Lesson 11: Writing equations for lines

Goals

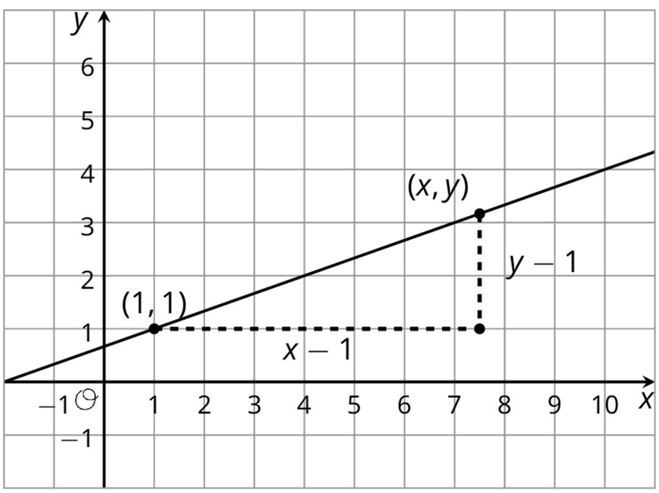
- Create an equation relating the quotient of the vertical and horizontal side lengths of a gradient triangle to the gradient of a line.
- Justify (orally) whether a point is on a line by finding quotients of horizontal and vertical distances.

Learning Targets

- I can decide whether a point is on a line by finding quotients of horizontal and vertical distances.

Lesson Narrative

The previous lesson introduces the idea of gradient for a line. In this lesson, the gradient is used to write a relationship satisfied by any point on a line. The key idea is to introduce a general or variable point on a line, that is a point with coordinates (x, y). These variables x and y can take any values as long as those values represent a point on the line. Because *all* gradient triangles lead to the same value of gradient, this general point can be used to write a relationship satisfied by all points on the line.



In this example, the gradient of the line is $\frac{1}{3}$ since the points (1,1) and (4,2) are on the line. The gradient triangle in the picture has vertical length $y - 1$ and horizontal length $x - 1$ so this gives the equation $\frac{y-1}{x-1} = \frac{1}{3}$ satisfied by *any* point on the line (other than (1,1)). This concise way of expressing which points lie on a line will be developed further in future units.

Building On

- Draw polygons in the coordinate plane given coordinates for the vertices; use coordinates to find the length of a side joining points with the same first coordinate or the same second coordinate. Apply these techniques in the context of solving real-world and mathematical problems.

- Recognise and represent proportional relationships between quantities.
- Understand congruence and similarity using physical models, transparencies, or geometry software.

Addressing

- Use similar triangles to explain why the gradient m is the same between any two distinct points on a non-vertical line in the coordinate plane; derive the equation $y = mx$ for a line through the origin and the equation $y = mx + b$ for a line intercepting the vertical axis at b.
- Understand congruence and similarity using physical models, transparencies, or geometry software.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect
- Stronger and Clearer Each Time
- Co-Craft Questions
- Think Pair Share

Student Learning Goals

Let's explore the relationship between points on a line and the gradient of the line.

11.1 Coordinates and Lengths in the Coordinate Plane

Warm Up: 5 minutes

The purpose of this warm-up is to ensure students understand that

- they can infer the coordinates of a point based on knowing the coordinates of points on the same horizontal and vertical lines;
- the length of a horizontal or vertical line segment can be determined based on the coordinates of its endpoints. Each of these is important background knowledge for this lesson.

Instructional Routines

- Think Pair Share

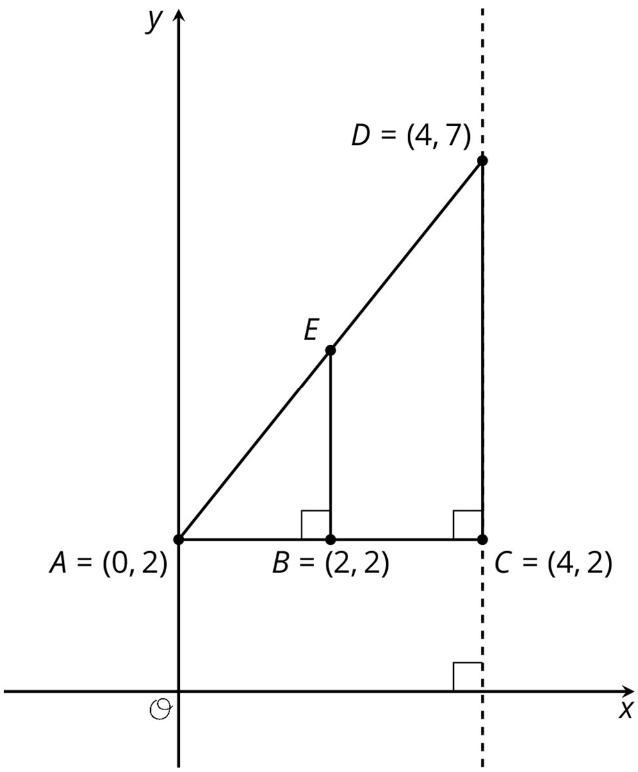
Launch

Give students 2 minutes of quiet think time. Ask them to share their reasoning with a partner followed by a whole-class discussion.

Anticipated Misconceptions

In order to find the length of line segment BE, it is first necessary to find the lengths of CD, AC, and AB, and reason about side lengths in similar triangles. If students have trouble getting started, scaffold the work by suggesting they first find any *other* line segment lengths in the diagram that they can find.

Student Task Statement



Find each of the following and explain your reasoning:

1. The length of line segment BE.

2. The coordinates of E.

Student Response

1. 2.5 or equivalent. Triangles ABE and ACD are similar by AA, so $\frac{AC}{AB} = \frac{CD}{BE}$.

2. (2,4.5). E has the same x-coordinate as B, and its y-coordinate is $2 + 2.5$.

Activity Synthesis

There are three important things students should understand or recall as a result of working on the warm-up:

- Points on the same vertical line have the same x-coordinate and points on the same horizontal line have the same y-coordinate.

- In order to find the length of a vertical line segment, you can subtract the y-coordinates of its endpoints.

- The side lengths of triangle ACD are the same as the side lengths of the similar triangle ABE multiplied by the scale factor $AC \div AB$.

Display the image from the task. For each question, invite a student to share their reasoning. The displayed image should be used as a tool for gesturing. For example, if you start at the origin and want to navigate to point D, you would move 4 units to the right. You move the same distance to the right to navigate from the origin to C. Therefore, C must have the same x-coordinate as D.

11.2 What We Mean by an Equation of a Line

10 minutes

Prior to this lesson, students have seen that right-angled triangles with a horizontal side, a vertical side, and a long side along the same line are all similar. This activity exploits this structure to examine the coordinates of points lying on a particular line. The discussion then produces an equation for the line. In the case where the line goes through $(0,0)$, the equation will be familiar from prior work with proportional relationships but in the next lesson similar triangles will be essential.

Monitor for different ways of answering the last question including

- With words and arithmetic: for example, divide the y-coordinate by the x-coordinate and see if it is equal to $\frac{3}{4}$.

- With words and proportional relationships: x and y are in a proportional relationship and when $x = 4$ we know $y = 3$.

- With an equation involving quotients of vertical and horizontal side lengths: for example, $\frac{y}{x} = \frac{3}{4}$.

Invite these students to share their reasoning during the discussion. If a student writes $y = \frac{3}{4}x$, this can be presented last but it is not essential that students see this now.

Instructional Routines

- Anticipate, Monitor, Select, Sequence, Connect

- Stronger and Clearer Each Time

Launch

Encourage students to think about what they know about gradient triangles from previous work. Give 2–3 minutes of quiet work time. Then ask students to share their responses and reasoning with a partner, followed by a whole-class discussion.

Action and Expression: Develop Expression and Communication. Invite students to talk about their ideas with a partner. Display sentence frames to support students when they describe their ideas. For example: "This point is/is not on the line because…."; "First I _____ because…."; and "Our strategies are the same/different because…."
Supports accessibility for: Language; Organisation Writing, Speaking, Listening: Stronger and Clearer Each Time. After students have had time to decide whether each of the three points lie on line j, ask students to write a brief explanation of their reasoning. Ask each student to meet with 2–3 other partners in a row for feedback. Provide students with prompts for feedback that will help them strengthen their ideas and clarify their language (e.g., "How did you determine the gradient of line j?", "How do you know this point is on line j?", and "How do you know this point is not on line j?", etc.). Students can borrow ideas and language from each partner to refine and clarify their original explanation. This will help students revise and refine both their reasoning and their verbal and written output.
Design Principles(s): Optimise output (for explanation); Maximise meta-awareness

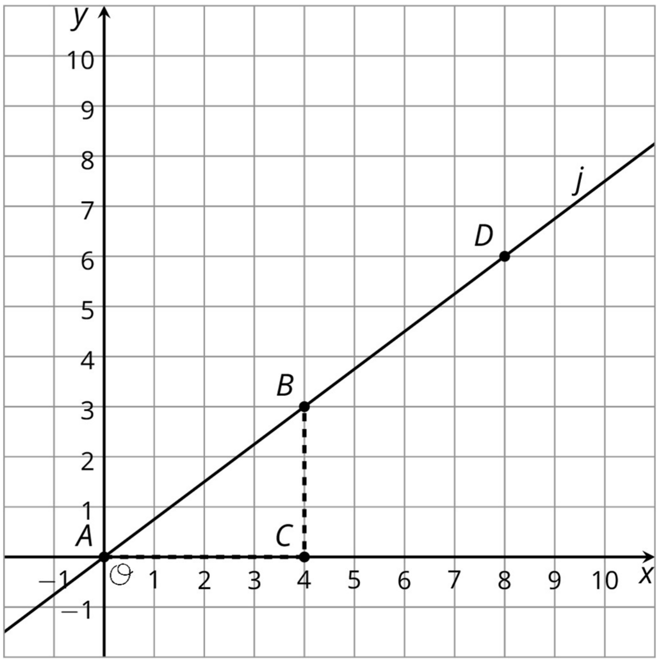
Anticipated Misconceptions

For question 5, there is a potential issue in saying all points on the line satisfy $\frac{y}{x} = \frac{3}{4}$, which is that this equation is not true for the point $(0,0)$. Unless a student notices this, it is not necessary to bring it up at this time. If necessary, we could say that the equation describes the relationship between the x- and y-coordinate for every point on the line except $(0,0)$.

Student Task Statement

Line j is shown in the coordinate plane.

1. What are the coordinates of B and D?
2. Is point $(20,15)$ on line j? Explain how you know.
3. Is point $(100,75)$ on line j? Explain how you know.
4. Is point $(90,68)$ on line j? Explain how you know.
5. Suppose you know the x- and y-coordinates of a point. Write a rule that would allow you to test whether the point is on line j.



Student Response

1. $B = (4,3)$ and $D = (8,6)$.

2. Yes. Explanations vary. Possible response: $\frac{15}{20}$ is equivalent to $\frac{3}{4}$ so the gradients are the same.

3. Yes. Explanations vary. Possible response: $\frac{75}{100} = \frac{3}{4}$ so the triangles are similar.

4. No. Explanations vary. Possible response: $\frac{68}{90}$ is not equal to $\frac{3}{4}$.

5. Answers vary. Students may write a rule in words like "the quotient of the y-coordinate and x-coordinate has to be 0.75."

Activity Synthesis

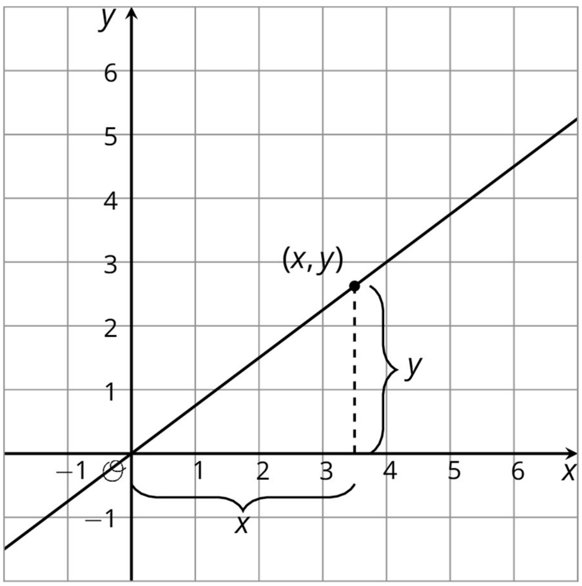
After reviewing how students answered the first four questions, have them share their responses for question 5. Sequence responses starting with the least abstract:

- Divide the y coordinate by the x coordinate and see if the result is equal to $\frac{3}{4}$. (Ask students to explain why this works: using similar triangles or gradient triangles, the key is that the quotient of the vertical length and horizontal length of these similar triangles always takes the same value.)
- The ratios $y: x$ and $3: 4$ are equivalent (since they represent corresponding sides of similar triangles *or* since the relationship between y and x is proportional and when $y = 3$ then $x = 4$)

- $\frac{y}{x} = \frac{3}{4}$. This is a different way of stating the relationship.

This structure of coordinates for points on a line will be examined in much greater detail in upcoming lessons. Students are just starting to learn what it means to write the equation of a line—it's not intended for them to understand everything there is to know, right now.

Make sure to draw a picture showing a point (x, y) on the line and label the vertical side with its length, x, and label the horizontal side with its length, y.



Ask students how the equation $\frac{y}{x} = \frac{3}{4}$ relates to this situation. According to the reasoning in this task, the point (x, y) is on the line when $\frac{y}{x} = \frac{3}{4}$. These values are equal because they both represent the quotient of the vertical and horizontal side of gradient triangles for the same line.

11.3 Writing Relationships from Gradient Triangles

15 minutes

In the previous activity, students found a rule which determines whether or not a point with coordinates (x, y) lies on a certain line: there were many ways to express this rule, including an equation such as $\frac{y}{x} = \frac{3}{4}$. In this activity, students find a rule to determine if a point (x, y) lies on a line no longer containing $(0,0)$. Proportional reasoning based on the graph of the line, which could be applied to the line in the previous activity, no longer applies here. But, proportional reasoning using similar gradient triangles does still apply and gives equations that look like $\frac{y}{x} = \frac{3}{4}$ except that the quotient on the left is a little more complex. Like in the previous activity, students are using the structure of a line and properties of similar triangles to investigate these rules relating coordinates of points on the line.

Monitor for different expressions students write in the second question. For example, for line k students may write

- $\frac{y-1}{x} = \frac{3}{4}$
- $4(y-1) = 3x$
- $4y = 3x + 4$

Invite students with different expressions to share during the discussion. Note that the rule or equation for the line is unlikely to come in the form $y = mx + b$. This is not important for now and will be addressed in future work. The important take away for this lesson is that we can write a criterion for when a point lies on a line by thinking about similar triangles and their properties.

Instructional Routines

- Co-Craft Questions

Launch

Tell students that next, we are going to find equations satisfied by points on some more lines. These lines do not represent proportional relationships, but we can still use what we know about similar triangles to find equations.

It really helps if students write the side lengths correctly. Therefore, ask students to complete the first question and then pause. Ensure that everyone knows and understands a correct expression for each side length before proceeding. Then, instruct students to complete the second question.

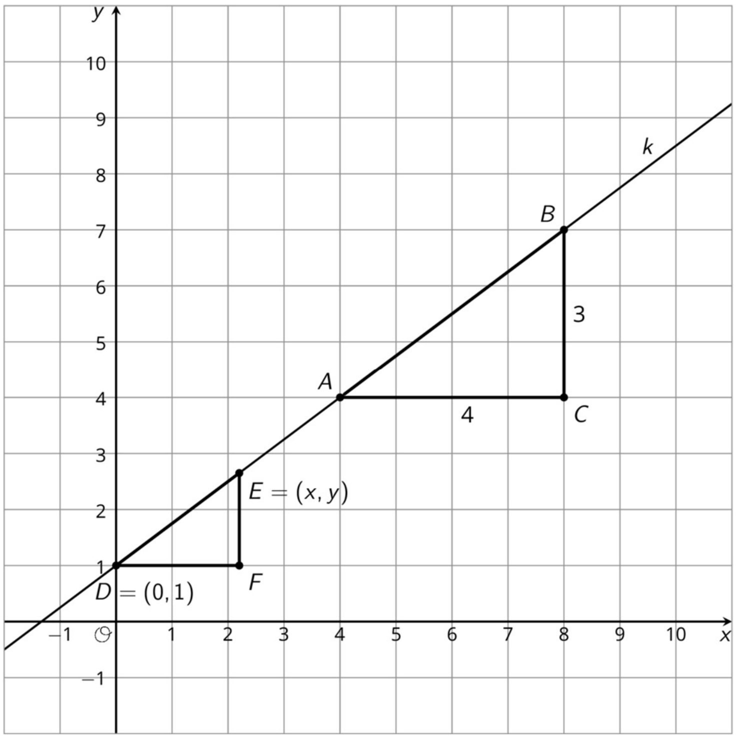
Engagement: Develop Effort and Persistence. Connect a new concept to one with which students have experienced success. For example, compare the graph in this activity to the previous one in which the line passed through the origin. Review the equation of a line going through $(0, 0)$ in order to provide an entry point into this activity.
Supports accessibility for: Social-emotional skills; Conceptual processing Conversing, Writing: Co-Craft Questions. Before presenting the questions for this activity, display the diagram of the line k and have students write possible mathematical questions about the gradient triangles along line k. Have students compare the questions they generated with a partner before sharing questions with the whole class. Listen for and amplify questions about creating a rule or an equation that determines whether a point with coordinates (x, y) lies on line k. Then reveal and ask students to work on the actual questions of the task. This routine will help develop students' meta-awareness of language as they generate questions about the gradient triangles of a line.
Design Principle(s): Maximise meta-awareness

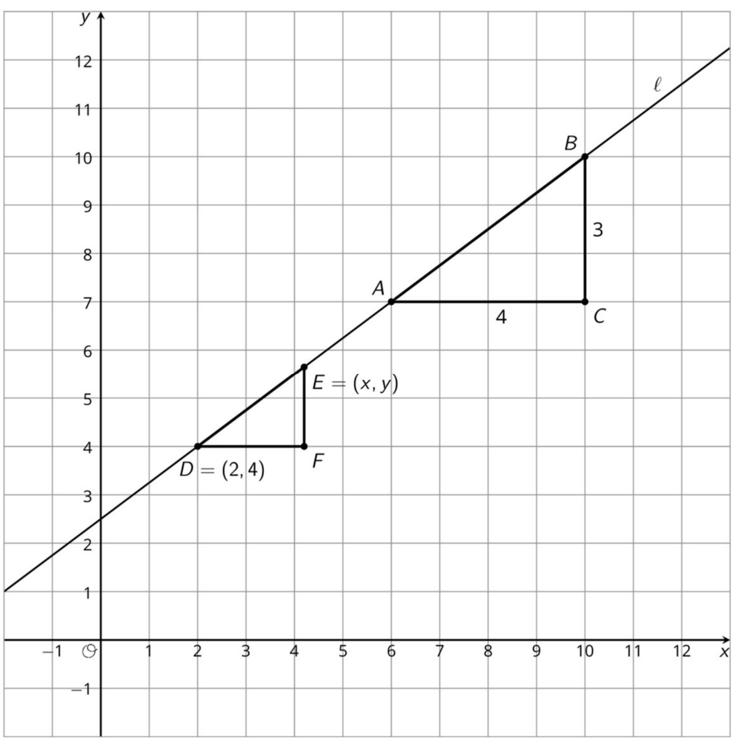
Student Task Statement

Here are two diagrams:

1. Complete each diagram so that all vertical and horizontal line segments have expressions for their lengths.

2. Use what you know about similar triangles to find an equation for the quotient of the vertical and horizontal side lengths of $\triangle DFE$ in each diagram.





Student Response

1. On graph of line k, line segment DF labelled x and line segment EF labelled $y-1$. On graph of line ℓ, line segment DF labelled $x-2$ and line segment EF labelled $y-4$.

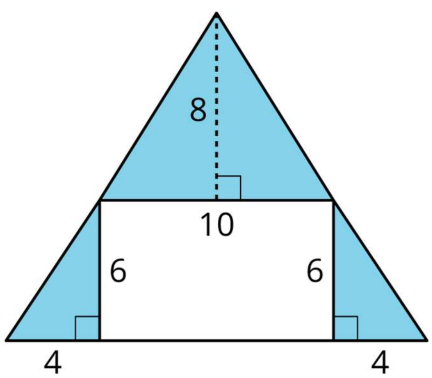
2. Answers vary. Possible responses: For line k, an equation satisfied by (x, y) is $\frac{y-1}{x}=\frac{3}{4}$. For line ℓ, an equation satisfied by (x, y) is $\frac{y-4}{x-2}=\frac{3}{4}$. These equations are true because the gradient of a line is always the same no matter which gradient triangle is used to calculate it.

Are You Ready for More?

1. Find the area of the shaded region by summing the areas of the shaded triangles.

2. Find the area of the shaded region by subtracting the area of the unshaded region from the large triangle.

3. What is going on here?



Student Response

1. 64 m^2

2. 66 m^2

3. The areas are different because the so-called "large triangle" is not actually a triangle. Each diagonal side is made up of two line segments with slightly different gradients. For example, the bottom left side has a gradient of $\frac{3}{2}$ while the top left side is slightly steeper, with a gradient of $\frac{8}{5}$. Therefore, the second method of finding the area was not valid for this shape.

Activity Synthesis

Ask selected students what they wrote for an equation satisfied by x and y for the two graphs, making sure to note different forms. One equation is $\frac{y-1}{x}=\frac{3}{4}$. There is no reason to manipulate this, but some students might rewrite this as $4y-4=3x$ or even $y=\frac{1}{4}(3x+4)$. For the second graph, students may write $\frac{y-4}{x-2}=\frac{3}{4}$ or $4(y-4)=3(x-2)$. At this point, the important thing to notice is that the coordinates for *any* point (x, y) on the line will satisfy this relationship. This is because we did not use any special properties of the point (x, y) (just that it lies on the line) to find the relationship. There is no reason to manipulate the equations $\frac{y-1}{x} = \frac{3}{4}$ or $\frac{y-4}{x-2} = \frac{3}{4}$ because these two equations contain all of the information from the similar triangles. (Once the equations are manipulated, this structure is lost, and it is this structure that is of central importance here.)

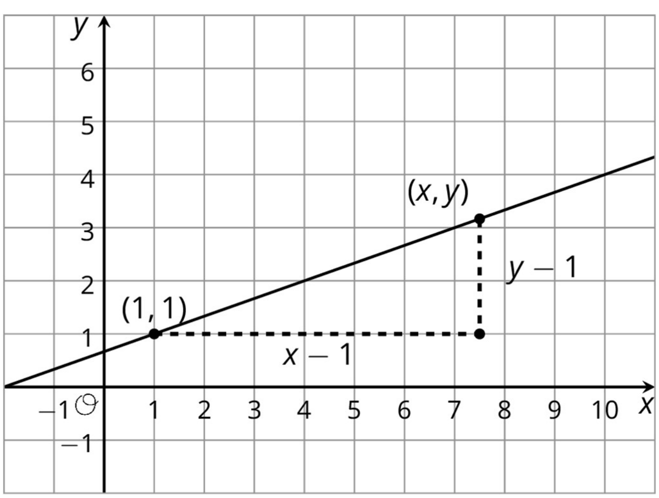
Make sure that students understand that the gradient of lines k and ℓ (as well as j from the previous activity) are all $\frac{3}{4}$. The location of the lines in the plane is different, and this is reflected in the different equations for the three lines. If the equations are in the form $\frac{y}{x} = \frac{3}{4}$, $\frac{y-1}{x} = \frac{3}{4}$, and $\frac{y-4}{x-2} = \frac{3}{4}$, we can see that what is the same in each is the $\frac{3}{4}$ and that there is a y in each numerator on the left and an x in the denominator.

Note, if needed, that the equation does not make sense if the denominator is equal to zero. For example, $\frac{y-1}{x} = \frac{3}{4}$ does not make sense if $x = 0$ and $y = 1$. The reason for this is that $(0,1)$ is the one point on the line where we can *not* build a gradient triangle together with the point $(0,1)$.

Time permitting, ask students how the equations for k, ℓ, and j are alike and how they are different. A key way they are alike is that they can all be written as some quotient involving y and x equal to $\frac{3}{4}$. The reason for the common value of $\frac{3}{4}$ is that the lines all have the same gradient, namely $\frac{3}{4}$. A key difference is the values subtracted from x and y on the left hand side of the equations: these are different because each line is in a different location on the coordinate plane.

Lesson Synthesis

Today, we used gradient triangles to find a relationship satisfied by the coordinates of *all* points on a line.



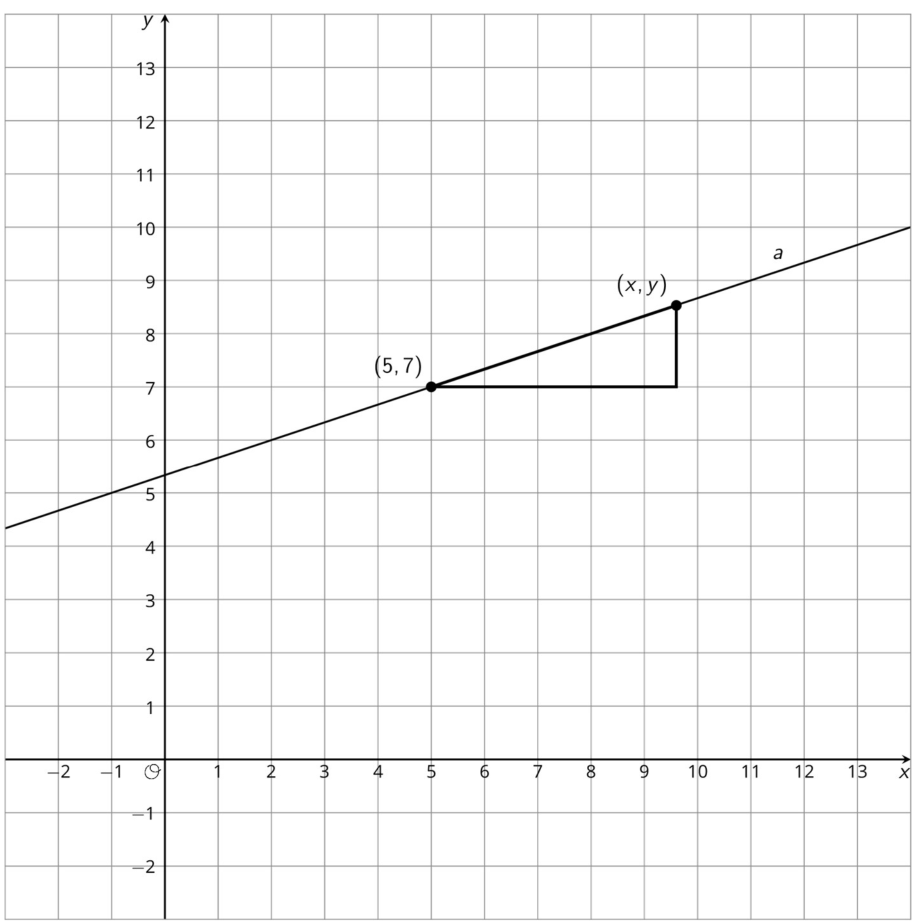
What is the gradient of this line? It's $\frac{2}{6}$ because the points (1,1) and (7,3) are on the line. The point (x, y) lies on this line so the gradient we calculate with the gradient triangle for (x, y) must also be equal to $\frac{2}{6}$. The gradient for this triangle is $\frac{y-1}{x-1}$ since the vertical side has length $y - 1$ and the horizontal side has length $x - 1$. This means that $\frac{y-1}{x-1} = \frac{2}{6}$ This relationship is true regardless of which point (x, y) we choose on the line!

11.4 Matching Relationships to Graphs

Cool Down: 5 minutes

Students use a gradient triangle one of whose points has variables as coordinates in order to write an equation satisfied by the variables (and therefore satisfied by the coordinates of any point on the line). The equation comes from using similar gradient triangles and identifying their gradients.

Student Task Statement



1. Explain why the gradient of line a is $\frac{2}{6}$.

2. Label the horizontal and vertical sides of the triangle with expressions representing their length.

3. Explain why $\frac{y-7}{x-5} = \frac{2}{6}$.

Student Response

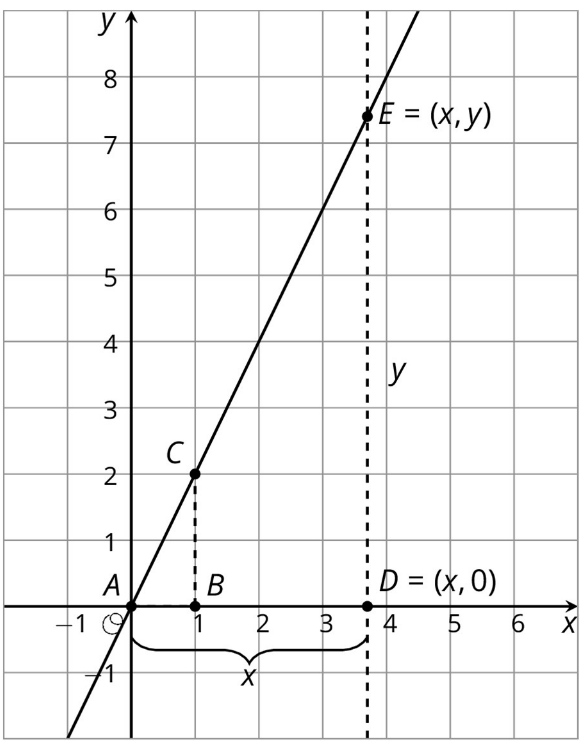
1. The points (5,7) and (11,9) are on the line. Using these to find the gradient gives a value of $\frac{2}{6}$.

2. The vertical side has length $y - 7$, and the horizontal side has length $x - 5$.

3. The triangle in the picture is a gradient triangle so the quotient of its vertical and horizontal sides has to be $\frac{2}{6}$. This means that $\frac{y-7}{x-5} = \frac{2}{6}$ is true for any point (x, y) on the line.

Student Lesson Summary

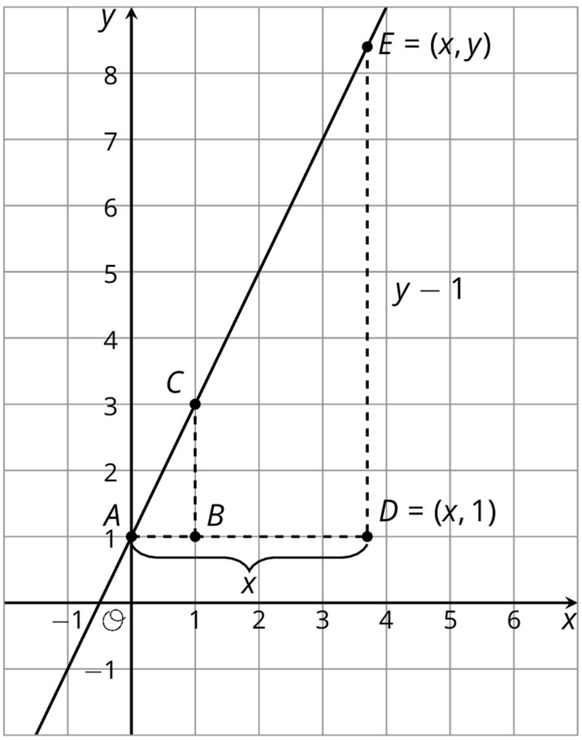
Here are the points A, C, and E on the same line. Triangles ABC and ADE are gradient triangles for the line so we know they are similar triangles. Let's use their similarity to better understand the relationship between x and y, which make up the coordinates of point E.



The gradient for triangle ABC is $\frac{2}{1}$ since the vertical side has length 2 and the horizontal side has length 1. The gradient we find for triangle ADE is $\frac{y}{x}$ because the vertical side has length y and the horizontal side has length x. These two gradients must be equal since they are from gradient triangles for the same line, and so: $\frac{2}{1} = \frac{y}{x}$.

Since $\frac{2}{1} = 2$ this means that the value of y is twice the value of x, or that $y = 2x$. This equation is true for any point (x, y) on the line!

Here are two different gradient triangles. We can use the same reasoning to describe the relationship between x and y for this point E.



The gradient for triangle ABC is $\frac{2}{1}$ since the vertical side has length 2 and the horizontal side has length 1. For triangle ADE, the horizontal side has length x. The vertical side has length $y - 1$ because the distance from (x, y) to the x-axis is y but the vertical side of the triangle stops 1 unit short of the x-axis. So the gradient we find for triangle ADE is $\frac{y-1}{x}$. The gradients for the two gradient triangles are equal, meaning: $\frac{2}{1} = \frac{y-1}{x}$

Since $y - 1$ is twice x, another way to write this equation is $y - 1 = 2x$. This equation is true for any point (x, y) on the line!

Lesson 11 Practice Problems

1. **Problem 1 Statement**

   For each pair of points, find the gradient of the line that passes through both points. If you get stuck, try plotting the points on graph paper and drawing the line through them with a ruler.

   a. (1,1) and (7,5)

   b. (1,1) and (5,7)

   c. (2,5) and (-1,2)

   d. (2,5) and (-7,-4)

   Solution

   a. $\frac{2}{3}$

   b. $\frac{3}{2}$

   c. 1

   d. 1

2. **Problem 2 Statement**

   Line ℓ is shown in the coordinate plane.

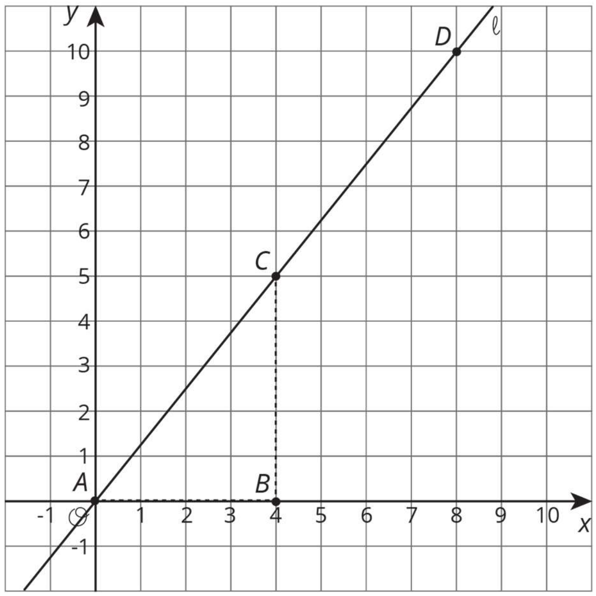
   a. What are the coordinates of points B and D?

   b. Is the point (16,20) on line ℓ? Explain how you know.

   c. Is the point (20,24) on line ℓ? Explain how you know.

   d. Is the point (80,100) on line ℓ? Explain how you know.

   e. Write a rule that would allow you to test whether (x, y) is on line ℓ.



Solution

a. $B = (4,0)$ and $D = (8,10)$

b. Yes, because $\frac{20}{16} = \frac{5}{4}$. The gradients are the same.

c. No, because $\frac{24}{20} \neq \frac{5}{4}$.

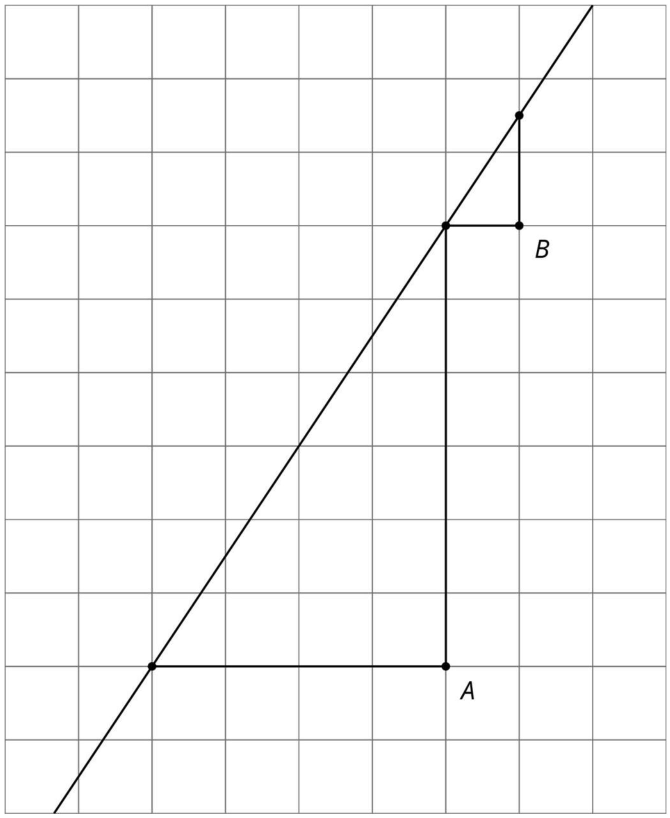
d. Yes, because $\frac{100}{80} = \frac{5}{4}$.

e. Answers vary. Sample response: $\frac{y}{x} = \frac{5}{4}$

3. **Problem 3 Statement**

Consider the graphed line.

Mai uses triangle A and says the gradient of this line is $\frac{6}{4}$. Elena uses triangle B and says no, the gradient of this line is 1.5. Do you agree with either of them? Explain.



Solution

They are both correct. The gradient of a line can be found using any right-angled triangle with shorter sides parallel to the axes and longest side on the line, as any two such triangles are similar. Numerically, this checks out as $\frac{6}{4}$ and 1.5 represent the same value.

4. **Problem 4 Statement**

A rectangle has length 6 and height 4.

Which of these would tell you that quadrilateral $ABCD$ is definitely *not* similar to this rectangle? Select **all** that apply.

a. $AB = BC$

b. $\angle ABC = 105°$

c. $AB = 8$

d. $BC = 8$

e. $BC = 2 \times AB$

f. $2 \times AB = 3 \times BC$

Solution ["A", "B", "E"]