Lesson 18: Comparing populations using samples

Goals

- Interpret a pair of box plots, including the amount of visual overlap between the two distributions.
- Justify (orally and in writing) whether there is likely to be a meaningful difference between two populations, based on a sample from each population.

Learning Targets

- I can calculate the difference between two medians as a multiple of the interquartile range.
- I can determine whether there is a meaningful difference between two populations based on a sample from each population.

Lesson Narrative

In previous lessons, students examined the distributions of two entire populations to decide whether or not they were very different. In this lesson, students use samples to make comparative inferences about populations.

Students see that if samples of two different populations have only a small difference between their measures of centre (relative to their variability), then we cannot say that there is a meaningful difference between the measures of centre of the populations. Due to sampling variability, it is possible that the two populations may not be very different. However, if samples from two different populations have a large difference between their measures of centre (relative to their variability), then we can say that there is likely to be a meaningful difference between the measures of centre of the two populations.

Building On

- Giving quantitative measures of centre (median and/or mean) and variability (interquartile range and/or range), as well as describing any overall pattern and any striking deviations from the overall pattern with reference to the context in which the data were gathered.

Addressing

- Informally assess the degree of visual overlap of two numerical data distributions with similar variabilities, measuring the difference between the centres by expressing it as a multiple of a measure of variability. For example, the mean height of players on the basketball team is 10 cm greater than the mean height of players on the soccer team, and compare this with the variability (range) on either team; on a dot plot, the separation between the two distributions of heights is noticeable.
- Use measures of centre and measures of variability for numerical data from random samples to draw informal comparative inferences about two populations. For example,

decide whether the words in a chapter of a year 8 science book are generally longer than the words in a chapter of a year 5 science book.

Building Towards

- Draw informal comparative inferences about two populations.

Instructional Routines

- Stronger and Clearer Each Time
- Co-Craft Questions
- Compare and Connect
- Think Pair Share

Student Learning Goals

Let's compare different populations using samples.

18.1 Same Mean? Same range?

Warm Up: 5 minutes

This warm-up reminds students of the meanings of mean and range by comparing two sets of data with similar but different values and asking whether they will have the same means or ranges or both.

Launch

Explain to students that the pairs of data sets are: A and B, X and Y, and P and Q.

Anticipated Misconceptions

For students who have a difficult time starting without calculating, help them to compare the values in the ones place for the first and third pairs of data.

Student Task Statement

Without calculating, tell whether each pair of data sets have the same mean and whether they have the same range.

1.

set A

1	3	3	5	6	8	10	14

set B

21	23	23	25	26	28	30	34

2.

set X

1	2	3	4	5

set Y

1	2	3	4	5	6

3.

set P

47	53	58	62

set Q

37	43	68	72

Student Response

1. Data sets A and B have different means, but the same ranges.
2. Data sets X and Y have different means and different ranges.
3. Data sets P and Q have the same means, but different ranges.

Activity Synthesis

The purpose of the discussion is to bring out methods students used to notice whether the pairs of data sets had the same mean or range or both.

Poll the class for each pair of data sets as to whether they had the same mean, range, both, or neither.

After students have had a chance to register their vote, ask some students to explain their reasoning for their answer.

18.2 With a Heavy Load

10 minutes

In a previous lesson, students compared heights of two teams of people when the entire populations were known. In this activity, students only have access to data from a sample of each population and are asked to determine if the populations are different based on the sample. Students construct informal arguments to explain why the different samples come from populations that are meaningfully different or not.

Instructional Routines

- Co-Craft Questions
- Think Pair Share

Launch

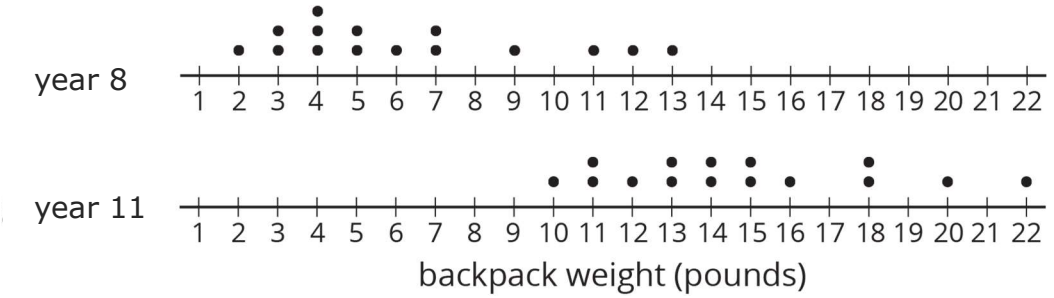
Arrange students in groups of 2. Allow students 3–5 minutes quiet work time followed by partner and whole-class discussions.

Writing, Conversing: Co-Craft Questions. Display the initial task statement and the dot plots for year 8 and year 11 without revealing the questions that follow. Ask pairs to write down possible mathematical questions that can be answered about the situation. As students share their questions with the class, listen for and amplify questions that ask about the mean weight of the backpacks and whether there is a meaningful difference between the weight of all year 8 and year 11 backpacks. This will help students write and verbalise their questions about the dot plots of the samples as they attempt to draw conclusions about the populations.
Design Principle(s): Optimise output; Cultivate conversation

Student Task Statement

Consider the question: Do year 11 students' backpacks generally weigh more than year 8 students' backpacks?

Here are dot plots showing the weights of backpacks for a random sample of students from these two year groups:



1. Did any year 8 backpacks in this sample weigh more than a year 11 backpack?
2. The mean weight of this sample of year 8 backpacks is 6.3 pounds. Do you think the mean weight of backpacks for *all* year 8 students is exactly 6.3 pounds?
3. The mean weight of this sample of year 11 backpacks is 14.8 pounds. Do you think there is a meaningful difference between the weight of all year 8 and year 11 students' backpacks? Explain or show your reasoning.

Student Response

1. Yes, three year 8 backpacks weighed more than the lightest year 11 backpack.

2. No. 6.3 pounds is the mean of a sample, and the population mean will probably be at least a little different.

3. Answers vary. Sample response: There is still probably a meaningful difference in the mean weights since there is very little overlap.

Activity Synthesis

The purpose of the discussion is for students to think about how comparing groups by using data from samples differs from comparing groups when the population is known.

Ask partners to share their decision about whether the groups had a meaningful difference with the class.

Consider asking these questions for discussion:

- "Compare the information in this activity to the information about team heights given in an earlier lesson." (In that lesson, we had data from the entire population and here it is only a sample.)
- "Is the overlap of the data more important when you only have a sample or when you have data from the population? Explain your reasoning." (It is more important when you only have a sample. If there is overlap with only some of the data, it's possible there is more overlap when we include more data from the population.)
- "Is it *possible* that the data in the two samples were drawn from population data that is identical?" (It is unlikely, but possible.)

Representation: Internalise Comprehension. Use colour and annotations to illustrate student thinking. As students describe their reasoning and the relationships they noticed for each sample, use colour and annotations to scribe their thinking on a display of each problem so that it is visible for all students.
Supports accessibility for: Visual-spatial processing; Conceptual processing

18.3 Do They Carry More?

15 minutes

The data in the previous activity came from only one sample for each year group. This may not be enough information to make a very good determination about the entire year 8 and year 11 populations. In this activity, students look at different samples from the same population to see that their means are relatively close based on the ranges of the samples. This concept can be reversed to say that if two samples have means that are *not* very close, then the samples likely came from populations that are quite different.

Instructional Routines

- Compare and Connect

Launch

Keep students in groups of 2. Allow students 5 minutes of partner work time, then pause the class to assign samples and explain the general rule.

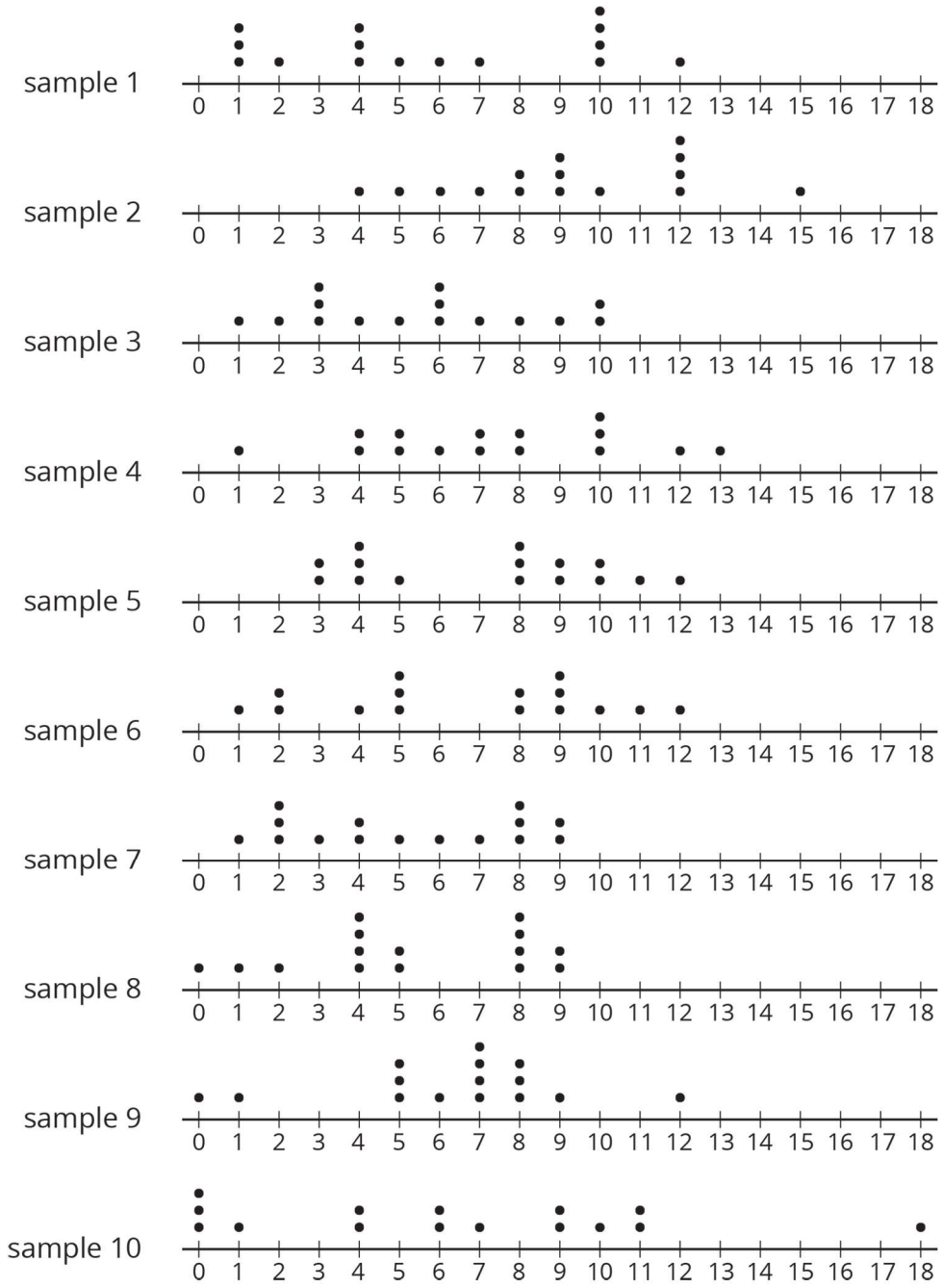
Ask students to pause after the third question in order to explain the general rule and assign a sample to each group. After all students have paused, assign each group one of the 10 samples to work with for the last 2 questions. Further, explain to students:

- As a general rule, if two populations have the same median and similar variability, the sample medians should be within 2 IQRs of one another.
- If the sample medians are more than 2 IQRs apart, it is very likely that the population medians are different. We will say that there is a *meaningful difference* between the two population medians.
- If the sample medians are less than or equal to 2 IQRs apart, it is more difficult to say that the two population medians are very different. In this case we will say that the samples do not provide evidence that the population medians differ.

Give students 5 more minutes of partner work time followed by a whole-class discussion.

Student Task Statement

Here are 10 more random samples of year 8 students' backpack weights.



sample number	mean weight (pounds)
1	5.8
2	9.2
3	5.5
4	7.3
5	7.2

6	6.6
7	5.2
8	5.3
9	6.3
10	6.4

1. a. Which sample has the highest mean weight?

   b. Which sample has the lowest mean weight?

   c. What is the difference between these two sample means?

2. Are these samples very different? Explain or show your reasoning.

3. Remember our sample of year 11 students' backpacks had a mean weight of 14.8 pounds. The range for this sample is 12 pounds. Your teacher will assign you one of the samples of year 8 students' backpacks to use.
   What is the difference between the sample means for the year 11 students' backpacks and the year 8 students' backpacks?

4. Do you think there is a meaningful difference between the weights of all year 8 and year 11 students' backpacks? Explain or show your reasoning.

Student Response

1. a. Sample 2

   b. Sample 7

   c. 4 pounds $(9.2 - 5.2 = 4)$

2. No. All of the samples came from the same population and have a similar distribution.

3. Answers vary. Possible responses:

sample	1	2	3	4	5	6	7	8	9	10
difference in means	9	5.6	9.3	7.5	7.6	8.2	9.6	9.5	8.5	8.4
range	11	11	9	12	9	11	8	9	12	11(18)

4. Yes. The difference in the means is large in comparison with the ranges, so the means have a meaningful difference.

Activity Synthesis

The purpose of the discussion is for students to understand the general ideas for determining if two samples suggest a meaningful difference between their populations.

Ask students, "Based only on the dot plots for the 10 samples, would you have guessed that they all might have come from the same population? Explain your reasoning." (Maybe. There is a lot of overlap among all of the samples.)

Representation: Develop Language and Symbols. Create a display of important terms and vocabulary. Invite students to suggest language or diagrams to include that will support their understanding. Include the following term and maintain the display for reference throughout the unit: meaningful difference.
Supports accessibility for: Language; Conceptual processing Representing, Speaking, Listening: Compare and Connect. Invite students to share their written responses to the last two questions with 2–3 other students. As students investigate each other's work, ask students to make observations about the difference in sample means for the year 11 and year 8 students' backpacks. Ask students how this observation helps them determine whether there is a meaningful difference between the weights of the backpacks. Listen for and amplify the language students use for determining whether two samples suggest a meaningful difference between their populations.
Design Principle(s): Optimise output (for representation); Maximise meta-awareness

18.4 Steel from Different Regions

15 minutes

In previous lessons, students used sample data to estimate population means and proportions and determined if there is a meaningful difference in population means based on sample means. In this activity, students practise using a general rule by estimating the measure of centre for a population and comparing populations based on those estimates as well as the associated measure of variability.

Instructional Routines

- Stronger and Clearer Each Time

Launch

Keep students in groups of 2.

Explain to students that different regions had different raw materials and techniques for constructing metal. One way of testing ancient metal is by looking at the carbon content in the steel. In some cases, this content could determine the region where the metal was made.

Explain to students how a general rule can be used with median and interquartile range (IQR). That is, if the difference between the medians is greater than 2 × IQR there is a meaningful difference between the distributions.

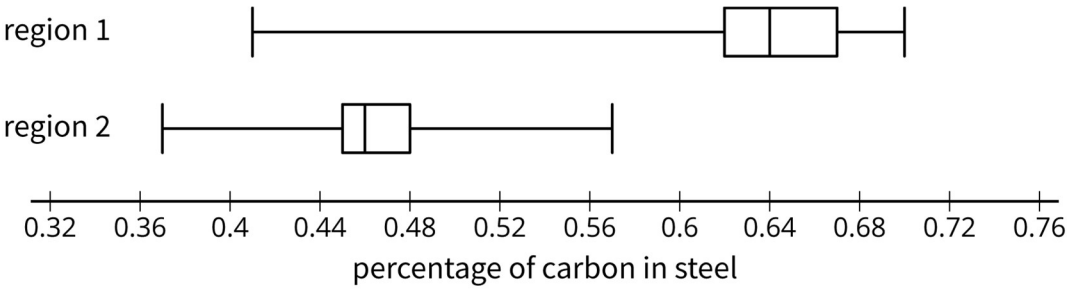
Allow students 10 minutes of partner work time followed by a whole-class discussion.

Action and Expression: Internalise Executive Functions. Chunk this task into more manageable parts to support students who benefit from support with organisational skills

in problem solving. For example, pause to check for understanding after 3–5 minutes of work time.
Supports accessibility for: Organisation; Attention Writing, Conversing: Stronger and Clearer Each Time. Ask students to write a response to the final question: "The anthropologists who conducted the study concluded there was a meaningful difference between the steel from these regions. Do you agree? Explain or show your reasoning." Ask each student to meet with 2–3 other partners for feedback. Provide students with prompts for feedback that will help them strengthen their ideas and clarify their language. For example, "What is the general rule for determining a meaningful difference between populations?" and "How do you know there is a meaningful difference between the steel from these regions?" Students can borrow ideas and language from each partner to refine and clarify their original explanation. This will help students revise and refine their ideas about how to use the general rule with median and IQR.
Design Principle(s): Optimise output (for explanation); Cultivate conversation

Student Task Statement

When anthropologists find steel artefacts, they can test the amount of carbon in the steel to learn about the people that made the artefacts. Here are some box plots showing the percentage of carbon in samples of steel that were found in two different regions:



1. Was there any steel found in region 1 that had:
   a. *more* carbon than some of the steel found in region 2?
   b. *less* carbon than some of the steel found in region 2?
2. Do you think there is a meaningful difference between all the steel artefacts found in regions 1 and 2?
3. Which sample has a distribution that is *not* approximately symmetrical?
4. What is the difference between the sample medians for these two regions?

	sample median (%)	IQR (%)
region 1	0.64	0.05
region 2	0.47	0.03

5. Express the difference between these two sample medians as a multiple of the larger interquartile range.

6. The anthropologists who conducted the study concluded that there was a meaningful difference between the steel from these regions. Do you agree? Explain or show your reasoning.

Student Response

1. a. Yes. Most of the steel from region 1 had more carbon in it than steel from region 2.

   b. Yes. Since the left end of the region 1 box plot overlaps with the box plot for region 2, there was at least 1 piece of steel that had less carbon in it than some of the steel from region 2.

2. Answers vary. Sample response: Based on the box plots, there is some overlap, but the boxes look so far apart that I think there will be a meaningful difference.

3. The distribution for region 1 is not symmetrical with the very long segment on the left.

4. 0.17%, since $0.64 - 0.47 = 0.17$

5. $0.17 \approx 0.05 \times 3.4$

6. I agree with the anthropologists. There is evidence of a meaningful difference because the difference in sample medians is greater than 2 IQRs.

Activity Synthesis

The purpose of the discussion is for students to understand how to adapt the general rule for determining a meaningful difference between populations to median and IQR.

Consider asking these questions for discussion:

- "Why did this problem use median and IQR instead of mean and range?" (Since the distribution for region 1 is not symmetrical, it makes more sense to use the median. Also the box plots will show the median and IQR, but there is not a good way to know the mean and range.)
- "Is there any overlap in the data from the two regions?" (Yes. The smallest percentage of carbon from the region 1 was well below the median from region 2 while the typical percentage of carbon from region 1 is much greater than from region 2.)
- "A piece of steel is found in a place between the two regions sampled. Would testing the percentage of carbon from this metal be useful in determining the region from which it came?" (Yes. Since there is a meaningful difference in the percentage of carbon in the steel from the two regions, it should give a good indication which region created the metal.)

Lesson Synthesis

Consider asking these discussion questions to emphasise the main ideas from this lesson:

- "When is it useful to use a median rather than a mean?" (It is useful when the distribution is not approximately symmetrical.)
- "What values do you need to calculate from a sample to use the general rule for determining if the measures of centre of two populations are meaningfully different?" (The measure of centre and measure of variation for each sample should be calculated to compare the groups.)
- "What is the general rule used to determine if the medians of two populations are meaningfully different?" (If the difference between the medians for the two samples is greater than twice the greater of the IQRs, then the medians are meaningfully different.)

18.5 Teachers Watching Films

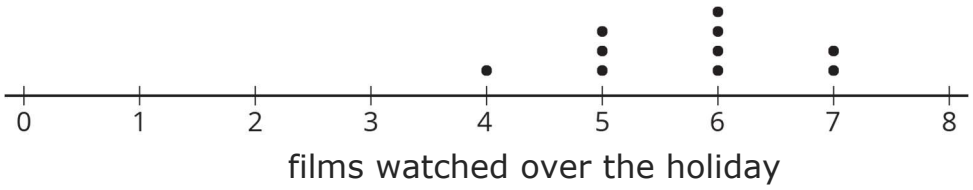
Cool Down: 5 minutes

This cool-down assesses whether students understand the general rule set out to identify whether the measures of centre for two groups are meaningfully different based on a sample of data from each group.

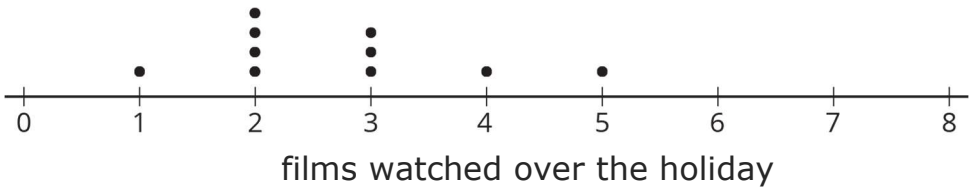
Student Task Statement

Noah is interested in comparing the number of films watched by students and teachers over the winter holiday. He takes a random sample of 10 students and 10 teachers and makes a dot plot of their responses.

Students:



Teachers:



Noah then calculates the measures of centre and variability for each group:

- Students: Mean = 5.7 films, range = 3 films

- Teachers: Mean = 2.7 films, range = 4 films

1. Is Noah's choice of mean and range appropriate for the data he has? Explain your reasoning.

2. Should Noah conclude that there is a difference in the mean number of films watched over the winter holiday between the two groups? Explain your reasoning.

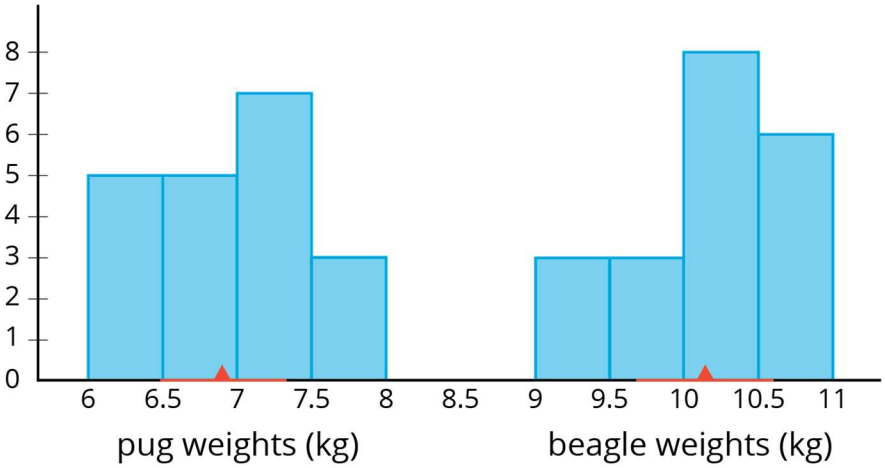
Student Response

1. Yes. Since both samples are approximately symmetrical, using the mean is a good choice.

2. Yes. Since the difference in the means is large compared with the ranges, there is a meaningful difference in the mean number of films watched.

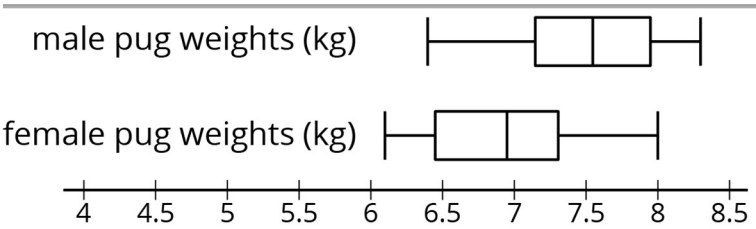
Student Lesson Summary

Sometimes we want to compare two different populations. For example, is there a meaningful difference between the weights of pugs and beagles? Here are histograms showing the weights for a sample of dogs from each of these breeds:



The red triangles show the mean weight of each sample, 6.9 kg for the pugs and 10.1 kg for the beagles. We can think of these as "typical" weights for the breed. These typical weights do not overlap. In fact, the distance between the means is $10.1 - 6.9$ or 3.2 kg, larger than either range. So, we can say there *is* a meaningful difference between the weights of pugs and beagles.

Is there a meaningful difference between the weights of male pugs and female pugs? Here are box plots showing the weights for a sample of male and female pugs:



We can see that the medians are different, but the weights between the first and third quartiles overlap. Based on these samples, we would say there is *not* a meaningful difference between the weights of male pugs and female pugs.

In general, if the measures of centre for two samples are at least two measures of variability apart, we say the difference in the measures of centre is meaningful. Visually, this means the range of typical values does not overlap. If they are closer, then we don't consider the difference to be meaningful.

Lesson 18 Practice Problems

Problem 1 Statement

Lin wants to know if students in primary school generally spend more time playing outdoors than students in KS3. She selects a random sample of size 20 from each population of students and asks them how many hours they played outdoors last week. Suppose that the ranges for each of her samples are similar.

Select **all** pairs of sample means for which Lin could conclude there is a meaningful difference between the two populations.

a. primary school: 12 hours, KS3: 10 hours

b. primary school: 14 hours, KS3: 9 hours

c. primary school: 13 hours, KS3: 6 hours

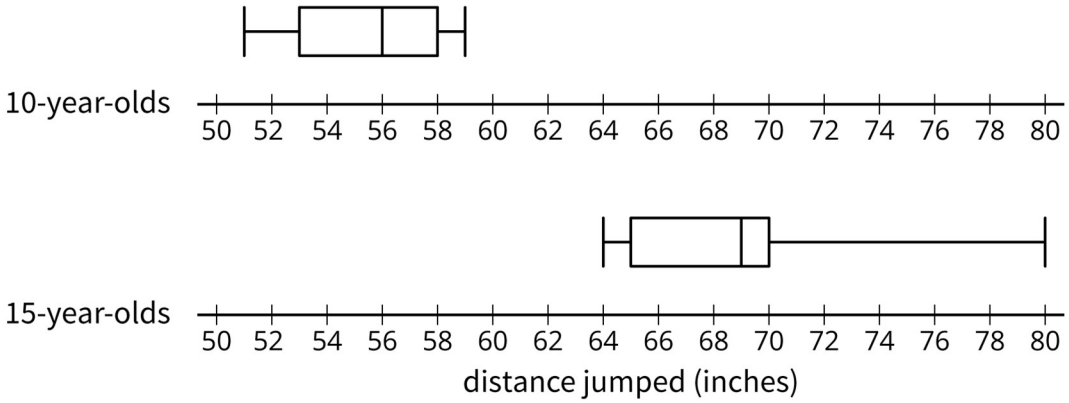
d. primary school: 13 hours, KS3: 10 hours

e. primary school: 7 hours, KS3: 15 hours

Solution ["C", "E"]

Problem 2 Statement

These two box plots show the distances of a standing jump, in inches, for a random sample of 10-year-olds and a random sample of 15-year-olds.



Is there a meaningful difference in median distance for the two populations? Explain how you know.

Solution

Yes, the difference in medians is 13 inches. This difference is more than 2 IQRs (the IQR is 5 and $13 \div 5 = 2.6$), so there is a meaningful difference in the median distances for 10-year-olds and 15-year-olds.

Problem 3 Statement

The median income for a sample of people from Bristol is about £60 000 and the median income for a sample of people from Taunton is about £46 000, but researchers have determined there is not a meaningful difference in the medians. Explain why the researchers might be correct.

Solution

The medians differ by £14 000, but if the IQR is larger than about £7 000, there will not be a meaningful difference between the median salaries in the two cities.

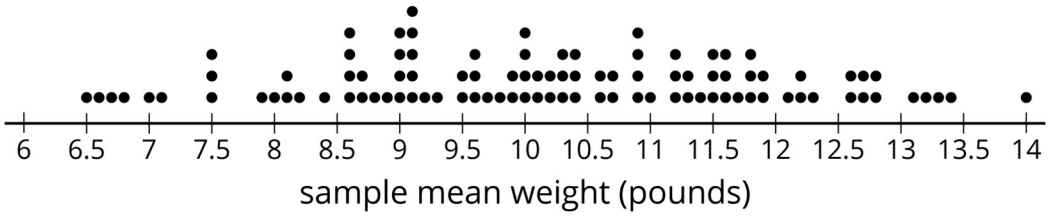
Problem 4 Statement

A farmer grows 5 000 pumpkins each year. The pumpkins are priced according to their weight, so the farmer would like to estimate the mean weight of the pumpkins he grew this year. He randomly selects 8 pumpkins and weighs them. Here are the weights (in pounds) of these pumpkins:

2.9 6.8 7.3 7.7 8.9 10.6 12.3 15.3

a. Estimate the mean weight of the pumpkins the farmer grew.

This dot plot shows the mean weight of 100 samples of eight pumpkins, similar to the one above.



b. What appears to be the mean weight of the 5 000 pumpkins?

c. What does the dot plot of the sample means suggest about how accurate an estimate based on a single sample of 8 pumpkins might be?

d. What do you think the farmer might do to get a more accurate estimate of the population mean?

Solution

a. 8.975 pounds

b. About 10 pounds

c. The sample means ranged from about 6.5 to 14 pounds. If the actual population mean is about 10 pounds, this shows that a sample mean based on a sample of size 8 might not be very close to the actual population value.

d. Use a larger sample size.