INTERNATIONAL BACCALAUREATE

Mathematics: applications and interpretation

MAI

EXERCISES [MAI 1.16]

EIGENVALUES - EIGENVECTORS

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A. Paper 1 questions (SHORT)

1. [Maximum mark: 8]

Let $\boldsymbol{A} = \begin{pmatrix} 3 & -2 \\ -3 & 4 \end{pmatrix}$ and $\boldsymbol{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(a) Find the characteristic polynomial $\det(\boldsymbol{A} - \lambda \boldsymbol{I})$ in the form $a\lambda^2 + b\lambda + c$ [2]

(b) Hence find the eigenvalues of matrix $\boldsymbol{A}$. [2]

(c) Find the corresponding eigenvectors. [4]

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2. [Maximum mark: 7]

Let $\boldsymbol{A} = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$

(a) Find the eigenvalues of matrix $\boldsymbol{A}$. [3]

(b) Find the corresponding eigenvectors. [4]

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3. [Maximum mark: 7]

Let $\boldsymbol{A} = \begin{pmatrix} 2 & 0 \\ 1 & 4 \end{pmatrix}$

(a) Find the eigenvalues of matrix $\boldsymbol{A}$. [3]

(b) Find the corresponding eigenvectors. [4]

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4. [Maximum mark: 9]

Let $\boldsymbol{M} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

(a) Find the eigenvalues of matrix $\boldsymbol{M}$. [3]

(b) Find the corresponding eigenvectors. [3]

The matrix $\boldsymbol{M}$ can be expressed in the form $\boldsymbol{M} = \boldsymbol{PDP}^{-1}$, where $\boldsymbol{D}$ is a diagonal matrix.

(c) Write down the matrices $\boldsymbol{D}$ and $\boldsymbol{P}$ [2]

(d) Write down an expression for $\boldsymbol{D}$ in terms of $\boldsymbol{P}$ and $\boldsymbol{M}$. [1]

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5. [Maximum mark: 9]

Let $\boldsymbol{A} = \begin{pmatrix} 2 & 1 \\ -3 & 2 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$

(a) Show that $\boldsymbol{A}$ has no real eigenvalues. [2]

(b) Find the eigenvalues of $\boldsymbol{B}$ in the form $a \pm \sqrt{b}$, where $a, b \in Z$. [3]

(c) Find the corresponding eigenvectors of $\boldsymbol{B}$. [4]

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B. Paper 2 questions (LONG)

6. [Maximum mark: 14]

Let $\boldsymbol{M} = \begin{pmatrix} 2 & -1 \\ -3 & 4 \end{pmatrix}$, and $\boldsymbol{O} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

(a) Given that $\boldsymbol{M}^2 - 6\boldsymbol{M} + k\boldsymbol{I} = \boldsymbol{O}$, find k. [4]

(b) Find the characteristic polynomial $\det(\boldsymbol{A} - \lambda \boldsymbol{I})$. [3]

(c) Comment on the results (a) and (b). [1]

(d) Write down the eigenvalues of $\boldsymbol{M}$. [2]

(e) Find the corresponding eigenvalues. [4]

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7. [Maximum mark: 18]

For three square matrices $\boldsymbol{A}$, $\boldsymbol{P}$ and $\boldsymbol{D}$ it is given that $\boldsymbol{A} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1}$

(a) Show that $\boldsymbol{A}^2 = \boldsymbol{P}\boldsymbol{D}^2\boldsymbol{P}^{-1}$ [2]

Let $\boldsymbol{A} = \begin{pmatrix} 3 & 2 \\ a & 6 \end{pmatrix}$

(b) Given that $\lambda = 4$ is an eigenvalue of $\boldsymbol{A}$ show that $a = -1$. [4]

(c) Find the second eigenvalue of matrix $\boldsymbol{A}$. [2]

(d) Find the corresponding eigenvectors. [4]

(e) Find a diagonal matrix $\boldsymbol{D}$ and an invertible matrix $\boldsymbol{P}$, such that $\boldsymbol{A} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1}$ [2]

(f) **Hence** show that $\boldsymbol{A}^n = \begin{pmatrix} 2\times 4^n - 5^n & 2\times 5^n - 2\times 4^n \\ 4^n - 5^n & 2\times 5^n - 4^n \end{pmatrix}$ [4]

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8. [Maximum mark: 14]

Let $\boldsymbol{M} = \begin{pmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{pmatrix}$

(a) Find the eigenvalues and the corresponding eigenvectors of matrix $\boldsymbol{M}$. [6]

(b) **Hence** show that $\boldsymbol{M}^n = \frac{1}{3}\begin{pmatrix} 1+2\times 0.7^n & 1-0.7^n \\ 2-2\times 0.7^n & 2+0.7^n \end{pmatrix}$ [6]

(c) Deduce the result of $\boldsymbol{M}^n$ as n tends to infinity. [2]

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