# Lesson 15: Solving equations with rational numbers

## Goals

- Explain (orally and in writing) how to solve an equation of the form $x + p = q$ or $px = q$, where $p$, $q$, and $x$ are rational numbers.
- Generalise (orally) the usefulness of additive inverses and multiplicative inverses for solving equations of the form $x + p = q$ or $px = q$.
- Generate an equation of the form $x + p = q$ or $px = q$ to represent a situation involving rational numbers.

## Learning Targets

- I can solve equations that include rational numbers and have rational solutions.

## Lesson Narrative

The purpose of this lesson is to get students thinking about how to solve equations involving rational numbers. Earlier in KS3, students solved equations of the form $px = q$ and $x + p = q$ and saw that additive and multiplicative inverses (opposites and reciprocals) were useful for solving them. However, that work did not include equations with negative values of $p$ or $q$ or with negative solutions. This lesson builds on the ideas of the last lesson and brings together the work on equations done so far in KS3.

### Building On

- Understand solving an equation or inequality as a process of answering a question: which values from a specified set, if any, make the equation or inequality true? Use substitution to determine whether a given number in a specified set makes an equation or inequality true.
- Apply and extend previous understandings of operations with fractions to add, subtract, multiply, and divide rational numbers.

### Addressing

- Use variables to represent quantities in a real-world or mathematical problem, and construct simple equations and inequalities to solve problems by reasoning about the quantities.
- Solve word problems leading to equations of the form $px + q = r$ and $p(x + q) = r$, where $p$, $q$, and $r$ are specific rational numbers. Solve equations of these forms fluently. Compare an algebraic solution to an arithmetic solution, identifying the sequence of the operations used in each approach. For example, the perimeter of a rectangle is 54 cm. Its length is 6 cm. What is its width?

- Solve real-world and mathematical problems involving the four operations with rational numbers. Calculations with rational numbers extend the rules for manipulating fractions to complex fractions.

## Building Towards

- Solve real-life and mathematical problems using numerical and algebraic expressions and equations.

## Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Three Reads
- Compare and Connect
- Discussion Supports
- Number Talk
- Take Turns

## Required Preparation

Print and cut up cards from the Card Sort: Matching Inverses blackline master. Prepare 1 set of cards for every 2 students.

| | | | |
|---|---|---|---|
| Card Sort: Matching Inverses $\frac{2}{1}$ | Card Sort: Matching Inverses $-2$ | Card Sort: Matching Inverses $3$ | Card Sort: Matching Inverses $-3$ |
| Card Sort: Matching Inverses $\frac{5}{10}$ | Card Sort: Matching Inverses $-0.5$ | Card Sort: Matching Inverses $\frac{1}{3}$ | Card Sort: Matching Inverses $-\frac{1}{3}$ |
| Card Sort: Matching Inverses $4$ | Card Sort: Matching Inverses $-4$ | Card Sort: Matching Inverses $\frac{10}{2}$ | Card Sort: Matching Inverses $-5$ |
| Card Sort: Matching Inverses $0.25$ | Card Sort: Matching Inverses $-\frac{1}{4}$ | Card Sort: Matching Inverses $0.2$ | Card Sort: Matching Inverses $-0.2$ |
| Card Sort: Matching Inverses $10$ | Card Sort: Matching Inverses $-\frac{10}{1}$ | Card Sort: Matching Inverses $25$ | Card Sort: Matching Inverses $-25$ |

| | | | |
|---|---|---|---|
| Card Sort: Matching Inverses<br>$0.1$ | Card Sort: Matching Inverses<br>$-\frac{1}{10}$ | Card Sort: Matching Inverses<br>$\frac{4}{100}$ | Card Sort: Matching Inverses<br>$-\frac{4}{100}$ |

### Student Learning Goals

Let's solve equations that include negative values.

## 15.1 Number Talk: Opposites and Reciprocals

**Warm Up: 5 minutes**

The purpose of this number talk is to:

- Remind students that the sum of a number and a number of the same magnitude with the opposite sign is zero.
- Remind students that the product of a number and its reciprocal is one.
- Establish common vocabulary for referring to these numerical relationships.

There may not be time for students to share every possible strategy. Consider gathering only one strategy for the equations with one variable, and a few strategies for the equations with two variables, since these have many possible answers, and they serve to generalise the relationship and provide an opportunity to introduce or reintroduce relevant vocabulary.

### Instructional Routines

- Discussion Supports
- Number Talk

### Launch

Display one equation at a time. (When you get to $c \times d = 1$, ensure students understand that $c$ and $d$ represent different numbers.) Give students 30 seconds of quiet think time per problem and ask them to give a signal when they have an answer and a strategy. Follow with a whole-class discussion.

*Representation: Internalise Comprehension.* To support working memory, provide students with sticky notes or mini whiteboards.
*Supports accessibility for: Memory; Organisation*

### Anticipated Misconceptions

Some students may use the strategy of guess and check for equations with two variables and use inverse operations to solve equations with one variable. Ask these students what they notice about the numbers in the equation and encourage them to find the value of the variable without using inverse operations.

### Student Task Statement

The **variables** $a$ through $h$ all represent *different* numbers. Mentally find numbers that make each equation true.

$\frac{3}{5} \times \frac{5}{3} = a$

$7 \times b = 1$

$c \times d = 1$

$-6 + 6 = e$

$11 + f = 0$

$g + h = 0$

### Student Response

- $a = 1$
- $b = \frac{1}{7}$
- Answers for $c$ and $d$ vary. Possible response: Possible response: $c = 3$ and $d = \frac{1}{3}$
- $e = 0$
- $f = -11$
- Answers for $g$ and $h$ vary. Possible response: $g = 5$ and $h = -5$

### Activity Synthesis

Ask students to share their reasoning for each problem. Record and display the responses for all to see.

If the following ideas do not arise as students share their reasoning, make these ideas explicit:

- The sum of a number and its opposite is 0.
- The product of a number and its reciprocal is 1.

- If you want to find a number that you can add to something and get 0 as a sum, use its opposite.
- If you want to find a number that you can multiply something by and get 1 as a product, use its reciprocal.

To involve more students in the conversation, ask some of the following questions:

- "Who can restate ___'s reasoning in a different way?"
- "Does anyone want to add on to _____'s strategy?"
- "Do you agree or disagree? Why?"

*Speaking: Discussion Supports.*: Display sentence frames to support students when they explain their strategy. For example, "First, I _____ because . . ." or "I noticed _____ so I . . . ." Some students may benefit from the opportunity to rehearse what they will say with a partner before they share with the whole class.
*Design Principle(s): Optimise output (for explanation)*

## 15.2 Match Solutions

**10 minutes (there is a digital version of this activity)**

Students solved equations of the form $x + p = q$ and $px = q$ earlier in KS3, but the equations only involved positive values. This activity bridges their understanding of a solution to an equation as a value that makes the equation true with their understanding of operations involving negative numbers from this unit. This activity builds on the work students have done in this lesson evaluating expressions at different values.

Monitor for students who:

- take an arithmetic approach by substituting in values and evaluating. (Does -2 × (-4.5) = -9?)
- take an algebraic approach by writing an equivalent equation using an inverse operation (If -2 × $x$ = -9, then $x$ = -9 ÷ -2.)

### Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Compare and Connect

### Launch

Allow students 5 minutes quiet work time followed by whole class discussion.

The digital version has an applet that allows students to see how many correct answers they have at any time.

*Action and Expression: Internalise Executive Functions.* To support development of organisational skills, check in with students within the first 2-3 minutes of work time. Check to make sure students matched the correct solution to the first equation and can explain their reasoning with an arithmetic or algebraic approach.
*Supports accessibility for: Memory; Organisation*

### Student Task Statement

1. Match each equation to its solution.

    a. $\frac{1}{2}x = -5$

    b. $-2x = -9$

    c. $-\frac{1}{2}x = \frac{1}{4}$

    d. $-2x = 7$

    e. $x + -2 = -6.5$

    f. $-2 + x = \frac{1}{2}$

1. $x = -4.5$
2. $x = -\frac{1}{2}$
3. $x = -10$
4. $x = 4.5$
5. $x = 2\frac{1}{2}$
6. $x = -3.5$

Be prepared to explain your reasoning.

### Student Response

a. 3

b. 4

c. 2

d. 6

e. 1

f. 5

Activity Synthesis

Select an equation for which there is a student who took an arithmetic approach and a student who took an algebraic approach. Ask students to share their reasoning for why a solution is correct. Sequence arithmetic approaches before algebraic approaches. An example of an arithmetic approach: “I know that -3.5 is the solution to $-2x = 7$, because I know that $-2 \times (-3.5) = 7$. An example of an algebraic approach: “If $-2x = 7$, then I know that $x = 7 \div -2$.) Record their work, side by side, for all to see. If there are no equations for which students took both approaches, present both approaches anyway. Tell students that either approach is valid, but that in the next unit they will see some more complicated equations for which one approach might be simpler than the other.

*Representing, Conversing, Listening: Compare and Connect.* Use this routine to support whole-class discussion. Ask students, “What is the same and what is different?” about how they matched the expressions. Connect strategies by showing the different ways operations are used in each approach (e.g., arithmetic method by substituting in values and evaluating; algebraic method by using an inverse operation). Use gestures, and colour on the display to highlight these connections. This helps students use mathematical language as they reason about their strategies to evaluate equivalent equations.
*Design Principle(s): Maximise meta-awareness; Support sense-making*

## 15.3 Trip to the Mountains

**10 minutes**

In this activity, students interpret equations that represent situations. The purpose is for students to see that equations of the form $x + p = q$ can be solved by adding the opposite of $p$ to the equation, regardless of whether $p$ is positive or negative. Students also see that equations of the form $px = q$ can be solved by multiplying the equation by the reciprocal of $p$. Through this work, students see that the structure of equations can be used to reason about a path to a solution even when negative values are included or when a variable can represent a negative number.

### Instructional Routines

- Three Reads

### Launch

Give students 5–6 minutes of quiet work time followed by whole-class discussion.

*Action and Expression: Internalise Executive Functions.* Chunk this task into more manageable parts to support students who benefit from support with organisational skills in problem solving. For example, present one question at a time and monitor students to ensure they are making progress throughout the activity.
*Supports accessibility for: Organisation; Attention Reading: Three Reads.* Use this routine to help students interpret the representations of the situation in this activity. Use the first read to help students comprehend the different stages of situation (e.g., hiking a mountain,

temperature falls, cost of the trip) without using numbers. Use the second read to identify the important quantities by asking students what can be counted or measured (e.g., increased height above sea level 290 feet, now at 450 feet; temperature fell 4 degrees, now is at 32 degrees; 3 times as many students this year, now 42 students; cost is $\frac{2}{3}$ of last year's, cost £32 this year). For the third read, ask students to brainstorm possible strategies to answer the given questions.
*Design Principle(s): Support sense-making*

### Anticipated Misconceptions

Students may be misled by words to add or multiply (or subtract or divide) by the wrong numbers. For example, the word "increased" in question 1 may lead students to simply add the numbers they see, while the words "three times as many" in question 4 may lead students to multiply the numbers in the problem. Encourage students to make sense of the situations by acting them out or using visual diagrams, which will help them understand the actions and relationships in the stories.

### Student Task Statement

The Hiking Club is on a trip to hike up a mountain.

1. The members increased their height above sea level 290 feet during their hike this morning. Now they are at a height of 450 feet.
    a. Explain how to find their height above sea level before the hike.
    b. Han says the equation $e + 290 = 450$ describes the situation. What does the variable $e$ represent?
    c. Han says that he can rewrite his equation as $e = 450 + -290$ to solve for $e$. Compare Han's strategy to your strategy for finding the beginning height above sea level.
2. The temperature fell 4 degrees in the last hour. Now it is 21 degrees. Write and solve an equation to find the temperature it was 1 hour ago.
3. There are 3 times as many students participating in the hiking trip this year than last year. There are 42 students on the trip this year.
    a. Explain how to find the number of students that came on the hiking trip last year.
    b. Mai says the equation $3s = 42$ describes the situation. What does the variable $s$ represent?
    c. Mai says that she can rewrite her equation as $s = \frac{1}{3} \times 42$ to solve for $s$. Compare Mai's strategy to your strategy for finding the number of students on last year's trip.

4. The cost of the hiking trip this year is $\frac{2}{3}$ of the cost of last year's trip. This year's trip cost £32. Write and solve an equation to find the cost of last year's trip.

### Student Response

1. Answers vary. Sample responses:

    a. Subtract the change of 290 from the current height above sea level of 450.

    b. $e$ represents the starting height above sea level

    c. Han used a variable to represent the unknown quantity, wrote an equation to describe the situation, and then solved by adding the opposite.

2. 25 degrees. $t - 4 = 21, t - 4 + 4 = 21 + 4, t = 25$

3. Answers vary. Sample responses:

    a. Divide this year's number by 3, or multiply this year's number by $\frac{1}{3}$.

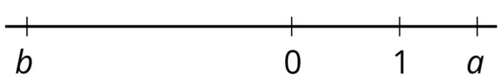
    b. $s$ represents the number of students on the trip last year.

    c. Mai used a variable to represent the unknown quantity, wrote an equation to describe the situation, and then solved by multiplying by the reciprocal.

4. 48. $\frac{2}{3}c = 32, c = 32 \times \frac{3}{2}, c = 48$

### Are You Ready for More?

A number line is shown below. The numbers 0 and 1 are marked on the line, as are two other rational numbers $a$ and $b$ .




Decide which of the following numbers are positive and which are negative.

$a - 1$

$a - 2$

$-b$

$a + b$

$a - b$

$ab + 1$

### Student Response

1. $a - 1$ is positive because $a$ is bigger than 1

2. $a - 2$ is negative because the distance between 1 and $a$ is less than the distance between 0 and 1. So $a$ is less than 2.

3. -$b$ is positive because $b$ is a negative number.

4. $a + b$ is negative because the distance from $b$ to zero is greater than the distance from 0 to $a$.

5. $a - b$ is positive because $b$ is a negative value.

6. $ab + 1$ is negative because both $a$ and $b$ have magnitudes greater than 1, so $ab$ will have magnitude greater than 1. Also, $ab$ will be negative as $a$ is positive and $b$ is negative. So $ab <$ -1.

### Activity Synthesis

Tell students: “We learned four things about the hiking trip in this activity: the students were climbing, the temperature was falling, there were more students this year than last, and the cost of the trip was less this year than last.” Then ask them:

- “Think about how you knew what operation described the rise in height above sea level, fall in temperature, rise in number of students, and fall in the cost. How did you know whether the situation used adding or multiplying?” (This conversation can highlight the problem with relying on “key words” . For example, when students see “times as many” , they might want to multiply the numbers they see in the problem. Encourage students to make sense of the situations by acting them out or drawing diagrams.)

- “How did you decide how to solve for the unknown quantity?”

- “What are some ways to know that a situation involves negative values?”

## 15.4 Card Sort: Matching Inverses

**Optional: 10 minutes**

The blackline master is a set of matching cards with fractions and integers. The students first recall the work from the previous section about additive inverses by matching them. They then match multiplicative inverses.

When matching multiplicative inverses, students should now use the fact that division follows the same structure as multiplication to identify that negative numbers require a negative inverse and positive numbers require a positive inverse. Monitor for students who use this step to make an initial sort.

## Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Discussion Supports
- Take Turns

## Launch

Remind students of the meaning of additive inverse and multiplicative inverse. If $x + y = 0$, then $x$ and $y$ are additive inverses. If $x \times y = 1$, then $x$ and $y$ are multiplicative inverses. Display these definitions for all to see and use as a reference while they work through this activity. Ask students for some examples of additive inverses, and add these to the display (for example, 7 and -7, because $7 + \text{-}7 = 0$). Ask students for some examples of multiplicative inverses, and add these to the display (for example, $\frac{1}{7}$ and 7, because $\frac{1}{7} \times 7 = 1$).

If necessary, demonstrate productive ways for partners to communicate during a matching activity. For example, partners take turns identifying a match and explaining why they think it is a match. The other partner either accepts their explanation, or explains why they don't think it's a match. Then they change roles for the next match.

Arrange students in groups of 2. Distribute one set of paper slips per group. Instruct students to first match the numbers that are additive inverses. Then, they will re-sort the same cards into pairs of multiplicative inverses. Consider asking students to pause their work after matching the additive inverses for discussion before proceeding to match multiplicative inverses.

*Conversing: Discussion Supports.* Display sentence frames to support students as they explain their reasoning for each match. For example, "___ matches ___ because . . .", and "I know that ___ and ___ are additive/multiplicative inverses because . . ." Encourage students to respond to the matches their partner makes using, "I agree/disagree, because . . ."
*Design Principle(s): Support sense-making; Cultivate conversation*

## Anticipated Misconceptions

Students might need a reminder of the difference between the additive inverse and multiplicative inverse.

## Student Task Statement

Your teacher will give you a set of cards with numbers on them.

1. Match numbers with their additive inverses.
2. Next, match numbers with their multiplicative inverses.
3. What do you notice about the numbers and their inverses?

### Student Response

Additive Inverses:

$\frac{2}{1}$ and -2; $\frac{5}{10}$ and -0.5; 3 and -3; $\frac{1}{3}$ and -$\frac{1}{3}$; 4 and -4; 0.25 and -$\frac{1}{4}$; $\frac{10}{2}$ and -5; 0.2 and -0.2; 10 and -$\frac{10}{1}$; 0.1 and -$\frac{1}{10}$; 25 and -25; $\frac{4}{100}$ and -$\frac{4}{100}$

Multiplicative Inverses:

2 and $\frac{5}{10}$; -2 & -0.5; 3 and $\frac{1}{3}$; -3 and -$\frac{1}{3}$; 4 and 0.25; -4 and -$\frac{1}{4}$; $\frac{10}{2}$ and 0.2; -5 and -0.2; 10 and 0.1; -$\frac{10}{1}$ and -$\frac{1}{10}$; 25 and $\frac{4}{100}$; -25 and -$\frac{4}{100}$

### Activity Synthesis

The most important thing to recognise is that multiplicative inverses require that the numbers have the same sign in order for the product to be positive, and so negative numbers require a negative multiplicative inverse and positive numbers require a positive inverse. Contrast this with additive inverses, which must have opposite signs in order for their sum to be 0. Select students that used that strategy, as well as some who used calculation, to share their thinking and draw out this conclusion.

## Lesson Synthesis

In this lesson students represented situations with equations and used inverses as a strategy to solve them. Bring the activities together by asking students:

- How can we solve an equation like $x + (-9.2) = 7.5$? (We can add the opposite of -9.2 to 7.5.)
- How can we solve an equation like $x \times (-9.2) = 7.5$? (We can multiply 7.5 by the reciprocal of -9.2.)
- Suppose we know that 60 is $\frac{4}{5}$ of a number. What is the difference between writing the equation $\frac{4}{5}x = 60$ and writing the equation $x = 60 \times \frac{5}{4}$? (The first equation describes the situation while the second shows a way to rewrite the equation to solve for the unknown.)

# 15.5 Hiking Trip

**Cool Down: 5 minutes**

### Student Task Statement

The Hiking Club is taking another trip. The hike leader's watch shows that they gained 296 feet in altitude from their starting position.

Their altitude is now 285 feet, but there is no record of their starting altitude.

Write and solve an equation to represent this situation and find their starting altitude.

### Student Response

$x + 296 = 285$; $x = -11$

## Student Lesson Summary

To solve the equation $x + 8 = -5$, we can add the opposite of 8, or -8, to each side:

Because adding the opposite of a number is the same as subtracting that number, we can also think of it as subtracting 8 from each side.

$$\begin{aligned} x + 8 &= -5 \\ (x+8) + -8 &= (-5) + -8 \\ x &= -13 \end{aligned}$$

We can use the same approach for this equation:

$$\begin{aligned} -12 &= t + -\frac{2}{9} \\ (-12) + \frac{2}{9} &= \left(t + -\frac{2}{9}\right) + \frac{2}{9} \\ -11\frac{7}{9} &= t \end{aligned}$$

To solve the equation $8x = -5$, we can multiply each side by the reciprocal of 8, or $\frac{1}{8}$:

Because multiplying by the reciprocal of a number is the same as dividing by that number, we can also think of it as dividing by 8.

$$\begin{aligned} 8x &= -5 \\ \frac{1}{8}(8x) &= \frac{1}{8}(-5) \\ x &= -\frac{5}{8} \end{aligned}$$

We can use the same approach for this equation:

$$\begin{aligned} -12 &= -\frac{2}{9}t \\ -\frac{9}{2}(-12) &= -\frac{9}{2}\left(-\frac{2}{9}t\right) \\ 54 &= t \end{aligned}$$

## Glossary

- variable

## Lesson 15 Practice Problems

1. **Problem 1 Statement**

   Solve.

   a. $\frac{2}{5}t = 6$

   b. $\text{-}4.5 = a - 8$

   c. $\frac{1}{2} + p = \text{-}3$

   d. $12 = x \times 3$

   e. $\text{-}12 = \text{-}3y$

   **Solution**

   a. $t = 15$

   b. $a = 3.5$

   c. $p = \text{-}3\frac{1}{2}$

   d. $x = 4$

   e. $y = 4$

2. **Problem 2 Statement**

   Match each equation to a step that will help solve the equation.

   a. $5x = 0.4$

   b. $\frac{x}{5} = 8$

   c. $3 = \frac{\text{-}x}{5}$

   d. $7 = \text{-}5x$

   1. Multiply each side by 5.
   2. Multiply each side by -5.
   3. Multiply each side by $\frac{1}{5}$.
   4. Multiply each side by $\frac{\text{-}1}{5}$.

   **Solution**

- A: 3
- B: 1
- C: 2
- D: 4

3. **Problem 3 Statement**

   Evaluate each expression if $x$ is $\frac{2}{5}$, $y$ is -4, and $z$ is -0.2.

   a. $x + y$

   b. $2x - z$

   c. $x + y + z$

   d. $y \times x$

   **Solution**

   a. $-3\frac{3}{5}$ (or equivalent)

   b. 1

   c. -3.8 (or equivalent)

   d. $\frac{-8}{5}$ (or equivalent)

4. **Problem 4 Statement**

   a. Write an equation where a number is added to a variable, and a solution is -8.

   b. Write an equation where a number is multiplied by a variable, and a solution is $\frac{-4}{5}$.

   **Solution**

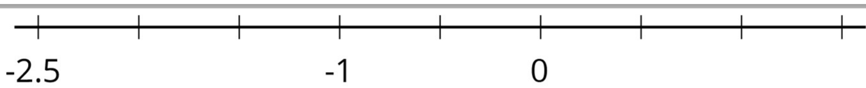
   Answers vary. Sample responses:

   a. $x + 2 = -6$

   b. $-5x = 4$

5. **Problem 5 Statement**

   The markings on the number line are evenly spaced. Label the other markings on the number line.




**Solution**

-2.5, -2, -1.5, -1, -0.5, 0, 0.5, 1, 1.5

6. **Problem 6 Statement**

In 2012, James Cameron descended to the bottom of Challenger Deep in the Marianas Trench; the deepest point in the ocean. The vessel he rode in was called DeepSea Challenger.

Challenger Deep is 35 814 feet deep at its lowest point

a. DeepSea Challenger's descent was a change in depth of (-4) feet per second. We can use the equation $y = -4x$ to model this relationship, where $y$ is the depth and $x$ is the time in seconds that have passed. How many seconds does this model suggest it would take for DeepSea Challenger to reach the bottom?

b. To end the mission DeepSea Challenger made a one-hour ascent to the surface. How many seconds is this?

c. The ascent can be modelled by a different proportional relationship $y = kx$. What is the value of k in this case?

**Solution**

a. 8 953.5 seconds, because $-35\,814 \div -4 = 8\,953.5$

b. 3 600 seconds, because $60 \times 60 = 3\,600$

c. It took 3 600 seconds to go 35 814 feet up. This means the proportional relationship is $y = \frac{35\,814}{3\,600}x$.