INTERNATIONAL BACCALAUREATE

Mathematics: analysis and approaches

MAA

EXERCISES [MAA 5.11]

DEFINITE INTEGRALS – AREAS

Compiled by Christos Nikolaidis

DEFINITE INTEGRALS – PROPERTIES

O. Practice questions

1. [Maximum mark: 36] ***[without GDC]***

Calculate the following definite integrals

$\int_0^1 (2x+3)\mathrm{d}x$
$\int_1^2 (2x+3)\mathrm{d}x$
$\int_0^2 (2x+3)\mathrm{d}x$
$\int_{-2}^2 (2x+3)\mathrm{d}x$
$\int_0^1 (\mathrm{e}^x+2)\mathrm{d}x$
$\int_0^\pi (\sin x+\cos x)\mathrm{d}x$
$\int_1^{\mathrm{e}} \frac{7}{x}\mathrm{d}x$
$\int_0^1 \mathrm{e}^{2x+3}\mathrm{d}x$
$\int_0^4 \frac{1}{x+1}\mathrm{d}x$
$\int_0^{10} x\,\mathrm{d}x$
$\int_0^{10} 5\,\mathrm{d}x$
$\int_4^{10} \mathrm{d}x$

2. [Maximum mark: 6] ***[without GDC]***

Let $f(x) = x\ln x - x$

(a) Find $f'(x)$ [3]

(b) **Hence** find $\int_1^3 \ln x\mathrm{d}x$ [3]

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3. [Maximum mark: 18] ***[without GDC]***

Let $\int_5^7 f(x)\mathrm{d}x = 8$ and $\int_5^7 g(x)\mathrm{d}x = 2$

Calculate the following expressions

$\int_5^7 3f(x)\mathrm{d}x$
$\int_7^5 f(x)\mathrm{d}x$
$\int_5^7 (f(x)+1)\mathrm{d}x$
$\int_5^7 (f(x)+x)\mathrm{d}x$
$\int_5^7 [f(x)-4g(x)]\mathrm{d}x$
$\int_5^8 f(x)\mathrm{d}x - \int_7^8 f(x)\mathrm{d}x$
$3\int_5^6 f(x)\mathrm{d}x + \int_6^7 3f(x)\mathrm{d}x$
$\int_8^{10} f(x-3)\mathrm{d}x$
$\int_{2.5}^{3.5} f(2x)\mathrm{d}x$

4. [Maximum mark: 9] ***[without GDC]***

(a) Find $\int \frac{1}{2x+3}\,\mathrm{d}x$. [2]

(b) Given that $\int_0^3 \frac{1}{2x+3}\,\mathrm{d}x = \ln\sqrt{P}$, find the value of P. [3]

(c) Given that $\int_0^m \frac{1}{2x+3}\,\mathrm{d}x = 1$, calculate the value of m. [4]

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A. Exam style questions (SHORT)

5. [Maximum mark: 5] ***[without GDC]***

Given $\int_3^k \frac{1}{x-2}\,\mathrm{d}x = \ln 7$, find the value of k.

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6. [Maximum mark: 4] ***[without GDC]***

Find the real number $k > 1$ for which $\int_1^k \left(1+\frac{1}{x^2}\right)\mathrm{d}x = \frac{3}{2}$.

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7. [Maximum mark: 3] ***[with GDC]***

Find the value of a such that $\int_0^a \cos^2 x\,\mathrm{d}x = 0.740$. Give your answer to 3 decimal places.

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8. [Maximum mark: 6] ***[without GDC]***

Given that $\int_1^3 g(x)\mathrm{d}x = 10$, deduce the value of (i) $\int_1^3 \frac{1}{2} g(x)\mathrm{d}x$; (ii) $\int_1^3 (g(x)+4)\mathrm{d}x$.

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9. [Maximum mark: 6] ***[without GDC]***

Let f be a function such that $\int_0^3 f(x)\,\mathrm{d}x = 8$.

(a) Deduce the value of (i) $\int_0^3 2f(x)\,\mathrm{d}x$ (ii) $\int_0^3 (f(x)+2)\mathrm{d}x$ [4]

(b) $\int_c^d f(x-2)\mathrm{d}x = 8$, write down the value of c and of d. [2]

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10. [Maximum mark: 6] ***[without GDC]***

Given that $\int_1^3 f(x)\mathrm{d}x = 5$, deduce the value of (i) $\int_1^3 2f(x)\mathrm{d}x$ (ii) $\int_1^3 \left(3x^2 + f(x)\right)\mathrm{d}x$.

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11. [Maximum mark: 6] ***[without GDC]***

The table shows some values of two functions, f, g and of their derivatives f', g':

x	1	2	3	4
$f(x)$	5	4	–1	3
$g(x)$	1	–2	2	–5
$f'(x)$	5	6	0	7
$g'(x)$	–6	–4	–3	4

(a) Calculate $\frac{\mathrm{d}}{\mathrm{d}x}\left(f(x)+g(x)\right)$, when $x=4$; [2]

(b) Calculate $\int_1^3 \left(g'(x)+6\right)\mathrm{d}x$. [4]

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12. [Maximum mark: 7] ***[without GDC]***

Let $\int_1^5 3f(x)\,\mathrm{d}x = 12$.

(a) Show that $\int_5^1 f(x)\,\mathrm{d}x = -4$ [3]

(b) Find the value of $\int_1^2 \left(x+f(x)\right)\mathrm{d}x + \int_2^5 \left(x+f(x)\right)\mathrm{d}x$ [4]

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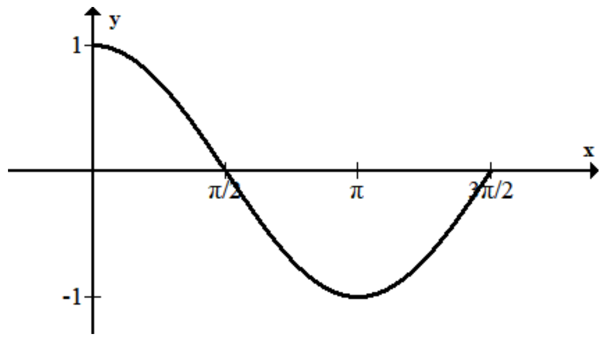
AREAS

O. Practice questions

13. [Maximum mark: 8] ***[with / without GDC]***

The following diagram shows part of the graph of $y=\cos x$ for $0 \le x \le \frac{3\pi}{2}$.



(a) Calculate (i) $\int_0^{\frac{\pi}{2}} \cos x\,dx$, ` (ii) $\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos x\,dx$ , `(iii) $\int_0^{\frac{3\pi}{2}} \cos x\,dx$ [5]

(b) Write down the area enclosed by the curve and x-axis

(i) between the vertical lines $x=0$ and $x=\frac{\pi}{2}$

(ii) between the vertical lines $x=\frac{\pi}{2}$ and $x=\frac{3\pi}{2}$

(iii) between the vertical lines $x=0$ and $x=\frac{3\pi}{2}$ [3]

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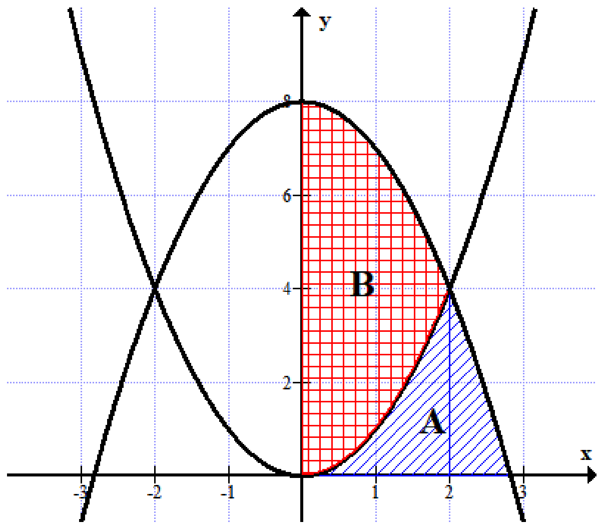
14. [Maximum mark: 8] ***[with / without GDC]***

Consider the two curves $y = x^2$, $y = 8 - x^2$



(a) Find the area A enclosed by the two curves and the ***x*-axis** in the first quadrant. [4]

(b) Find the area B enclosed by the two curves and the ***y*-axis** in the first quadrant [4]

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15. [Maximum mark: 21] ***[with GDC]***

Complete the following table

Region enclosed by	Expression for the area	Area
$f(x)=\cos(x^2)$ $g(x)=e^x$, for $-1.5\le x\le 0.5$.	$\int_{-1.11}^{0}\left(\cos(x^2)-e^x\right)dx$	
$y=\sin x$ $y=x^2-2x+1.5$, for $0\le x\le\pi$.		0.271
$y=\ln x$ $y=e^x-e$, for $x>0$.		
$y=\frac{2}{1+x^2}$ $y=e^{x/3}$, for $-3\le x\le 3$.		
$f(x)=4-x^2$ $g(x)=(x+1)\cos x$		
$y=e^{-x}-x+1$ and the coordinate axes		
$f: x\mapsto\frac{\sin x}{x}$, x-axis for $\pi\le x\le 3\pi$		
$y=x^3-3x^2-9x+27$ $y=x+3$		

A. Exam style questions (SHORT)

16. [Maximum mark: 7] ***[with GDC]***

The diagram shows part of the graph of the function $f(x) = 1 + 3\sin(x+2)$.

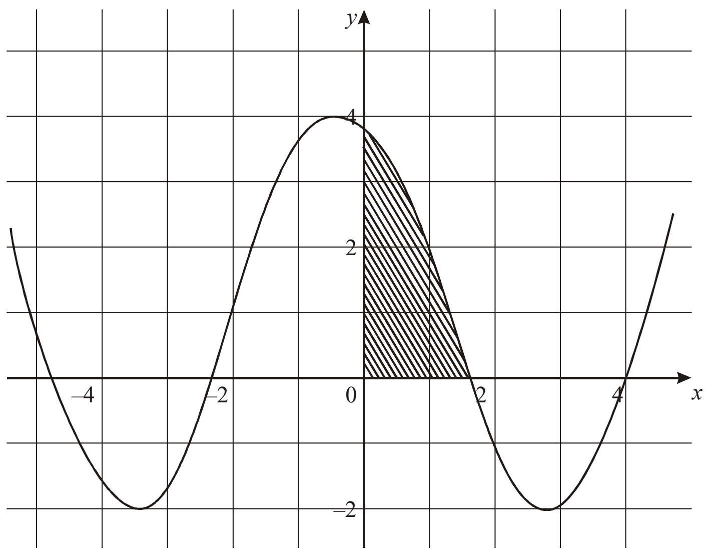
(a) Find $f'(x)$ [2]

(b) Find $\int f(x)\mathrm{d}x$ [2]

The area of the shaded region is given by $\int_0^a f(x)\mathrm{d}x$.

(c) (i) Find the value of a.

(ii) Find the area of the shaded region. [3]



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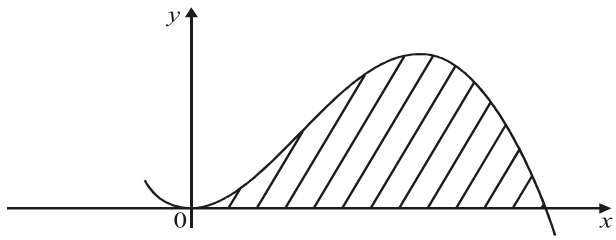
17. [Maximum mark: 4] ***[without GDC]***

The diagram shows part of the graph of $y = 12x^2(1-x)$.



(a) Write down an integral which represents the area of the shaded region. [1]

(b) Find the area of the shaded region. [3]

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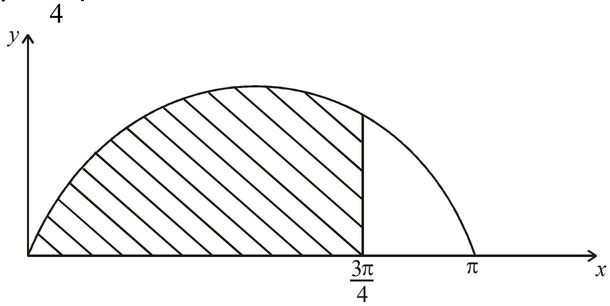
18. [Maximum mark: 6] ***[without GDC]***

The diagram shows part of the curve $y = \sin x$. The shaded region is bounded by the curve and the lines $y = 0$ and $x = \frac{3\pi}{4}$.



Given that $\sin\frac{3\pi}{4} = \frac{\sqrt{2}}{2}$ and $\cos\frac{3\pi}{4} = -\frac{\sqrt{2}}{2}$, calculate the area of the shaded region.

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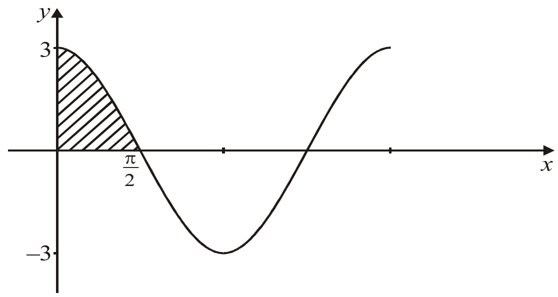
19. [Maximum mark: 4] ***[without GDC]***

The graph represents the function $f: x \mapsto p\cos x$, $p \in \mathbb{N}$.



(a) Write down the value of p; [1]

(b) Find the area of the shaded region. [3]

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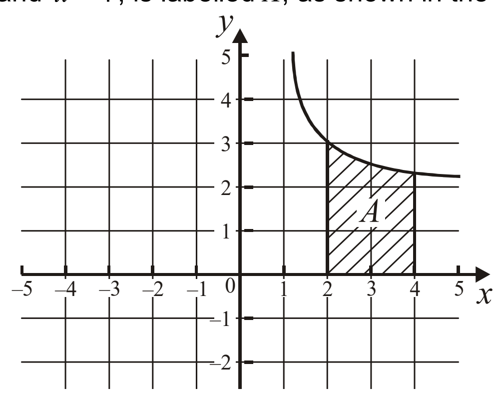
20. [Maximum mark: 6] ***[without GDC]***

Consider the function $f(x) = 2 + \dfrac{1}{x-1}$. The region enclosed by the graph of $f(x)$, the x-axis and the lines $x = 2$ and $x = 4$, is labelled A, as shown in the diagram below.



Find (i) $\int f(x)\,\mathrm{d}x$. (ii) the area of A.

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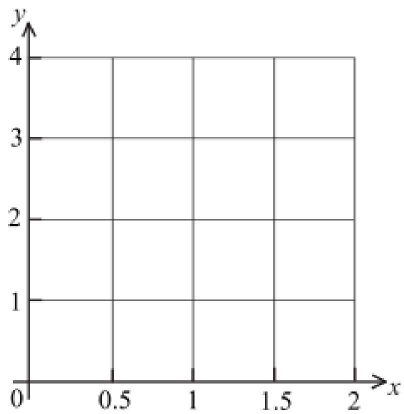
21. [Maximum mark: 4] ***[with GDC]***

For $x \geq \frac{1}{2}$, let $f(x) = x^2 \ln(x+1)$ and $g(x) = \sqrt{2x-1}$.

(a) Sketch the graphs of f and g on the grid below. [2]



(b) Let A be the region completely enclosed by the graphs of f and g.
Find the area of A. [2]

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22. [Maximum mark: 6] ***[with GDC]***

The function f is defined as $f(x) = \sin x \ln x$ for $x \in [0.5\ ,\ 3.5]$

(a) Write down the x-intercepts. [2]

(b) The area above the x-axis is A and the **total** area below the x-axis is B.
If $A = kB$, find k. [4]

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23. [Maximum mark: 7] ***[without GDC]***

Find the area between the curves $y = 2 + x - x^2$ and $y = 2 - 3x + x^2$

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24. [Maximum mark: 6] ***[without GDC]***

Find the area enclosed by the two curves $y = x^2$ and $y = 2a^2 - x^2$ where $a > 0$.

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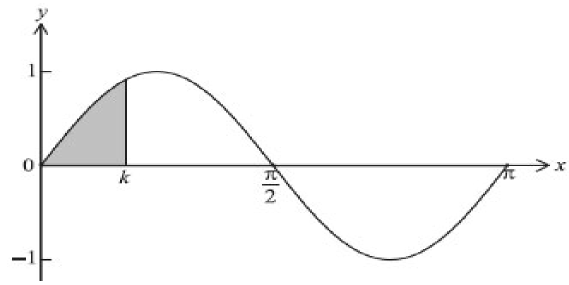
25. [Maximum mark: 5] ***[with GDC]***

Part of the graph of $y = \sin 2x$ is shown below. The area of the shaded region is 0.85. Find k.



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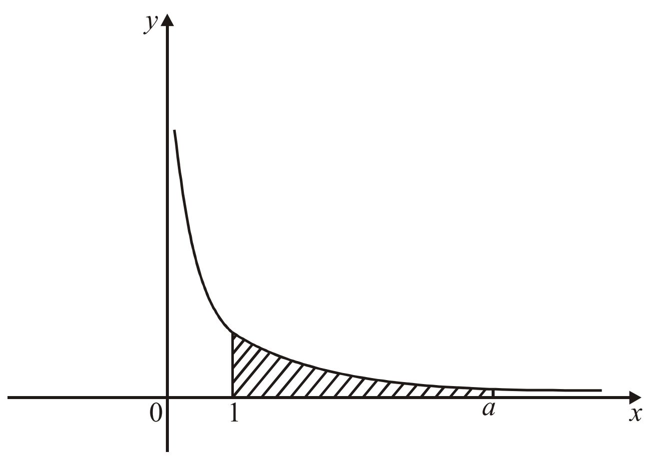
26. [Maximum mark: 4] ***[without GDC]***

The diagram shows part of the graph of $y = \dfrac{1}{x}$. The area of the shaded region is 2 units.



Find the exact value of a.

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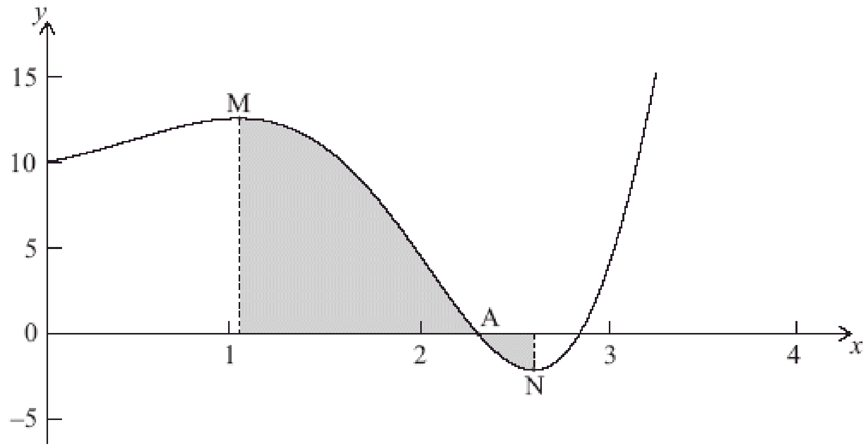
27. [Maximum mark: 6] ***[with GDC]***

Let $f(x) = e^x \sin 2x + 10$, for $0 \le x \le 4$. Part of the graph of f is given below.



There is an x-intercept at the point A, a local maximum point at M, where $x = p$ and a local minimum point at N, where $x = q$.

(a) Write down the x-coordinate of A. [1]

(b) Find the value of (i) p; (ii) q. [2]

(c) Find $\int_p^q f(x)\mathrm{d}x$. Explain why this is not the area of the shaded region. [3]

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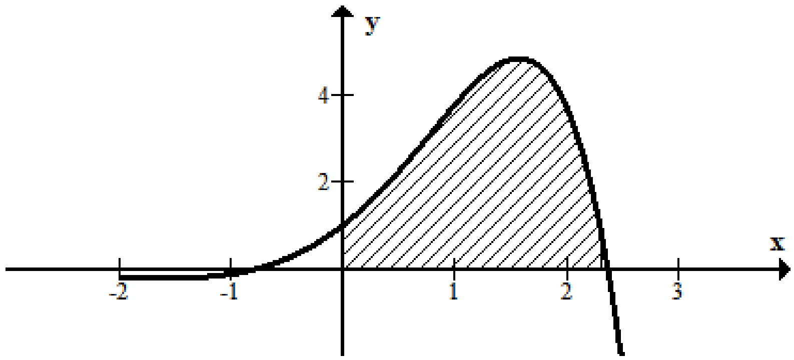
28. [Maximum mark: 6] ***[with GDC]***

Consider the function. $f(x) = \cos x + \sin x$.

(a) Find in terms of π, the smallest **positive** value of x such that $f(x) = 0$. [3]

The diagram shows the graph of $y = e^x(\cos x + \sin x)$, $-2 \le x \le 3$.



(b) Write down an expression for the area of the shaded region and find its value. [3]

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29. [Maximum mark: 5] ***[with GDC]***

The function f is defined as $f(x) = e^x \sin x$, where x is in radians.

Let A be the x-intercept corresponding to the smallest **positive** zero of f.

(a) Write down the x-coordinate of the point A. [1]

(b) Let R be the region enclosed by the curve and x-axis, between the origin and A.

(i) Write down an expression for the area of *R*.

(ii) Find the area of *R*. [4]

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30. [Maximum mark: 7] ***[without GDC]***

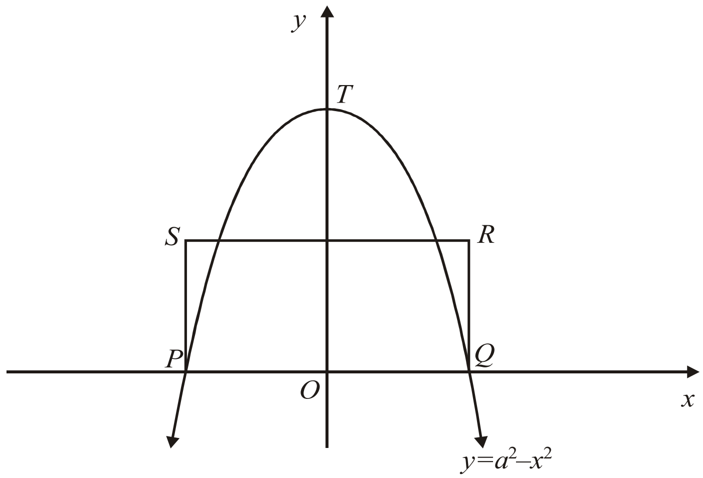
Let $f(x)=\dfrac{-3x}{x^2-1}$ and $g(x)=f'(x)$.

(a) Show that $g(x)=\dfrac{3\left(x^2+1\right)}{\left(x^2-1\right)^2}$. [3]

(b) Let A be the area of the region enclosed by the graph of g and the x-axis, between $x=0$ and $x=a$, where $a>0$. Given that $A=2$, find the value of a. [4]

31*. [Maximum mark: 4] ***[without GDC]***

In the diagram, PTQ is an arc of the parabola $y=a^2-x^2$, where *a* is a positive constant, and $PQRS$ is a rectangle. The area of the rectangle $PQRS$ is equal to the area between the arc PTQ of the parabola and the x-axis.



Find, in terms of a, the dimensions of the rectangle.

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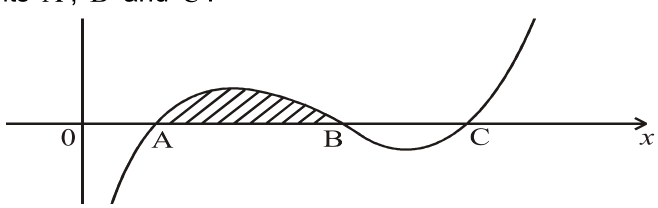
32. [Maximum mark: 4] ***[with GDC]***

The figure below shows part of the curve $y=x^3-7x^2+14x-7$. The curve crosses the x-axis at the points A, B and C.



(a) Find the x-coordinate (i) of A (ii) of B. [2]

(b) Find the area of the shaded region. [2]

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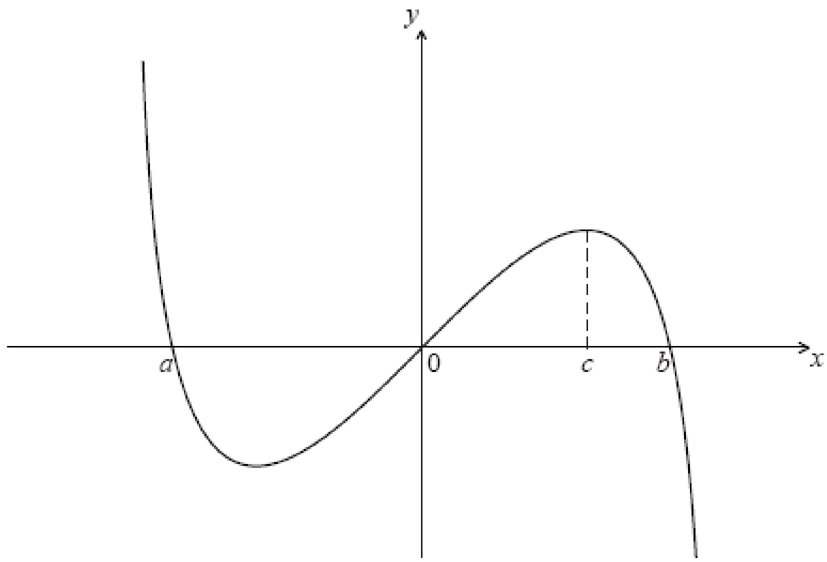
33. [Maximum mark: 7] ***[with GDC]***

Let $f(x) = x\ln(4-x^2)$, for $-2 \le x \le 2$. The graph of f is shown below.



The graph of f crosses the x -axis at $x = a$, $x = 0$ and $x = b$.

(a) Find the value of a and of b . [3]

The graph of f has a maximum value when $x = c$.

(b) Find the value of c . [2]

(c) Let *R* be the region enclosed by the curve, the x -axis and the line $x = c$, between $x = a$ and $x = c$. Find the area of *R*. [2]

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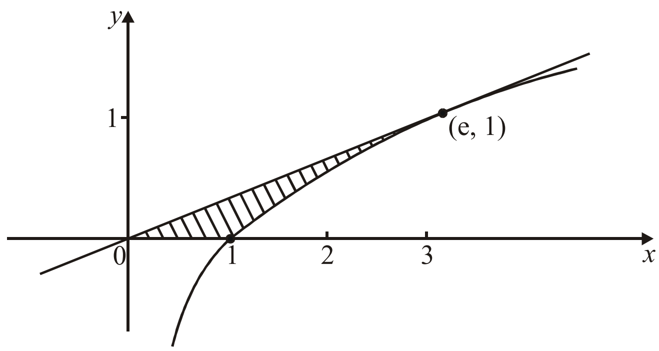
B. Exam style questions (LONG)

34. [Maximum mark: 10] ***[without GDC]***

(a) Find the equation of the tangent line to the curve $y = \ln x$ at the point $(\mathrm{e}, 1)$, and verify that the origin is on this line. [4]

(b) Show that $\frac{\mathrm{d}}{\mathrm{d}x}(x\ln x - x) = \ln x$ [2]

(c) The diagram shows the region enclosed by the curve $y = \ln x$, the tangent line in part (a), and the line $y = 0$.



Use the result of part (b) to show that the area of this region is $\frac{1}{2}\mathrm{e} - 1$. [4]

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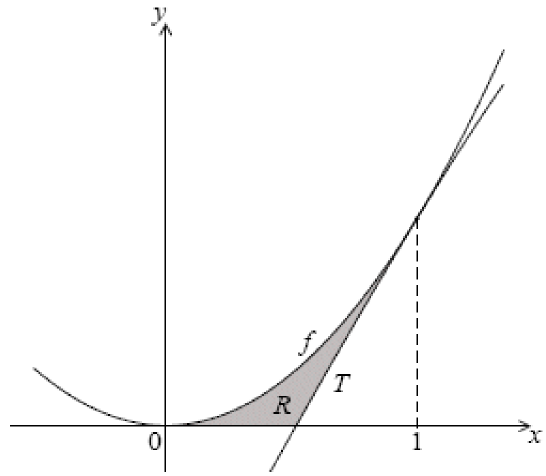
35. [Maximum mark: 16] ***[without GDC]***

The following diagram shows part of the graph of the function $f(x)=2x^2$.



The line *T* is the tangent to the graph of f at $x=1$.

(a) Show that the equation of *T* is $y=4x-2$. [5]

(b) Find the x-intercept of *T*. [2]

(c) The shaded region *R* is enclosed by the graph of f, the line *T*, and the x-axis.

(i) Write down an expression for the area of *R*.

(ii) Find the area of *R*. [9]

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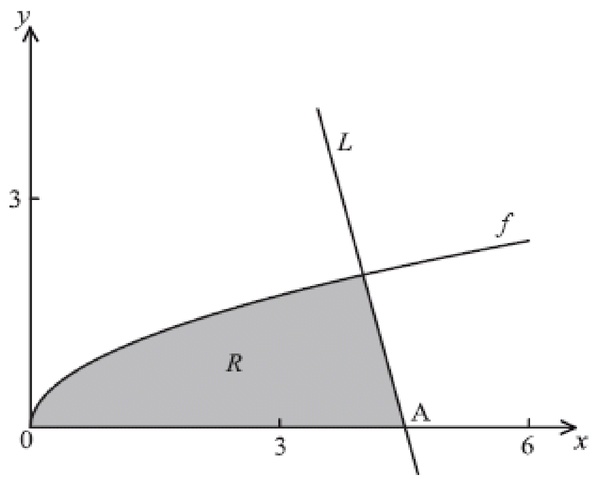
36. [Maximum mark: 12] ***[without GDC]***

Let $f(x)=\sqrt{x}$. Line *L* is the normal to the graph of f at the point (4, 2).

(a) Show that the equation of *L* is $y=-4x+18$. [4]

(b) Point A is the x-intercept of *L*. Find the x-coordinate of A. [2]

In the diagram below, the shaded region *R* is bounded by the x-axis, the graph of f and the line *L*.



(c) Find an expression for the area of *R*. [3]

(d) Find the area *R* [3]

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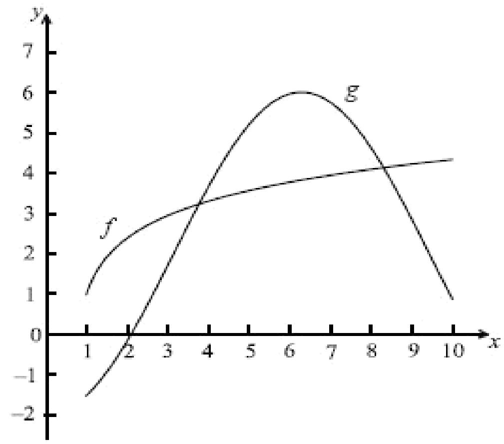
37. [Maximum mark: 14] ***[with GDC]***

The following diagram shows the graphs of $f(x)=\ln(3x-2)+1$ and $g(x)=-4\cos(0.5x)+2$, for $1\le x\le 10$.



(a) Let A be the area of the region **enclosed** by the curves of f and g.

(i) Find an expression for A; (ii) Calculate the value of A. [6]

(b) Find (i) $f'(x)$; (ii) $g'(x)$. [4]

(c) There are two values of x for which the gradient of f is equal to the gradient of g. Find both these values of x. [4]

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38*. [Maximum mark: 14] ***[without GDC]***

A curve has equation $y = x(x-4)^2$.

(a) For this curve find

(i) the x-intercepts;

(ii) the coordinates of the maximum point (by using derivatives);

(iii) the x-coordinate of the point of inflexion. [8]

(b) Use your answers to part (a) to sketch a graph of the curve for $0 \le x \le 4$, clearly indicating the features you have found in part (a). [3]

(c) (i) On your sketch indicate by shading the region whose area is given by the following integral: $\int_0^4 x(x-4)^2\,dx$

(ii) Explain, using your answer to part (a), why the value of this integral is greater than 0 but less than 40. [3]

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39. [Maximum mark: 10] ***[with GDC]***

(a) Sketch the graph of $y = \pi \sin x - x$, $-3 \le x \le 3$. Label and number both axes and indicate clearly the approximate positions of the x-intercepts and the local maximum and minimum points. [5]

(b) Find the solution of the equation $\pi \sin x - x = 0$, $x > 0$. [1]

(c) Find the indefinite integral $\int (\pi \sin x - x)\mathrm{d}x$ and hence, or otherwise, calculate the area of the region enclosed by the graph, the x-axis and the line $x = 1$. [4]

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40. [Maximum mark: 14] ***[with GDC]***

Note: Radians are used throughout this question.

(a) (i) Sketch the graph of $y = x^2 \cos x$, for $0 \le x \le 2$ making clear the approximate positions of the positive intercept, the maximum point and the end-points.

(ii) Write down the **approximate** coordinates of the positive x-intercept, the maximum point and the end-points. [7]

(b) Find the **exact value** of the positive x-intercept for $0 \le x \le 2$. [2]

Let R be the region in the first quadrant enclosed by the graph and the x-axis.

(c) (i) Shade *R* on your diagram.

(ii) Write down an integral which represents the area of *R.* [3]

(d) Evaluate the integral in part (c)(ii), [2]

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41. [Maximum mark: 14] ***[with GDC]***

Note: Radians are used throughout this question.

(a) Draw the graph of $y = \pi + x\cos x$, $0 \le x \le 5$. Make clear

(i) the integer values of x and y on each axis;

(ii) the approximate positions of the x-intercepts and the turning points. [5]

(b) **Without the use of a calculator**, show that π is a solution of the equation $\pi + x\cos x = 0$. [3]

(c) Find another solution of the equation $\pi + x\cos x = 0$ for $0 \le x \le 5$, giving your answer to **six** significant figures. [2]

(d) Let R be the region enclosed by the graph and the axes for $0 \le x \le \pi$. Shade R on your diagram, and write down an integral which represents the area of R. [2]

(e) Evaluate the integral in part (d) to an accuracy of **six** significant figures. [2]

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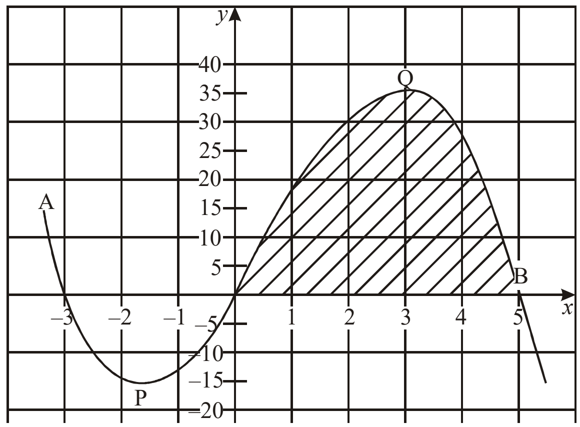
42. [Maximum mark: 15] ***[with GDC]***

The diagram below shows part of the graph of the function $f: x \mapsto -x^3 + 2x^2 + 15x$.



The graph intercepts the x-axis at A(–3,0), B(5,0) and the origin, O. There is a minimum point at P and a maximum point at Q.

(a) The function may also be written in the form $f: x \mapsto -x(x-a)(x-b)$, where $a < b$.

Write down the value of (i) a; (ii) b. [2]

(b) (i) Find the **exact** values of x at which $f'(x) = 0$;

(ii) Find the value of the function at Q. [7]

(c) (i) Find the equation of the tangent to the graph of f at O.

(ii) This tangent cuts the graph of f at point R. Give the x-coordinate of R. [4]

(d) Determine the area of the shaded region. [2]

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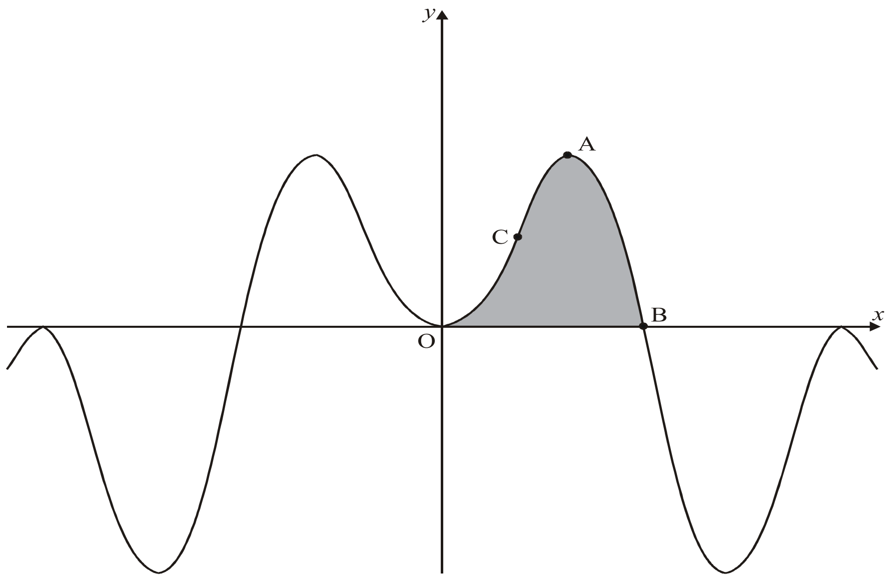
43. [Maximum mark: 18] ***[without GDC]***

The following diagram shows part of the graph of $f(x)=(\sin x)^2\cos x$ (x is in radians).



The point A is a maximum point and the point B lies on the x-axis.

(a) (i) Find $f'(x)$.

(ii) Hence show that at the point A, $\cos x=\sqrt{\frac{1}{3}}$.

(iii) Find the exact maximum value. [9]

(b) Find the exact value of the x-coordinate at the point B. [1]

(c) (i) Find $\int f(x)\,dx$.

(ii) Find the area of the shaded region in the diagram. [4]

(d) (i) Show that $f''(x)=9(\cos x)^3-7\cos x$,

(ii) Hence show that at the point C, $\cos x=\frac{\sqrt{7}}{3}$. [4]

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44. [Maximum mark: 13] ***[with GDC]***

Consider functions of the form $y = e^{-kx}$.

(a) Show that $\int_0^1 e^{-kx}\,dx = \frac{1}{k}(1 - e^{-k})$. [3]

(b) Let $k = 0.5$

(i) Sketch the graph of $y = e^{-0.5x}$, for $-1 \le x \le 3$, indicating the y-intercept.

(ii) Shade the region enclosed by this graph, the x-axis, y-axis and line $x = 1$.

(iii) Find the area of this region. [5]

(c) (i) Find $\frac{dy}{dx}$ in terms of k, where $y = e^{-kx}$.

(ii) Find the value of k given that the point $P(1, 0.8)$ lies on the graph of $y = e^{-kx}$.

(iii) Find the gradient of the tangent to the curve at P. [5]

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45. [Maximum mark: 12] ***[with GDC]***

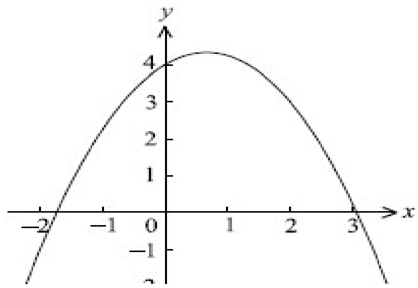
Let $f(x) = -\frac{3}{4}x^2 + x + 4$.

(a) (i) Find the equation of the normal to the curve of f at (2, 3).

(ii) This normal intersects the curve of f at (2, 3) and at one other point P.

Find the x-coordinate of P. [9]

Part of the graph of f is given below.



(b) Let *R* be the region under the curve of f from $x = -1$ to $x = 2$.

Write down an expression for the area of *R* and **hence** evaluate this area. [3]

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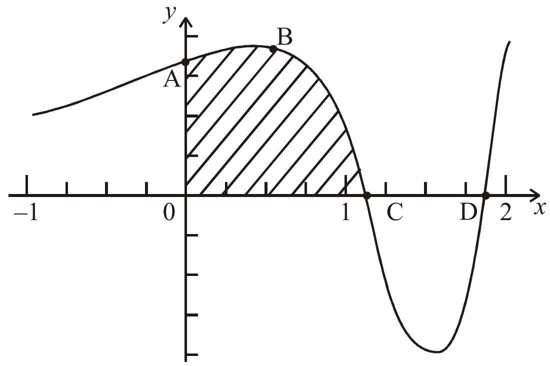
46. [Maximum mark: 15] ***[with GDC]***

The diagram below shows a sketch of the graph of the function $y = \sin(e^x)$ where $-1 \le x \le 2$, and x is in **radians**. The graph cuts the y-axis at A, and the x-axis at C and D. It has a maximum point at B.



(a) Find the coordinates of A. [2]

(b) The coordinates of C may be written as $(\ln k, 0)$. Find the **exact** value of k. [2]

(c) (i) Write down the y-coordinate of B.

(ii) Find $\frac{dy}{dx}$ and hence show that at B, $x = \ln\frac{\pi}{2}$. [6]

(d) Write down the integral which represents the shaded area; evaluate the integral. [5]

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47. [Maximum mark: 14] ***[with GDC]***

Consider the function $f(x)=1+e^{-2x}$.

(a) (i) Find $f'(x)$.

(ii) Explain briefly how this shows that $f(x)$ is a decreasing function. [2]

Let P be the point on the graph of f where $x=-\frac{1}{2}$.

(b) Find an expression in terms of e for

(i) the y-coordinate of P; (ii) the gradient of the tangent to the curve at P. [2]

(c) Find the equation of the tangent to the curve at P, in the form $y=ax+b$. [3]

(d) (i) Sketch the curve of f for $-1\le x\le 2$.

(ii) Draw the tangent at $x=-\frac{1}{2}$.

(iii) Shade the area enclosed by the curve, the tangent and the y-axis.

(iv) Find this area. [7]

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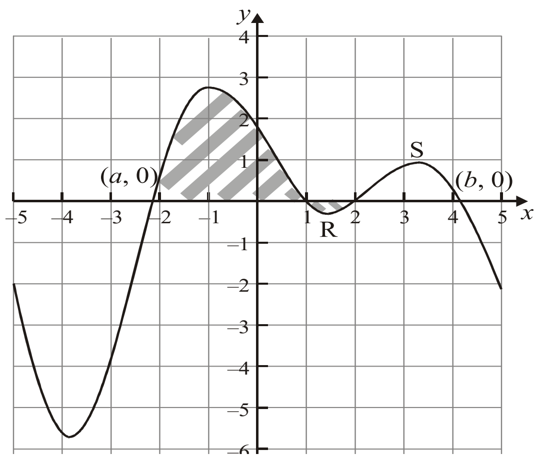
48. [Maximum mark: 11] ***[with GDC]***

Let $h(x) = (x-2)\sin(x-1)$ for $-5 \le x \le 5$. The curve of $h(x)$ is shown below. There is a minimum point at R and a maximum point at S. The curve intersects the x-axis at the points $(a, 0)$, $(1, 0)$, $(2, 0)$ and $(b, 0)$.



(a) Find the exact values of a and b. [2]

The regions between the curve and the x-axis are shaded for $a \le x \le 2$ as shown.

(b) (i) Write down an expression which represents the **total** area shaded.

(ii) Calculate this total area. [5]

(c) (i) The y-coordinate of R is -0.240. Find the y-coordinate of S.

(ii) Hence or otherwise, find the range of values of k for which the equation $(x-2)\sin(x-1) = k$ has **four** distinct solutions. [4]

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49. [Maximum mark: 16] ***[with GDC]***

The function f is defined by $f : x \mapsto -0.5x^2 + 2x + 2.5$.

(a) Write down (i) $f'(x)$; (ii) $f'(0)$ [2]

(b) Let *N* be the normal to the curve at the point where the graph intercepts the y-axis. Show that the equation of *N* may be written as $y = -0.5x + 2.5$. [3]

Let $g : x \mapsto -0.5x + 2.5$

(c) (i) Find the solutions of $f(x) = g(x)$.

(ii) Hence find the coordinates of the other point of intersection of the normal and the curve. [6]

(d) Let *R* be the region enclosed between the curve and *N*.

(i) Write down an expression for the area of *R*.

(ii) Hence write down the area of *R*. [5]

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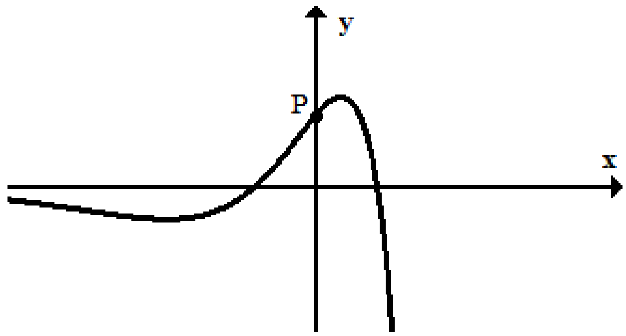
50. [Maximum mark: 13] ***[with GDC]***

Let $f(x) = e^x(1-x^2)$. Part of the graph of $y = f(x)$, for $-6 \le x \le 2$, is shown below.



(a) Write down an expression for the area enclosed by the curve and x-axis. [3]

(b) Find the coordinates of the y-intercept P. [1]

(c) Let *L* be the normal to the curve at P. Show that *L* has equation $x + y = 1$. [4]

(d) Let *R* be the region enclosed by the curve $y = f(x)$ and the line *L*.

(i) Find an expression for the area of *R*.

(ii) Calculate the area of *R*. [5]

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51. [Maximum mark: 19] ***[with GDC]***

The function f is defined as $f(x)=(2x+1)\mathrm{e}^{-x}$, $0\le x\le 3$. The point P(0, 1) lies on the graph of $f(x)$, and there is a maximum point at Q.

(a) Sketch the graph of $y=f(x)$, labelling the points P and Q. [3]

(b) (i) Show that $f'(x)=(1-2x)\mathrm{e}^{-x}$.

(ii) Find the **exact** coordinates of Q. [7]

(c) The equation $f(x)=k$, where $k\in\mathbb{R}$, has two solutions. Write down the range of values of k. [2]

(d) Let R be the point on the curve of f with x-coordinate 3. Find the area of the region enclosed by the curve and the line (PR). [7]

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52. [Maximum mark: 15] ***[with GDC]***

Let $f(x)=5\cos\frac{\pi}{4}x$ and $g(x)=-0.5x^2+5x-8$, for $0\le x\le 9$.

(a) On the same diagram, sketch the graphs of f and g. [3]

(b) Consider the graph of f. Write down

(i) the x-intercept between $x=0$ and $x=3$; (ii) the period; (iii) the amplitude. [4]

(c) Consider the graph of g. Write down

(i) the two x-intercepts; (ii) the equation of the axis of symmetry.

(d) Let *R* be the region enclosed by the graphs of f and g. Find the area of *R*. [6]

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53. [Maximum mark: 11] ***[without GDC]***

Let $h(x)=\dfrac{3x-5}{x-2}$, $x\neq 2$.

(a) (i) **Sketch** the graph of h for $-3\leq x\leq 7$ and $-2\leq y\leq 8$, including asymptotes.

(ii) Write down the **equations** of the asymptotes. [5]

(b) The expression $\dfrac{3x-5}{x-2}$ may also be written as $3+\dfrac{1}{x-2}$. Use this to find

(i) $\int h(x)\,\mathrm{d}x$.

(ii) the **exact** value of $\int_3^5 h(x)\mathrm{d}x$. [5]

(c) On your sketch, shade the region whose area is represented by $\int_3^5 h(x)\mathrm{d}x$. [1]

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54. [Maximum mark: 10] ***[with GDC]***

Let $f(x)=x(x-5)^2$, for $0\le x\le 6$. The diagram 1 shows the graph of f.

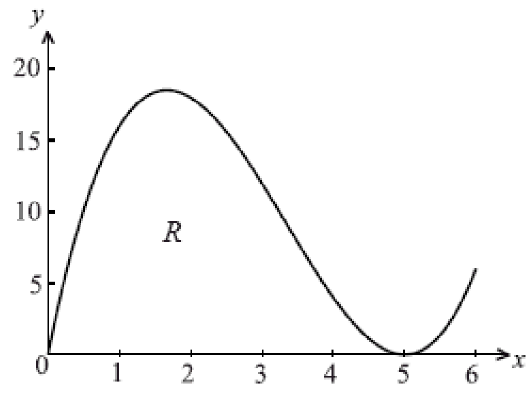

Diagram 1

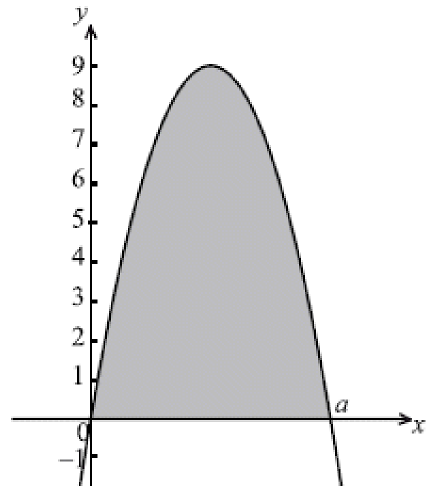

Diagram 2

Let *R* be the region enclosed by the x-axis and the curve of f.

(a) Find the area of *R*. [3]

(b) The diagram 2 shows a part of the graph of a quadratic function $g(x)=x(a-x)$.

The graph of g crosses the x-axis when $x=a$.

The area of the shaded region is equal to the area of *R*. Find the value of a. [7]

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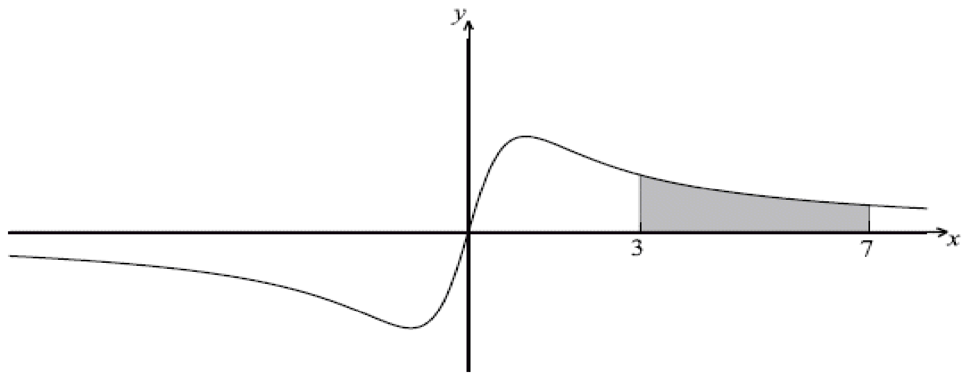
55. [Maximum mark: 16] ***[without GDC]***

Let $f(x)=\dfrac{ax}{x^2+1}$, $-8\le x\le 8$, $a\in\mathbb{R}$. The graph of f is shown below.



(a) Show that $f(-x)=f(x)$. [2]

(b) Given that $f''(x)=\dfrac{2ax(x^2-3)}{(x^2+1)^3}$, find the coordinates of all points of inflexion. [7]

(c) It is given that $\int f(x)\mathrm{d}x=\dfrac{a}{2}\ln(x^2+1)+C$.

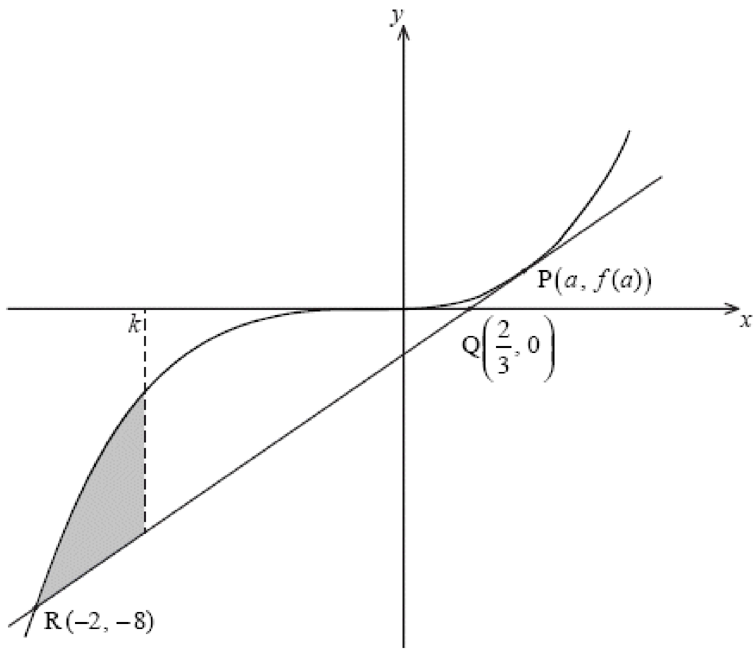
(i) Find the area of the shaded region, giving your answer in the form $p\ln q$.

(ii) Find the value of $\int_4^8 2f(x-1)\mathrm{d}x$. [7]

56. [Maximum mark: 16] ***[without GDC]***

Let $f(x)=x^3$.

The point P $(a, f(a))$, where $a>0$, lies on the graph of f. The tangent at P crosses the x-axis at the point $\mathrm{Q}\left(\frac{2}{3}, 0\right)$ and intersects the graph of f at the point $\mathrm{R}(-2,-8)$ as shown in the diagram below.



(a) (i) Show that the gradient of [PQ] is $\dfrac{a^3}{a-\frac{2}{3}}$.

(ii) Find $f'(x)$ and hence $f'(a)$ in terms of a.

(iii) Hence show that $a=1$. [7]

The equation of the tangent at P is $y=3x-2$.

Let *T* be the region enclosed by the graph of f, the tangent [PR] and the line $x=k$, between $x=-2$ and $x=k$ where $-2<k<1$. This is shown in the diagram above.

(b) Given that the area of *T* is $2k+4$, show that k satisfies the equation $k^4-6k^2+8=0$. [9]

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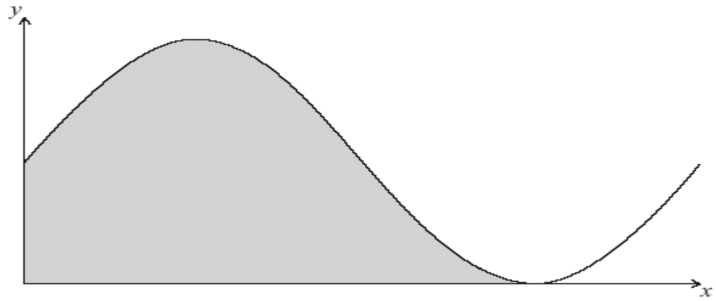
57. [Maximum mark: 17] ***[without GDC]***

Let $f(x) = 6 + 6\sin x$. Part of the graph of f is shown below.



The shaded region is enclosed by the curve of f, the x-axis, and the y-axis.

(a) Solve for $0 \le x < 2\pi$. (i) $6 + 6\sin x = 6$; (ii) $6 + 6\sin x = 0$. [5]

(b) Write down the exact value of the x-intercept of f, for $0 \le x < 2\pi$. [1]

(c) The area of the shaded region is k. Find the value of k, in terms of π. [6]

Let $g(x) = 6 + 6\sin\left(x - \frac{\pi}{2}\right)$. The graph of f is transformed to the graph of g.

(d) Give a full geometric description of this transformation. [2]

(e) Given that $\int_p^{p+\frac{3\pi}{2}} g(x)\mathrm{d}x = k$ and $0 \le p < 2\pi$, write down the two values of p. [3]

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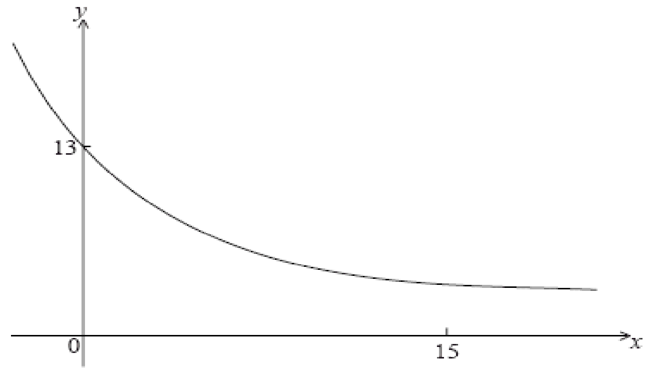
58. [Maximum mark: 16] ***[with GDC]***

Let $f(x) = Ae^{kx} + 3$. Part of the graph of f is shown below. The y-intercept is at (0,13).



(a) Show that $A = 10$. [2]

(b) Given that $f(15) = 3.49$ (correct to 3 significant figures), find the value of k. [3]

(c) (i) Using your value of k, find $f'(x)$.

(ii) Hence, explain why f is a decreasing function.

(iii) Write down the equation of the horizontal asymptote of the graph f. [6]

Let $g(x) = -x^2 + 12x - 24$.

(d) Find the area enclosed by the graphs of f and g. [6]

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59. [Maximum mark: 14] ***[with GDC]***

(a) On the same axes sketch the graphs of the functions, $f(x)$ and $g(x)$, where

$f(x) = 4 - (1 - x)^2$, for $-2 \le x \le 4$,

$g(x) = \ln(x + 3) - 2$, for $-3 < x \le 5$. [2]

(b) (i) Write down the equation of any vertical asymptotes.

(ii) State the x-intercept and y-intercept of $g(x)$. [3]

(c) Find the values of x for which $f(x) = g(x)$. [2]

(d) Let A be the region where $f(x) \ge g(x)$ and $x \ge 0$.

(i) On your graph shade the region A.

(ii) Write down an integral that represents the area of A.

(iii) Evaluate this integral. [4]

(e) In the region A find the maximum vertical distance between $f(x)$ and $g(x)$. [3]

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60. [Maximum mark: 12] ***[with GDC]***

(a) Sketch and label the graphs of $f(x)=\mathrm{e}^{-x^2}$ and $g(x)=\mathrm{e}^{x^2}-1$ for $0\le x\le 1$, and shade the region A which is bounded by the graphs and the y-axis. [3]

(b) Let the x-coordinate of the point of intersection of the curves $y=f(x)$ and $y=g(x)$ be p. Without finding the value of p, show that

$$\frac{p}{2}<\text{(area of region } A\text{)}< p\,.$$ [4]

(c) Find the value of p correct to four decimal places. [2]

(d) Express the area of region A as a definite integral and calculate its value. [3]

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