belong, attend to students' explanations and ensure the reasons given make sense.

14.2 Draining and Filling a Tank

10 minutes

The purpose of this activity is for students to use the four operations on rational numbers to solve a problem about water in a tank. The activity presents another example where negative time is used; this time to describe before a sensor starts working. Students examine the change as a separate column before using the starting point to model the draining of the tank.

Students who see that they are doing the same calculations over and over and see that the structure of the expression is the same every time are expressing generality in repeated reasoning. Monitor for different explanations for the last question.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Compare and Connect
- Think Pair Share

Launch

Arrange students in groups of 2. Give students 4 minutes of quiet work time, followed by partner and whole-class discussion.

Action and Expression: Internalise Executive Functions. Chunk this task into more manageable parts. For example, after students have completed the first 2-3 rows of the first table, check-in with either select groups of students or the whole class. Invite students to share how they have applied generalisations from previous lessons about using the four operations on rational numbers so far.
Supports accessibility for: Conceptual processing; Organisation; Memory

Anticipated Misconceptions

The last question might be hard for students because they have had the table to support calculating the answers. Ask students how can they use the previous entries in the table to help them calculate the answer? What is something they see changing in the expressions in the table that would change in this question?

Student Task Statement

A tank of water is being drained. Due to a problem, the sensor does not start working until some time into the draining process. The sensor starts its recording at time zero when there are 770 litres in the tank.

1. Given that the drain empties the tank at a constant rate of 14 litres per minute, complete the table:

time after sensor starts (minutes)	change in water (litres)	expression	water in the tank (litres)
0	0	770 + (0)(-14)	770
1	-14	770 + (1)(-14)	756
5	-70		
10			

2. Later, someone wants to use the data to find out how long the tank had been draining before the sensor started. Complete this table:

time after sensor starts (minutes)	change in water (litres)	expression	water in the tank (litres)
1	-14	770 + (1)(-14)	756
0	0	770 + (0)(-14)	770
-1	14	770 + (-1)(-14)	784
-2	28		
-3			
-4			
-5			

3. If the sensor started working 15 minutes into the tank draining, how much was in the tank to begin with?

Student Response

1.

time after sensor starts (minutes)	change in water (litres)	expression	water in the tank (litres)
0	0	770 + (0)(-14)	770
1	-14	770 + (1)(-14)	756
5	-70	770 + (5)(-14)	700
10	-140	770 + (10)(-14)	630

2.

time after sensor starts (minutes)	change in water (litres)	expression	water in the tank (litres)
1	-14	770 + (1)(-14)	756
0	0	770 + (0)(-14)	770
-1	14	770 + (-1)(-14)	784
-2	28	770 + (-2)(-14)	798
-3	42	770 + (-3)(-14)	812
-4	56	770 + (-4)(-14)	826
-5	70	770 + (-5)(-14)	840

3. 980 litres

Activity Synthesis

Make sure students have filled out the tables appropriately. Select students to share their reasoning for the last one.

Speaking, Representing: Compare and Connect. Use this routine when students share their strategies for completing the table. Ask students to consider what is the same and what is different about the structure of each expression. Draw students' attention to the connection between representations (e.g., "Where do you see the change in water in your expression?", "How are do repeated calculations appear in the expression?"). These exchanges strengthen students' mathematical language use and reasoning with different representations.
Design Principle(s): Support sense-making; Maximise meta-awareness

14.3 Buying and Selling Power

15 minutes

The purpose of this activity is for students to use the four operations on rational numbers to solve real-world problems. Monitor for students who solved the problem using different representations and approaches.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Three Reads

Launch

Arrange students in groups of 2–4. Given them 4 minutes of quiet work time, followed by small group and then whole-class discussion.

Engagement: Develop Effort and Persistence. Encourage and support opportunities for peer interactions. Invite students to talk about their ideas with a partner before writing them down. Display sentence frames to support students when they explain their strategy. For example, "First, I _____ because _____. Then, I…," "I noticed _____ so I…" and "I tried _____ and what happened was…"
Supports accessibility for: Language; Social-emotional skills Reading: Three Reads. Use this routine to support reading comprehension of this word problem, without solving it for students. In the first read, students read the problem with the goal of comprehending the situation (e.g., A utility company charges for energy use.). In the second read, ask students to identify the quantities and relationships, without focusing on specific values. Listen for, and amplify, the quantities that vary in relation to each other in this situation: electricity used, in kilowatt-hours; amount spent on electricity used, in pounds; electricity generated and not used, in kilowatt-hours; amount credited for electricity generated and not used, in pounds. Invite students to draw a diagram to represent the relationships among these quantities. After the third read, ask students to brainstorm possible strategies to answer the questions. This helps students connect the language in the word problem and the reasoning needed to solve the problem, maintaining the intended level of cognitive demand in the task.
Design Principle(s): Support sense-making

Student Task Statement

A utility company charges £0.12 per kilowatt-hour for energy a customer uses. They give a credit of £0.025 for every kilowatt-hour of electricity a customer with a solar panel generates that they don't use themselves.

A customer has a charge of £82.04 and a credit of -£4.10 on this month's bill.

1. What is the amount due this month?
2. How many kilowatt-hours did they use?

3. How many kilowatt-hours did they generate that they didn't use themselves?

Student Response

1. £77.94
2. About 684 kilowatt-hours
3. 164 kilowatt-hours

Are You Ready for More?

1. Find the value of the expression without a calculator.

   $(2)(-30) + (-3)(-20) + (-6)(-10) - (2)(3)(10)$

2. Write an expression that uses addition, subtraction, multiplication, and division and only negative numbers that has the same value.

Student Response

1. 0
2. Answers vary. Sample response: $-10 \div -2 + (-2)(-2)(-2) - -3$

Activity Synthesis

Select students to share their solution. Help them make connections between different solution approaches.

Lesson Synthesis

In this lesson we saw that directed numbers can be used to represent situations where amounts are changing different ways.

In the activity about the water tank,

- What did a positive amount represent? (water added to the tank)
- What did a negative amount represent? (water drained from the tank)

In the activity about the price of electricity,

- What did a positive amount represent? (money the customer owed the company)
- What did a negative amount represent? (money the company owed the customer)

14.4 Charges and Checks

Cool Down: 5 minutes

Student Task Statement

Lin's sister has a current account. If the account balance ever falls below zero, the bank charges her a fee of £5.95 per day. Today, the balance in Lin's sister's account is -£2.67.

1. If she does not make any deposits or withdrawals, what will be the balance in her account after 2 days?

2. In 14 days, Lin's sister will be paid £430 and will deposit it into her current account. If there are no other transactions besides this deposit and the daily fee, will Lin continue to be charged £5.95 each day after this deposit is made? Explain or show your reasoning.

Student Response

1. -£14.57

2. No. Reasoning varies. Sample explanation: even if the fee was £10 per day, that would total £140, which is much less than what she will deposit.

Student Lesson Summary

We can apply the rules for arithmetic with rational numbers to solve problems.

In general: $a - b = a + \text{-}b$

If $a - b = x$, then $x + b = a$. We can add $\text{-}b$ to both sides of this second equation to get that $x = a + \text{-}b$

Remember when multiplying or dividing:

- The sign of a positive number multiplied or divided by a negative number is always negative.
- The sign of a negative number multiplied or divided by a positive number is always negative.
- The sign of a negative number multiplied or divided by a negative number is always positive.

Lesson 14 Practice Problems

1. **Problem 1 Statement**

   A bank charges a service fee of £7.50 per month for a current account.

   A bank account has £85.00. If no money is deposited or withdrawn except the service charge, how many months until the account balance is negative?

   Solution

12, because $85 \div 7.50 = 11\frac{1}{3}$ which means it will take 12 months to have a negative balance in the account

2. **Problem 2 Statement**

The table shows transactions in a current account.

January
-38.50
126.30
429.40
-265.00
February
250.00
-135.20
35.50
-62.30
March
-14.00
99.90
-82.70
-1.50
April
-86.80
-570.00
100.00
-280.10

a. Find the total of the transactions for each month.

b. Find the mean total for the four months.

Solution

a. January: 252.20; February: 88; March 1.70; April: -836.90

b. -123.75, because $[252.20 + 88 + 1.70 + (-836.9)] \div 4 = -123.75$

3. **Problem 3 Statement**

A large aquarium of water is being filled with a hose. Due to a problem, the sensor does not start working until some time into the filling process. The sensor starts its

recording at the time zero minutes. The sensor initially detects the tank has 225 litres of water in it.

a. The hose fills the aquarium at a constant rate of 15 litres per minute. What will the sensor read at the time 5 minutes?

b. Later, someone wants to use the data to find the amount of water at times before the sensor started. What should the sensor have read at the time -7 minutes?

Solution

a. 300 litres, because $225 + 15 \times 5 = 300$

b. 120 litres, because $225 + 15 \times -7 = 120$

4. **Problem 4 Statement**

A furniture store pays a wholesale price for a mattress. Then, the store marks up the retail price to 150% of the wholesale price. Later, they put the mattress on sale for 50% off of the retail price. A customer just bought the mattress on sale and paid £1 200.

a. What was the retail price of the mattress, before the discount?

b. What was the wholesale price, before the markup?

Solution

a. £2 400, because $1\,200 \div 0.5 = 2\,400$.

b. £1 600, because $2\,400 \div 1.5 = 1\,600$.

5. **Problem 5 Statement**

a. A restaurant bill is £21. You leave a 15% tip. How much do you pay including the tip?

b. Which of the following represents the amount a customer pays including the tip of 15% if the bill was b pounds? Select **all** that apply.

- $15b$
- $b + 0.15b$
- $1.15b$
- $1.015b$
- $b + \frac{15}{100}b$

- $b + 0.15$
- $0.15b$

Solution

a. £24.15

b. $b + 0.15b$, $1.15b$, $b + \frac{15}{100}b$