Lesson 12: How much will fit?

Goals

- Draw a cylinder and label its height and radius, describe (in writing) the shape of the "base" of the shape.
- Estimate the volumes of various containers using different units of measure, and explain (orally) the reasoning.

Learning Targets

- I know that volume is the amount of space contained inside a three-dimensional shape.
- I recognise the 3D shapes cylinder, cone, cuboid, and sphere.

Lesson Narrative

The purpose of this lesson is to remind students of the tangible meaning of volume: that it's the amount of space contained in a three-dimensional shape. Students estimate the amount of stuff different containers hold, recalling units of measurement commonly used for volume, like fluid ounces, cups, litres, gallons, cubic feet, and cubic centimetres (also known as millilitres). They revisit the names of shapes learned prior to this unit: cylinders, cones, cuboids, and spheres, and see some physical containers that can be modelled with these. It is important for students to make these connections between physical and mathematical objects so that, later on, real-world objects can be modelled with idealised shapes.

Students also learn a method for quickly drawing a cylinder. Later in the unit, they also learn quick methods for sketching a cone and a sphere. This skill was included both because it is a handy thinking tool to have access to in problem solving and also because it helps students better understand the meaning of terms like radius and height as they apply to these mathematical objects.

Addressing

- Solve real-world and mathematical problems involving volume of cylinders, cones, and spheres.

Instructional Routines

- Compare and Connect
- Discussion Supports
- Poll the Class

Required Preparation

Consider bringing in containers, dried rice, and measuring tools for the What's Your Estimate activity. It is also recommended that you have various-sized solid objects for students to pass around during the Do You Know These Shapes activity.

Student Learning Goals

Let’s reason about the volume of different shapes.

12.1 Two Containers

Warm Up: 5 minutes

In the previous lesson, students studied the relationship between volume of liquid and the height of the liquid when poured into a cylindrical container. The purpose of this warm-up is to shift students’ attention toward other types of containers and to consider how the volume of two containers differs. This warm-up is direct preparation for the first activity of the lesson in which students reason about volumes of several container types and re-familiarise themselves with the language of three-dimensional objects.

Instructional Routines

- Poll the Class

Launch

Tell students to close their books or devices. Arrange students in groups of 2. Display the image of the two containers filled with beans for all to see.

Give partners 1 minute to estimate how many beans are in each container. Poll the class for their estimates and display these values for all to see, in particular the range of values expressed.

Tell students that the smaller container holds 200 beans. Ask students to open their books or devices and reconsider their estimate for the large container now that they have more information. Give 1–2 minutes for students to write down a new estimate. Follow with a whole-class discussion.

Anticipated Misconceptions

Some students may not be sure how to start estimating the amount of beans in the larger jar once the number of beans in the smaller jar is known. Encourage them to start by estimating how the dimensions (e.g., height, width) of the two containers compare. For example, since the larger jar is more than twice the height of the smaller and has a greater width, then it must have at least twice as many beans.

Student Task Statement

Your teacher will show you some containers. The small container holds 200 beans. Estimate how many beans the large jar holds.

Student Response

Sample response: approximately 1 000 beans.

Activity Synthesis

Poll the class for their new estimates for the number of beans in the larger container and display these next to the original estimates for all to see. Tell the class that the large container actually holds about 1 000 beans.

Discuss:

- “How did you and your partner calculate your estimate for the large jar?” (We estimated the large jar holds 900 beans since the large jar is about 3 times taller than the smaller jar, and it’s about 1.5 times wider and $200 \times 3 \times 1.5 = 900$.)
- “Is there a more accurate way to measure the difference in volume between the two containers than ‘number of beans.’” (Yes, we could use something smaller than beans so there is less air, such as rice or water.)
- “What are some examples of units used to measure volume? Where have you seen them used in your life?” (Cups, tablespoons, gallons, litres, cubic centimetres, etc. Drinks often have fluid ounces, gallons, or litres written on them. Recipes may use cups or tablespoons.)

12.2 What’s Your Estimate?

15 minutes

The purpose of this activity is for students to practise using precise language to describe how they estimated volumes of objects. Starting from an object of known volume, students must consider the difference in dimensions between the two objects. The focus here is on strategies to estimate the volume and units of measure used, not on exact answers or calculating volume using a formula (which will be the focus of later lessons). Notice students who:

- have clear strategies to estimate volume of contents inside container

- have an estimate that is very close to the actual volume

Instructional Routines

- Discussion Supports

Launch

Arrange students in groups of 2.

Option 1: Bring in real containers and have students estimate how much rice each would hold, one at a time, preferably with one container whose volume is stated so students have a visual reference for their estimates. Also bring plenty of dried rice and measuring tools, such as tablespoons or cups. After collecting students' estimates, you can demonstrate how much rice each container holds using whichever units of measure the class deems reasonable. Note that 1 tablespoon is 0.5 ounces or around 15 millilitres. 1 cup is 8 ounces or around 240 millilitres. 1 millilitre is the same as 1 cubic centimetre.

Option 2: Display images one at a time for all to see. Give students 1–2 minutes to work with their partner and write down an estimate for the objects of unknown volumes in the picture. Follow with a whole-class discussion.

1. 

2. 
100% NATURAL
old fashioned
rolled oats
ORGANIC
Pumpkin
CHUNK WHITE
ALBACORE TUNA

3. 

4. 

Representation: Internalise Comprehension. Provide access to 3-D models of the objects depicted in the images for students to manipulate. Ask students to use the 3-D models to estimate the quantities in the other shapes. Encourage students to place objects with known volumes inside or next to others to ensure an accurate estimate.
Supports accessibility for: Visual-spatial processing; Conceptual processing

Anticipated Misconceptions

Students may think there is a single right answer. Measurements are always approximate. Some of the measurements given by the authors of this activity were calculated using estimates from the photos, and may not be very precise. Measurements listed on the sides of packages are more accurate, but actual contents may vary slightly.

Student Task Statement

Your teacher will show you some containers.

1. If the pasta box holds 8 cups of rice, how much rice would you need for the other cuboids?

2. If the pumpkin can holds 15 fluid ounces of rice, how much do the other cylinders hold?

3. If the small **cone** holds 2 fluid ounces of rice, how much does the large cone hold?

4. If the golf ball were hollow, it would hold about 0.2 cups of water. If the baseball were hollow, how much would the **sphere** hold?

Student Response

Some answers are more exact than others.

1. pudding: 1.5 cups; pasta: 8 cups; chicken stock: 32 fluid ounces (or 4 cups); answers are from the information on the container.

2. tuna: 5.5 fluid ounces; oatmeal: $5\frac{2}{3}$ cups (or about 45 fluid ounces); pumpkin: 15 fluid ounces; answers are from the information on the container.

3. cone with uneven top: approximately 3 fluid ounces (holds more if tilted); cone with smooth top: approximately 2 fluid ounces.

4. baseball: approximately 0.9 cup; golf ball: approximately 0.2 cup.

Activity Synthesis

For each set of containers, display the image and select previously identified students to share their strategies for estimating the volume. Once strategies for each set of containers are shared, discuss:

- "How do the estimates differ if we measure using water verses rice?" (Measuring with rice leaves a bit of empty space between the grains, while water, being liquid, leaves no empty space, so it's more accurate.)

- "If the containers we used were much larger (like a water tank), would our units of measure change? Why?" (If we were measuring larger volumes, we might want to use a larger unit, like gallons. 4 000 ml sounds big, but it's only a bit more than 1 gallon, which isn't that much water.)

Conclude the discussion by asking students to compare some other units of measure for volume that they know of. Have students recall what they know about unit conversion between some units of measure. Example:

- Fluid ounces, quarts, cups, litres, millilitres

- Cubic feet, cubic metres, cubic yards

- Note that cubic centimetres are special, because 1 cc = 1 ml

If it comes up, here is the scoop on *ounces*: units called "ounces" are used to measure both volume and weight. It is important to be clear about what quantity you are measuring! To differentiate between them, people refer to the units of measure for volume as "fluid ounces." For water, 1 fluid ounce is very close to 1 ounce by weight. This is not true for other substances! For example, mercury is much denser than water. 1 fluid ounce of mercury weighs about 13.6 ounces! Motor oil is less dense than water (that's why it floats), so 1 fluid ounce of oil weighs only about 0.8 ounces. The metric system is not so confusing for quantities that would be measured in ounces, since it's common to measure mass instead of weight and measure it in grams, whereas volume is measured in millilitres.

Speaking: Discussion Supports. Use this routine to support whole-class discussion. For each strategy that is shared, ask students to restate and/or revoice what they heard using mathematical language. Consider providing students time to restate what they hear to a partner, before selecting one or two students to share with the class. Ask the original speaker if their peer was accurately able to restate their thinking. Call students' attention to any words or phrases that helped to clarify the original statement. This will provide more students with an opportunity to speak as they make sense of the reasoning of others.
Design Principle(s): Support sense-making

12.3 Do You Know These Shapes?

10 minutes

The purpose of this activity is for students to learn or remember the names of the shapes they worked with in the previous activity and learn a quick method for sketching a cylinder. Students start by determining the shapes that are the faces of the four shapes. They also determine which shape would be considered the base of each shape shown. This allows students to connect previously learned two-dimensional shapes to new three-dimensional shapes introduced here. The last question introduces students to a way to sketch a cylinder. This is a skill they will continue to use throughout the unit when working on problems that do not provide a visual example of a situation. Identify students who sketch cylinders that are different sizes or drawn sideways.

Instructional Routines

- Compare and Connect

Launch

It is strongly recommended that you provide physical, solid objects for students to hold and look at. If using physical objects, pass around the objects for students to see and feel before starting the activity.

Give students 3–5 minutes of quiet work time, followed by a whole-class discussion.

Anticipated Misconceptions

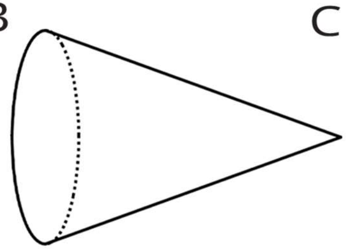
If students struggle to visualise the shapes of the faces, position the object so that students can only see two dimensions. Ask students what two-dimensional shape they see.

Student Task Statement

- What shapes are the faces of each type of object shown here? For example, all six faces of a cube are squares.

A

B



C

D

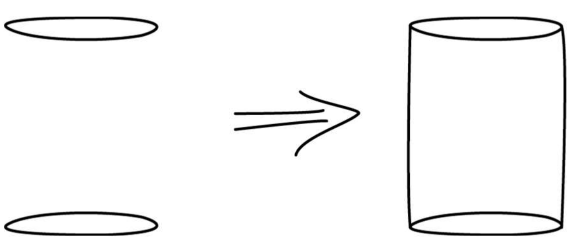
1. Which faces could be referred to as a “base” of the object?

2. Here is a method for quickly sketching a cylinder:

- Draw two ovals.

- Connect the edges.

- Which parts of your drawing would be hidden behind the cylinder? Make these parts dashed lines.



- Practise sketching some cylinders. Sketch a few different sizes, including short, tall, narrow, wide, and sideways. Label the radius r and height h on each cylinder.

Student Response

cuboid: rectangles; cone: circle and a curved surface; cylinder: 2 circles and a curved surface; sphere: one curved surface.

1. cuboid: any rectangular face can be called the base; cone: the circular face is the base; cylinder: either circular face can be called the base; sphere: has no base.

2. Answers vary.

Are You Ready for More?

A soccer ball is a polyhedron with 12 black pentagonal faces and 20 white hexagonal faces. How many edges in total are on this polyhedron?

Student Response

Since each pentagonal face has 5 edges and there are 12 pentagonal faces, there are 60 edges on pentagonal faces. Similarly, there are $20 \times 6 = 120$ edges on the 20 hexagonal faces. Combined, this would make 180 edges, except that every edge is counted twice by this process. Dividing by 2, we conclude that there are 90 edges on a soccer ball.

Activity Synthesis

If using physical objects, display each object one at a time for all to see. If using images, display the images for all to see, and refer to each object one at a time. Ask students to identify:

- which shape the object is an example of
- the different shapes that make up the faces of the shape
- the shape that is the base of the shape

Select previously identified students to share their sketches of cylinders. The goal is to ensure that students see a variety of cylinders: short, tall, sideways, narrow, etc. If no student drew a "sideways" cylinder, sketch one for all to see and make sure students understand that even though it is sideways, the height is still the length perpendicular to the base.

Tell students that we will be working with these different three-dimensional shapes for the rest of this unit. Consider posting a display in the classroom that shows a diagram of each object labelled with its name, and where appropriate, with one side labelled "base." As volume formulas are developed, the formulas can be added to the display.

Representing: Compare and Connect. Invite students to do a gallery walk, to observe each other's sketches. As students circulate, ask students to describe what is similar and what is different about the cylinders to a partner. Draw students' attention to the different

characteristics of the cylinders (e.g., laying sideways, short, narrow, long, etc.). In this discussion, demonstrate precise ways to describe shapes, such as using the terms cylinders, cones, spheres, and cuboids. These exchanges strengthen students' mathematical language use and reasoning about cylinders.
Design Principle(s): Optimise output (for comparison); Cultivate conversation

Lesson Synthesis

The volume of a three-dimensional shape is the amount of space it encloses. Ask students:

- "What are some shapes you worked with in today's lesson?" (Cylinders, cones, spheres, cuboids)
- "What are some different units of measure we use to calculate volume of these shapes?" (Cubic feet, fluid ounces, gallons, cubic nanometres, rice)
- "What are some examples of objects we see in our world that are very similar to these shapes that you didn't see in the pictures earlier?" (A basketball is like a sphere. A mobile phone is like a cuboid.)

12.4 Rectangle to Round

Cool Down: 5 minutes

Student Task Statement

Here is a box of pasta and a cylindrical container. The two objects are the same height, and the cylinder is just wide enough for the box to fit inside with all 4 vertical edges of the box touching the inside of the cylinder. If the box of pasta fits 8 cups of rice, estimate how many cups of rice will fit inside the cylinder. Explain or show your reasoning.

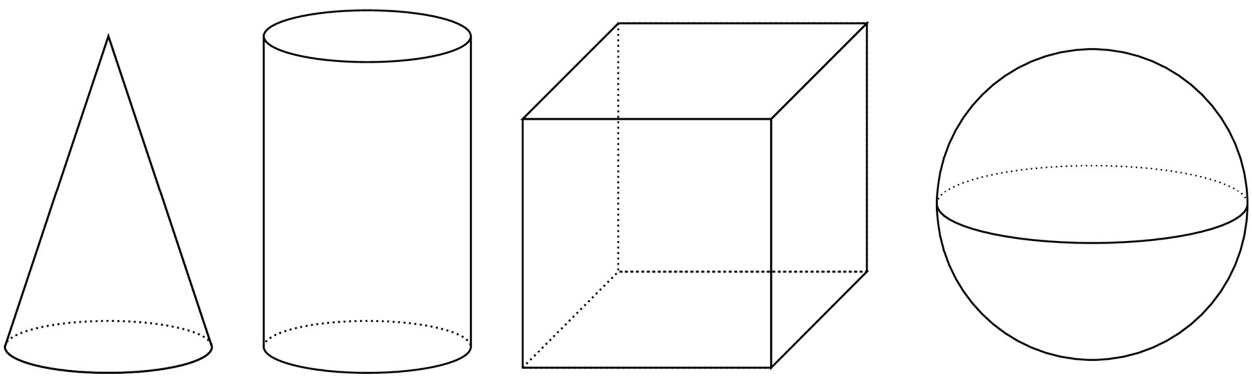
Student Response

Answers vary. Sample response: About 11 cups of rice since it should be a little more than the box, but not a lot.

Student Lesson Summary

The volume of a three-dimensional shape, like a jar or a room, is the amount of space the shape encloses. We can measure volume by finding the number of equal-sized volume units that fill the shape without gaps or overlaps. For example, we might say that a room has a volume of 1 000 cubic feet, or that a bucket can carry 5 gallons of water. We could even measure volume of a jar by the number of beans it could hold, though a bean count is not really a measure of the volume in the same way that a cubic centimetre is because there is space between the beans. (The number of beans that fit in the jar do depend on the volume of the jar, so it is an okay estimate when judging the relative sizes of containers.)

In earlier years, we studied three-dimensional shapes with flat faces that are polygons. We learned how to calculate the volumes of cuboids. Now we will study three-dimensional shapes with circular faces and curved surfaces: cones, cylinders, and spheres.



To help us see the shapes better, we can use dotted lines to represent parts that we wouldn't be able to see if a solid physical object were in front of us. For example, if we think of the cylinder in this picture as representing a tin can, the dotted arc in the bottom half of that cylinder represents the back half of the circular base of the can. What objects could the other shapes in the picture represent?

Glossary

- cone
- sphere

Lesson 12 Practice Problems

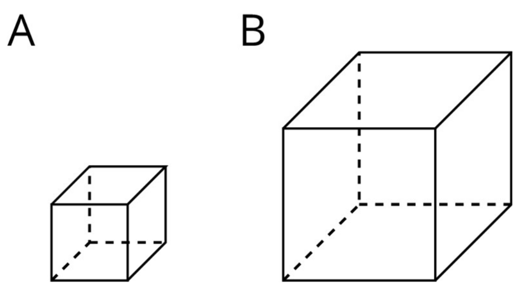
1. **Problem 1 Statement**

   a. Sketch a cube and label its side length as 4 cm (this will be cube A).

   b. Sketch a cube with sides that are twice as long as cube A and label its side length (this will be cube B).

   c. Find the volumes of cube A and cube B.

   Solution



   a. Cube A, with a labelled side length of 4 cm

   b. Cube B, with a labelled side length of 8 cm

   c. Cube A: 64 cm^3, cube B: 512 cm^3

2. **Problem 2 Statement**

   Two paper drink cups are shaped like cones. The small cone can hold 6 oz of water. The large cone is $\frac{4}{3}$ the height and $\frac{4}{3}$ the diameter of the small cone. Which of these could be the amount of water the large cone holds?

   a. 8 cm

   b. 14 oz

   c. 4.5 oz

   d. 14 cm

   Solution B

3. **Problem 3 Statement**

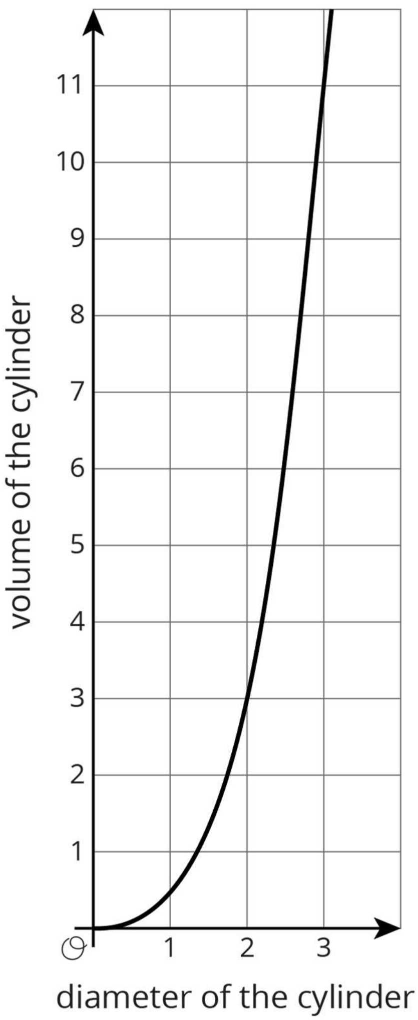
   The graph represents the volume of a cylinder with a height equal to its radius.

   a. When the diameter is 2 cm, what is the radius of the cylinder?

   b. Express the volume of a cube of side length s as an equation.

c. Make a table for volume of the cube at $s = 0$ cm, $s = 1$ cm, $s = 2$ cm, and $s = 3$ cm.

d. Which volume is greater: the volume of the cube when $s = 3$ cm, or the volume of the cylinder when its diameter is 3 cm?



Solution

a. The radius is 1 cm.

b. s^3

c.

s	volume of cube
0 cm	0 cm^3
1 cm	1 cm^3

2 cm	8 cm^3
3 cm	27 cm^3

d. The volume of the cube at $s = 3$ cm. (Its volume is 27 cm^3, while the volume of the cylinder when its diameter is 3 cm is $\frac{27\pi}{8}$ cm^3, or about 10.6 cm^3.)

4. **Problem 4 Statement**

Select **all** the points that are on a line with gradient 2 that also contains the point (2,-1).

a. (3,1)

b. (1,1)

c. (1,-3)

d. (4,0)

e. (6,7)

Solution ["A", "C", "E"]

5. **Problem 5 Statement**

Several glass aquariums of various sizes are for sale at a pet shop. They are all shaped like cuboids. A 15-gallon tank is 24 inches long, 12 inches wide, and 12 inches tall. Match the dimensions of the other tanks with the volume of water they can each hold.

A. Tank 1: 36 inches long, 18 inches wide, and 12 inches tall

B. Tank 2: 16 inches long, 8 inches wide, and 10 inches tall

C. Tank 3: 30 inches long, 12 inches wide, and 12 inches tall

D. Tank 4: 20 inches long, 10 inches wide, and 12 inches tall

1. 5 gallons

2. 10 gallons

3. 20 gallons

4. 30 gallons

Solution

- A: 4
- B: 1
- C: 3

- D: 2

6. **Problem 6 Statement**

   Solve: $\begin{cases} y = \text{-}2x - 20 \\ y = x + 4 \end{cases}$

   Solution

   (-8,-4)